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Modeling and Stability Analysis of an Axially Moving Beam With Frictional Contact¹

This paper considers a moving beam in frictional contact with pads, making the system susceptible for self-excited vibrations. The equations of motion are derived and a stability analysis is performed using perturbation techniques yielding analytical approximations to the stability boundaries. Special attention is given to the interaction of the beam and the rod equations. The mechanism yielding self-excited vibrations does not only occur in moving beams, but also in other moving continua such as rotating plates, for example.
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1 Introduction

In many engineering applications, self-excited vibrations are an unwanted phenomenon. They occur when instabilities arise in a system. There can be various reasons for these instabilities. In this paper, we are particularly interested in self-excited vibrations caused by friction. An example for oscillations of this type is the squealing of disk brakes considered by some of the authors in a previous paper [1], where a discretization approach was used. In the present paper, we give a continuous approach for the traveling beam with clamped boundary conditions that is in frictional contact with two idealized brake pads. An engineering application for the model might be traveling belts, band saws, etc.

Many contributions on axially moving media can be found in the literature. A fundamental work is by Wickert and Mote [2] who investigate the moving string and the moving beam showing mathematical properties and calculating numerically the spectrum of the beam for simply supported and clamped boundary conditions. In a second paper [3], they develop a complex modal analysis for continuous systems using a first-order partial differential equation with respect to time. A similar approach has been developed for continuous systems by Meirovitch in [4], which can also be extended to continuous gyroscopic systems as done in [5]. In [6], Parker investigates the eigenvalues of gyroscopic continua in the vicinity of the critical speeds using perturbation techniques on the first-order system. In particular, he analytically calculates the critical speeds for the simply supported moving beam. In a paper by Seyranian and Kliem [7], the splitting of the double zero eigenvalues at the critical speeds of the beam is investigated using perturbation techniques directly on the operator polynomial.

In all the papers cited above, the boundary conditions were self-adjoint. Only a few authors consider the influence of nonconservative forces on axially moving continua. For example, Cheng and Perkins study the stability of a string sliding through an elastically supported dry friction guide [8]. However, in their model, the friction forces only affect the tension of the string and, therefore, no instability occurs before the first critical speed. A related problem is the stationary beam under moving frictional forces discussed in [9]. In the literature about rotating plates, more papers dealing with nonconservative forces can be found, for example, we mention [10,11].

For the traveling beam considered in our paper, the frictional contact makes the problem nonconservative and introduces intermediate transition conditions into the boundary value problem. Special attention is given to the coupling of beam and rod equations using the assumptions of the Euler-Bernoulli theory in the linear elasticity problem and taking into account the exact contact kinematics of the beam and the pads.

We obtain a distributed gyroscopic system with dissipative and nonconservative positional forces originating from the pads. Since the gyroscopic stability is highly sensitive to the influence of the dissipative and especially nonconservative positional forces (see [12–22]), the stability analysis needs more sophisticated tools than the ones used in the previous papers. Recently, Kirillov and Seyranian [23–25] developed an effective method of analyzing stability boundaries and its singularities for distributed nonconservative systems based on the bifurcation theory of eigenvalues of two point non-self-adjoint boundary value problems with the differential expression and boundary conditions depending on the spectral parameter and multiple physical parameters. We develop this approach further to study boundary value problems with intermediate transition conditions.

The outline of the paper is as follows. We first derive the model from the theory of linear elasticity using the principle of virtual work. The stability of the system is then investigated by interpreting damping and nonconservative forces as perturbations. We use a discretization approach for the stability analysis of a nonperturbed conservative gyroscopic system; then, based on numerically obtained data, we perform a perturbation analysis directly on the boundary value problem of the nonconservatively loaded beam.

2 Derivation of the Mathematical Model

We consider an axially moving Euler-Bernoulli beam sliding through two idealized massless brake pads with constant velocity $\dot{q}_0$; see Fig. 1. We introduce a spatially fixed frame with unit vectors e_x , e_y , and e_z and a frame with unit vectors $e_{\tilde{x}}$, $e_{\tilde{y}}$, and $e_{\tilde{z}}$ moving with the undeformed configuration of the beam. The beam is pretensioned with the force κ before applying the pads. As usual in Euler-Bernoulli theory, we neglect the stresses σ_y , σ_z , and τ_{yz} , and assume that the cross sections of the beam stay planar and perpendicular to the neutral plane. The mass of a cross section is assumed to be concentrated on the neutral plane and is assumed to be constant between the two supports. Each point on the neutral plane has a displacement $u(x, t)$ in the x direction and $w(x, t)$ in the z direction counted out of the prestressed configuration with no pads in contact with the beam.

¹Dedicated to Professor Franz Ziegler on the occasion of his 70th birthday.

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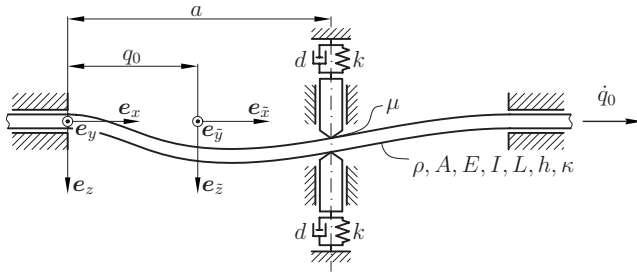


Fig. 1 Axially moving beam

To derive the equations of motion, we use the principle of virtual work with the assumptions stated above giving

$$\int_0^L \int_A \left[\rho \frac{d^2}{dt^2} \mathbf{p}_M \cdot \delta \mathbf{p}_M + (\sigma_0 + E\epsilon) \delta \epsilon \right] dA dx = \sum_i (\mathbf{F}_i \cdot \delta \mathbf{p}_i) \quad (1)$$

for an extensible Euler-Bernoulli beam, where

$$\epsilon = u' + \frac{1}{2} w'^2 - zw'' \quad (2)$$

is the strain and σ_0 is the pretension of the beam. The forces $\mathbf{F}_i$ are the contact forces between the pads and the beam and $\delta \mathbf{p}_i$ are the virtual velocities/displacements of the contact points on the beam. In order to calculate the contact forces, we have to consider the contact kinematics.

2.1 Kinematics. A point on the neutral fiber is

$$\mathbf{p}_M(x, t) = [x + u(x, t)] \mathbf{e}_x + w(x, t) \mathbf{e}_z \quad (3)$$

where $x = q_0(t)$ due to the kinematic constraint. When differentiating $\mathbf{p}_M(x, t)$ with respect to time, we therefore have

$$\frac{d}{dt} \mathbf{p}_M(x, t) = [\dot{q}_0 + u_t(x, t) + \dot{q}_0 u_x(x, t)] \mathbf{e}_x + [w_t(x, t) + \dot{q}_0 w_x(x, t)] \mathbf{e}_z \quad (4)$$

Since cross sections stay planar and perpendicular to the neutral plane, it is possible to describe points on the upper surface of the beam through points on the neutral fiber, which is parametrized by

$$f(x, z, t) = z - w(x, t) = 0 \quad (5)$$

The position vector of a point on the upper surface of the beam is given by

$$\mathbf{p}(x, t) = [x + u(x, t)] \mathbf{e}_x + w(x, t) \mathbf{e}_z - \frac{h}{2} \mathbf{e}_y(x, t) \quad (6)$$

where

$$\mathbf{e}_y(x, t) = \frac{\nabla f(x, z, t)}{|\nabla f(x, z, t)|} \quad (7)$$

is the gradient vector to the point x on the neutral plane. The position vector of the point currently in contact with the upper pad is given by

$$(x_p + u(x_p, t)) \mathbf{e}_x + w(x_p, t) \mathbf{e}_z - \frac{h}{2} \mathbf{e}_y(x_p, t) \quad (8)$$

where since we work in Lagrangian coordinates $x_p = a + \Delta x_p$ is the position of the point on the neutral plane corresponding to the point currently in contact with the upper pad, as shown in Fig. 2.

From geometrical considerations, it is seen that $\Delta x_p = \bar{\Delta} x_p + u(a + \Delta x_p, t)$, where

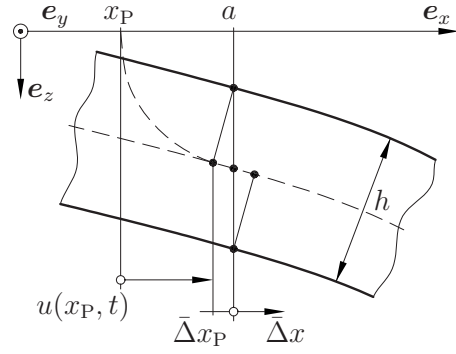


Fig. 2 Contact kinematics

$$\bar{\Delta} x_p = -\frac{h}{2} \sin[\arctan w'(a + \bar{\Delta} x_p, t)] \quad (9)$$

$$= -\frac{h}{2} \frac{w'(a + \bar{\Delta} x_p, t)}{\sqrt{1 + w'(a + \bar{\Delta} x_p, t)^2}} \quad (10)$$

that is a fixed point equation of the type

$$\bar{\Delta} x_p^{k+1} = g(\bar{\Delta} x_p^k) \quad \bar{\Delta} x_p^0 = 0 \quad (11)$$

Since $|w'(x, t)| \ll 1$ we obtain

$$|g(x) - g(y)| = \left| \frac{h}{2} \frac{w'(x, t)}{\sqrt{1 + w'(x, t)^2}} - \frac{h}{2} \frac{w'(y, t)}{\sqrt{1 + w'(y, t)^2}} \right| < 1 \quad (12)$$

and therefore the mapping g is contracting. The Banach fixed point theorem is therefore applicable and $\bar{\Delta} x_p$ can be iteratively determined to arbitrary precision. The quantity Δx_p is now determined from $\Delta x_p = \bar{\Delta} x_p + u(a + \Delta x_p, t)$, which is also a fixed point equation that can be solved using the Banach fixed point theorem. For the lower contact point, we proceed similarly. The position vectors and, hence, the virtual velocities of the contact points, can thus be determined to arbitrary precision.

2.2 Contact Forces. The contact forces between the surface of the beam and the pads have already been stated in [1], but are restated here for the convenience of the reader.

The normal force is perpendicular to the surface of the beam and is therefore given by:

$$\mathbf{N}_p = -N_p \mathbf{e}_\tau(x_p, t) \quad (13)$$

We assume the applicability of Coulomb's law of friction, and therefore the friction force has the magnitude $R_p = \mu N_p$, and is directed against the relative velocity between the point P on the beam and the point $\bar{P}$ on the pad (see Fig. 3), so that

$$\mathbf{R}_p = -R_p \frac{\mathbf{v}_{\bar{P}} - \mathbf{v}_P}{|\mathbf{v}_{\bar{P}} - \mathbf{v}_P|} \quad (14)$$

Throughout the paper we exclude stick slip, which in the linear case, is assured by the condition

$$|\mathbf{v}_{\bar{P}} - \mathbf{v}_P| > 0 \Rightarrow |\dot{u}(a, t)| + |\dot{w}(a, t)| \frac{h}{2} < \dot{q}_0 \quad (15)$$

From a force balance at the upper pad

$$0 = N_p \cos \varphi - R_p \sin \varphi - N_0 + k \bar{z}_p + d \dot{\bar{z}}_p \quad (16)$$

where $\bar{z}_p$ is the vertical displacement of the point $\bar{P}$ on the pad, with

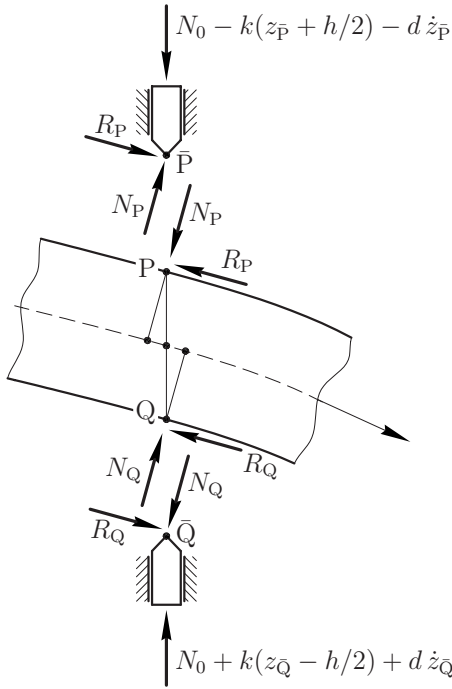


Fig. 3 Contact forces

$$\varphi = \arctan[w'(x_P, t)] \quad (17)$$

we can now calculate the magnitude of N_P , so that the contact forces are completely defined. Since in Euler-Bernoulli theory the cross sections of the beam stay planar, we can replace N_P and R_P by an equivalent system of loads consisting of the force $B_P e_x + A_P e_z$ and a torque M_P , both acting on the neutral fiber. Again, we can proceed similarly for the lower contact point Q. The linearized expressions for the forces and torques are then

$$B_P = -\mu(N_0 - kw(x_P, t) - d\dot{w}(x_P, t)) - N_0(1 + \mu^2)w'(x_P, t) \quad (18)$$

$$A_P = N_0 - kw(x_P, t) - d\dot{w}(x_P, t) \quad (19)$$

$$M_P = \frac{1}{2}h[\mu^2 N_0 w'(x_P, t) + \mu N_0 - k\mu w(x_P, t) - d\mu \dot{w}(x_P, t)] \quad (20)$$

$$B_Q = -\mu[N_0 + kw(x_Q, t) + d\dot{w}(x_Q, t)] + N_0(1 + \mu^2)w'(x_Q, t) \quad (21)$$

$$A_Q = -N_0 - kw(x_Q, t) - d\dot{w}(x_Q, t) \quad (22)$$

$$M_Q = \frac{1}{2}h[\mu^2 N_0 w'(x_Q, t) - \mu N_0 - k\mu w(x_Q, t) - d\mu \dot{w}(x_Q, t)] \quad (23)$$

2.3 Boundary Value Problem. Due to the contact forces, the derivatives w'' and w''' will not be continuous at the points x_P and x_Q , i.e., their left and right limits do not coincide; for example $w''(x_P^-, t) \neq w''(x_P^+, t)$. Therefore, we have to consider three different segments of the beam as shown in Fig. 4.

Carrying out the variations in (1) requiring that the functions satisfy the geometric boundary conditions and applying the main theorem of variational calculus, we obtain a boundary value problem for the beam in w

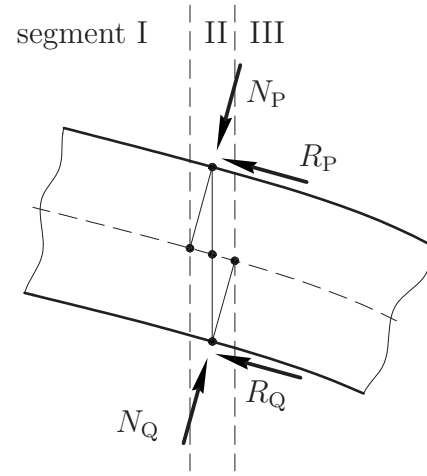


Fig. 4 Segments of the beam

$$\rho A \left[\ddot{w}(x, t) + 2\dot{q}_0 \dot{w}'(x, t) + \left(\dot{q}_0^2 - \frac{\kappa}{\rho A} \right) w''(x, t) \right] + EI w^{IV}(x, t) = 0 \quad (24)$$

with boundary conditions

$$w(0, t) = w(L, t) = 0 \quad w'(0, t) = w'(L, t) = 0 \quad (25)$$

and transition conditions

$$A_{P,Q} + EI[w'''(x_{P,Q}^-, t) - w'''(x_{P,Q}^+, t)] = 0 \quad (26)$$

$$M_{P,Q} + EI[w''(x_{P,Q}^-, t) - w''(x_{P,Q}^+, t)] = 0 \quad (27)$$

Furthermore, we get a boundary value problem for the rod in u , which consists of the partial differential equation

$$\rho A \ddot{u}(x, t) + 2\dot{q}_0 \rho A \dot{u}'(x, t) + (\dot{q}_0^2 \rho A - EA)u''(x, t) = 0 \quad (28)$$

with boundary conditions

$$u(0, t) = u(L, t) = 0 \quad (29)$$

and transition conditions

$$EA[u'(x_{P,Q}^-, t) - u'(x_{P,Q}^+, t)] - B_{P,Q} = 0 \quad (30)$$

Two important facts are to be noted: first of all, from Eqs. (19) and (20) we observe that u does not occur in the boundary value problem of the beam and the beam equations can be solved independently of the boundary value problem for the rod. However, vibrations of the beam very well excite the rod as is seen from (18) and (21). That means the stability behavior of the system is determined by the beam equations, at least beyond the critical speed for the rod. At a first view, this one-sided coupling might be surprising, since in conservative problems, this phenomenon cannot occur. The present problem is, however, nonconservative due to the friction forces and it can be seen that the coupling vanishes for $\mu=0$. The second fact to be noted is that the boundary value problems are nonconservative because of the transition conditions through which the contact forces between beam and pads enter the system.

Since segment II of the beam is very small, it is possible to simplify the transition conditions by expanding terms containing x_P or x_Q around $x=a$; for example

$$u'(x_P^+, t) = u'(a, t) + \mathcal{O}(w^2) \quad (31)$$

The transition conditions at x_P and x_Q then simplify to a single transition condition at $x=a$. The transition conditions (26)–(30) are thus replaced by

$$A + EI[w'''(a^-, t) - w'''(a^+, t)] = 0 \quad (32)$$

$$M + EI[w''(a^-, t) - w''(a^+, t)] = 0 \quad (33)$$

$$B - EA[u'(a^-, t) - u'(a^+, t)] = 0 \quad (34)$$

where $A = A_P + A_Q = -2kw(a, t) - 2d\dot{w}(a, t)$, $M = M_P + M_Q = h[N_0\mu^2 w'(a, t) - k\mu w(a, t) - d\mu\dot{w}(a, t)]$ and $B = B_P + B_Q = -2\mu N_0$. The simplification of the transition conditions therefore leads to a complete uncoupling of the boundary value problems for the beam and the rod.

2.4 Discretization. In [1], the authors investigated the traveling beam using a Ritz discretization approach. We will use these discretized equations and compare them to the results obtained from the continuous approach taken in this paper. Using the Ritz expansion in (1)

$$w(x, t) = \sum_{i=1}^I W_i(x) q_i(t) \quad (35)$$

yields nonlinear equations of motion $\ddot{q} = f(q, \dot{q})$, which can be linearized to

$$M\ddot{q} + (G + D)\dot{q} + Kq = 0 \quad M = M^T \quad G = -G^T \quad D \neq D^T \quad K \neq K^T \quad (36)$$

where

$$m_{ji} = \rho A \int_0^L W_j W_i dx \quad (37)$$

$$g_{ji} = \rho A \dot{q}_0 \int_0^L (W_j W'_i - W'_j W_i) dx \quad (38)$$

$$d_{ji} = d[2W_j(a)W_i(a) + h\mu W'_j(a)W_i(a)]$$

$$k_{ji} = (\kappa - \rho A \dot{q}_0^2) \int_0^L W'_j W'_i dx + EI \int_0^L W''_j W''_i dx + 2kW_j(a)W_i(a) - h\kappa\mu W'_j(a)W_i(a) + hN_0(1 + \mu^2)W'_j(a)W'_i(a) - \frac{h^2 N_0 \mu}{2} W''_j(a)W'_i(a) \quad (39)$$

Using a result of Karapetjan [16] and Lakhadanov [19] in [1], it was concluded that in the undamped case $d=0$, the stability domain of the nonconservative gyroscopic system is a set of measure zero in the space of the system parameters. Provided that the Ritz expansion converges to the solution, which is the case choosing appropriate shape functions, this result carries over to the continuous problem. It would now be feasible to perform a perturbation analysis of these discretized equations. However, we prefer to perform a perturbation analysis directly on the continuous system, using the discretization only to calculate the spectrum of the unperturbed problem.

Before continuing with the investigation of the unperturbed problem, we draw the reader's attention to a difference between the continuous approach used to derive the boundary value problem, and the Ritz discretization approach taken from [1]. From the derivation of the simplified boundary value problem in Sec. 2.3, it is clear that, since $\Delta x_{P,Q}$ occur only in the arguments of u and w , and are expandable in u and w without having a constant part, they do in fact not enter the simplified boundary value problem. However, when considering the virtual work of the contact forces in the Ritz expansion, we calculated

$$\delta W_A = A_P \sum_{i=1}^I W_i(a + \Delta x_P) \delta q_i(t) + A_Q \sum_{i=1}^I W_i(a + \Delta x_Q) \delta q_i(t) \quad (40)$$

where we expanded

$$W_i(a + \Delta x_P) = W_i(a) + W'_i(a) \Delta x_P + \dots = W_i(a) + W'_i(a) \frac{h}{2} \sum_{j=1}^I (W'_j(a) q_j + \mathcal{O}(q_j^2)) \quad (41)$$

without considering the stretching of the beam. Due to the constant terms in A_P and A_Q , we get terms of the form $hN_0 W'_j(a) W'_i(a)$ in the discretized equation of motion, that would not have shown up neglecting $\Delta x_{P,Q}$. Similar terms arise from $M_{P,Q}$. The explanation for this lies in the fact that in the Ritz discretization, the energy expressions were considered up to second order in the q_i , whereas to derive the boundary value problem, a purely geometric linearization was performed with respect to w and u . To get comparable results from the perturbation approach on the discretized system, it would therefore be appropriate to neglect the influence of $\Delta x_{P,Q}$ and perform a geometric linearization. Since we only need the discretization for the unperturbed problem, the corresponding equations are not stated separately.

3 Stability Analysis

In this section, we perform a stability analysis of the beam with the simplified transition conditions (32) and (33). Since the transverse vibrations of the beam are of major interest in applications, we concentrate on their investigation.

We assume that as in many squeal problems, the friction and damping forces coming from the pads are small compared to inertia, gyroscopic, and restoring terms. Therefore, we multiply all forces coming from the pads with δ if they come from damping in the pad, and with γ otherwise, with γ and δ serving as weights for their contribution. Introducing the dimensionless time $\tau = \omega t$ and length $\bar{x} = x/L$ where $\omega^2 = EI/\rho AL^4$ yields the dimensionless parameters

$$\bar{a} = \frac{a}{L} \quad \bar{\rho} = \frac{\dot{q}_0}{L\omega} \quad \bar{h} = \frac{h}{L} \quad \bar{k} = \frac{kL^3}{EI} \quad \bar{d} = \frac{\omega dL^3}{EI} \quad \bar{N}_0 = \frac{N_0 L^2}{EI} \quad \bar{\kappa} = \frac{\kappa L^2}{EI} \quad (42)$$

Using the ansatz $w(\bar{x}, t) = w(\bar{x})e^{\lambda t}$ where, after separation of time, we use the same symbol w for notational simplicity, the boundary value problem can now be stated as

$$L(w) = \lambda^2 w + 2\bar{\rho}\lambda w' + (\bar{\rho}^2 - \bar{\kappa})w'' + w^{IV} = 0 \quad (43)$$

where L is a linear differential operator with boundary and transition conditions

$$U_1(w) = U_1^0(w) + \gamma U_1^1(w) + \delta U_1^{\delta}(w) = 0, \dots, U_8(w) = U_8^0(w) + \gamma U_8^1(w) + \delta U_8^{\delta}(w) = 0 \quad (44)$$

where

$$U_i^0(w) = \alpha_0 w(0) + \alpha_1 w'(0) + \dots + \alpha_7 w^{(7)}(1)$$

$$U_i^{1,\gamma,\delta}(w) = \beta_0^{\gamma,\delta} w(\bar{a}^-) + \beta_1^{\gamma,\delta} w'(\bar{a}^-) + \dots + \beta_7^{\gamma,\delta} w^{(7)}(\bar{a}^+) \quad (45)$$

are linear forms in $w(\bar{x})$ and its derivatives taken at $\bar{x}=0$, $\bar{x}=1$, $\bar{x}=\bar{a}^-$, and $\bar{x}=\bar{a}^+$. In our case, they read

$$U_1^0 = w(0) \quad U_2^0 = w'(0) \quad U_3^0 = w(\bar{a}^-) - w(\bar{a}^+) \quad (46)$$

$$U_4^0 = w'(\bar{a}^-) - w'(\bar{a}^+) \quad U_5^0 = w''(\bar{a}^-) - w''(\bar{a}^+) \quad U_6^0 = w'''(\bar{a}^-) - w'''(\bar{a}^+) \quad (47)$$

$$U_7^0 = w(1) \quad U_8^0 = w'(1) \quad (48)$$

for the unperturbed problem, and the perturbations are given by

$$U_5^{1\gamma} = -\bar{h}\bar{k}\mu w(\bar{a}) + \bar{h}\bar{N}_0\mu^2 w'(\bar{a}) \quad U_5^{1\delta} = -\bar{h}\bar{d}\mu\lambda w(\bar{a}) \quad (49)$$

$$U_6^{1\gamma} = -2\bar{k}w(\bar{a}) \quad U_6^{1\delta} = -2\bar{d}\lambda w(\bar{a}) \quad (50)$$

other forms $U_i^{1\gamma,\delta}$ being zero.

We now derive the boundary value problem adjoint to (43) and (44). We multiply the differential expression $L(w)$ by a function $v(\bar{x})$ and integrate over the intervals $(0, \bar{a})$ and $(\bar{a}, 1)$ since we are planning to use integration by parts and some of the derivatives of $w(\bar{x})$ are not continuous at $\bar{x}$. Using the notation $\langle w, v \rangle = \int_0^{\bar{a}-} w\bar{v}d\bar{x} + \int_{\bar{a}+}^1 w\bar{v}d\bar{x}$, $\bar{v}$ being the complex conjugate to v , we obtain

$$\langle L(w), v \rangle = \langle w, L^*(v) \rangle + \sum_{i=1}^{16} U_i(w) V_{17-i}(v) \quad (51)$$

where we expressed the boundary terms and terms occurring at $\bar{x}=\bar{a}^-$ and $\bar{x}=\bar{a}^+$ in terms of linear forms $U_1, \dots, U_8$, which are, in fact, the boundary and transition conditions, and supplementary forms $U_9, \dots, U_{16}$, such that we can express all of the boundary and transition terms uniquely in the U_i .

From (51) we obtain the differential equation for the adjoint problem

$$L^*(v) = \bar{\lambda}^2 v - 2\bar{\rho}\bar{\lambda}v' + (\bar{\rho}^2 - \bar{\kappa})v'' + v^{IV} = 0 \quad (52)$$

with boundary conditions

$$V_1(v) = V_1^0(v) + \gamma V_1^{1\gamma}(v) + \delta V_1^{1\delta}(v) = 0, \dots, V_8(v) = V_8^0(v) + \gamma V_8^{1\gamma}(v) + \delta V_8^{1\delta}(v) = 0$$

where

$$V_1^0 = v(1) \quad V_2^0 = -v'(1) \quad V_3^0 = v(\bar{a}^-) - v(\bar{a}^+) \quad V_4^0 = -v'(\bar{a}^-) + v'(\bar{a}^+) \quad (53)$$

$$V_5^0 = (\bar{\rho}^2 - \bar{\kappa})[v(\bar{a}^-) - v(\bar{a}^+)] + v''(\bar{a}^-) - v''(\bar{a}^+) \quad (54)$$

$$V_6^0 = -(\bar{\rho}^2 - \bar{\kappa})[v'(\bar{a}^-) - v'(\bar{a}^+)] - [v'''(\bar{a}^-) - v'''(\bar{a}^+)] + 2\bar{\rho}\bar{\lambda}[v(\bar{a}^-) - v(\bar{a}^+)] \quad (55)$$

$$V_7^0 = -v(0) \quad V_8^0 = v'(0) \quad (56)$$

and $V_i^{1\gamma,\delta}$ are skipped since for our analysis, only the adjoint to the unperturbed problem is required. The supplementary expressions read

$$V_9^0 = (\bar{\rho}^2 - \bar{\kappa})v(1) + v''(1) \quad (57)$$

$$V_{10}^0 = -(\bar{\rho}^2 - \bar{\kappa})v'(1) - v'''(1) + 2\bar{\rho}\bar{\lambda}v(1) \quad V_{11}^0 = v(\bar{a}^+) \quad (58)$$

$$V_{12}^0 = -v'(\bar{a}^+) \quad V_{13}^0 = (\bar{\rho}^2 - \bar{\kappa})v(\bar{a}^+) + v''(\bar{a}^+) \quad (59)$$

$$V_{14}^0 = -(\bar{\rho}^2 - \bar{\kappa})v'(\bar{a}^+) - v'''(\bar{a}^+) + 2\bar{\rho}\bar{\lambda}v(\bar{a}^+) \quad (60)$$

$$V_{15}^0 = -(\bar{\rho}^2 - \bar{\kappa})v(0) - v''(0)$$

$$V_{16}^0 = (\bar{\rho}^2 - \bar{\kappa})v'(0) + v'''(0) + 2\bar{\rho}\bar{\lambda}v(0) \quad (61)$$

3.1 Spectrum of the Unperturbed Problem. The unperturbed problem $\gamma=\delta=0$ is similar to the problem studied by Wickert and Mote in [2]. It can be written as an operator polynomial

$$L(u) = \lambda^2 M(u) + \lambda G(u) + K(u) = \lambda^2 u + 2\bar{\rho}\bar{\lambda}u' + (\bar{\rho}^2 - \bar{\kappa})u'' + u^{IV} = 0 \quad (62)$$

$$M = 1 \quad G = 2\bar{\rho}\frac{\partial}{\partial\bar{x}} \quad K = (\bar{\rho}^2 - \bar{\kappa})\frac{\partial^2}{\partial\bar{x}^2} + \frac{\partial^4}{\partial\bar{x}^4} \quad (63)$$

with boundary conditions

$$u(0) = u(1) = 0 \quad u'(0) = u'(1) = 0 \quad (64)$$

We now use the function u for the unperturbed problem, which is not to be confused with the axial displacement appearing in the rod equations. Note that we consider clamped boundary conditions in contrast to the simply supported boundary conditions studied in [2,6,7].

Because of the boundary conditions (64) the operators M and K are symmetric and the quantities $\langle M(u), u \rangle$ and

$$\begin{aligned} \langle K(u), u \rangle &= \int_0^1 [(\bar{\rho}^2 - \bar{\kappa})u'' + u^{IV}]\bar{u}d\bar{x} \\ &= \int_0^1 [-(\bar{\rho}^2 - \bar{\kappa})u'\bar{u}' + u''\bar{u}'']d\bar{x} \end{aligned} \quad (65)$$

are real, where $\bar{u}$ is the complex conjugate of u . The operator $G(u)$ is skew symmetric i.e., $\langle G(u), u \rangle = -\langle u, G(u) \rangle$ is a purely imaginary quantity.

The eigenvalues of the unperturbed boundary value problem are found setting $u(\bar{x}) = e^{\beta\bar{x}}$, which yields four different solutions β_i depending on λ . Hence the eigenfunctions of the unperturbed problem are given by

$$u(\bar{x}) = A_1 e^{\beta_1\bar{x}} + A_2 e^{\beta_2\bar{x}} + A_3 e^{\beta_3\bar{x}} + A_4 e^{\beta_4\bar{x}} \quad (66)$$

where the A_i are constants. To determine the A_i , we substitute $u(\bar{x})$ into the four boundary conditions, which yields a linear homogeneous system of equations $\mathbf{U}(\lambda)\mathbf{A} = \mathbf{0}$. For nontrivial solutions, the determinant of the coefficient matrix has to vanish. The values λ thus obtained are the eigenvalues of the problem (see [26], [27]). Having calculated the eigenvalues, the eigenfunctions $u(\bar{x})$ can be calculated. Eigenfunctions corresponding to different eigenvalues are linearly independent. If an eigenvalue of multiplicity m occurs and there are $p \leq m$ linearly independent eigenfunctions, then to every eigenfunction $u^k(\bar{x})$, $k=1, \dots, p$ corresponds a Jordan (Keldysh) chain $u^k \rightarrow u_1^k, \dots, u_{m_k}^k$ of linearly independent associated functions $u_1^k(\bar{x}), \dots, u_{m_k}^k(\bar{x})$ defined by

$$L(u^k) = 0 \quad (67)$$

$$L(u_1^k) + \frac{1}{1!} \frac{\partial L}{\partial \lambda}(u^k) = 0 \quad (68)$$

$$\dots \quad (69)$$

$$L(u_{m_k}^k) + \frac{1}{1!} \frac{\partial L}{\partial \lambda}(u_{m_k-1}^k) + \dots + \frac{1}{m_k!} \frac{\partial^{m_k} L}{\partial \lambda^{m_k}}(u^k) = 0 \quad (70)$$

with $m_1 + \dots + m_p = m - p$ [27].

Taking the scalar product of $L(u)$ with an eigenfunction of the adjoint problem, which is easily seen to be $v=u$, for purely imaginary eigenvalues λ we obtain

$$\langle L(u), u \rangle = \lambda^2 \langle M(u), u \rangle + \lambda \langle G(u), u \rangle + \langle K(u), u \rangle = 0 \quad (71)$$

from which we get (see e.g., [15])

$$\lambda = \frac{-\langle G(u), u \rangle \pm \sqrt{\Delta}}{2\langle M(u), u \rangle} \quad (72)$$

$$\Delta = \langle G(u), u \rangle^2 - 4\langle K(u), u \rangle \langle M(u), u \rangle \quad (73)$$

Note that only one of the λ in (72) is an eigenvalue of the system. It can, however, still be seen from (72) that on the divergence boundary $\langle K(u), u \rangle = 0$ holds, and that on the flutter boundary we

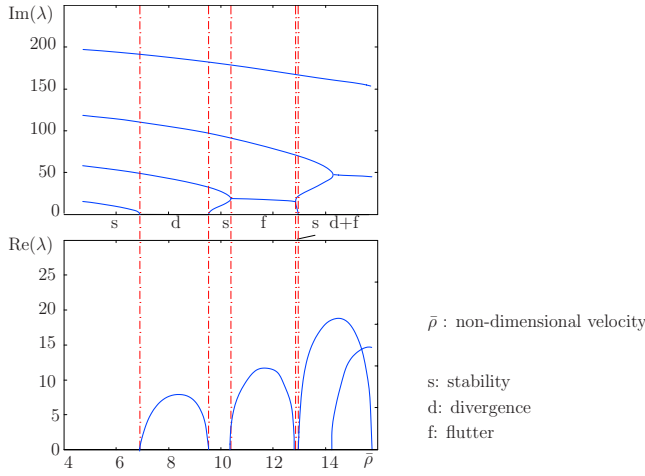


Fig. 5 Eigenvalue curves for $\bar{\kappa}=3\pi$

have $\Delta=0$. Note that for $\Delta=0$, it follows from (72) that

$$2\langle M(u), u \rangle \lambda + \langle G(u), u \rangle = 0 \quad (74)$$

which is, in fact, the necessary and sufficient condition for existence of the associated function u_1 following from (68), and its analog is known in aero-elasticity problems [28] as flutter condition. This reflects the fact (see e.g., [29]) also clear from the perturbation formulas derived in Secs. 3.2.1 and 3.2.2 that generically on the stability boundary of a gyroscopic system, we always find a double eigenvalue with a Jordan chain [30].

We can also observe the fact known for discrete systems (see e.g., [15]) that the gyroscopic system has to pass the divergence boundary before it can experience flutter, since from (72) it is seen that flutter can only occur with a negative definite stiffness operator.

Since the characteristic equation is highly nonlinear, we prefer to use the discretization approach of the previous section to calculate the eigenvalues of the problem. With $\delta=\gamma=0$, we obtain the eigenvalue curves shown in Fig. 5, which qualitatively agree with the curves obtained in [2].

In the special case that $\bar{\kappa}=0$, we can calculate the critical speeds analytically by substituting $\bar{\kappa}=0$ in (62) and using (66). For nontrivial solutions, we obtain the condition

$$0 = \sin \frac{\bar{\rho}}{2} \left(\sin \frac{\bar{\rho}}{2} - \frac{\bar{\rho}}{2} \cos \frac{\bar{\rho}}{2} \right) \quad (75)$$

which is satisfied for $\bar{\rho}=2\pi n$ and $\bar{\rho}/2 = \tan \bar{\rho}/2$. At the first critical speed $\bar{\rho}=2\pi$, we have a double zero eigenvalue with a Jordan chain. For $\bar{\kappa}=0$, we can also analytically obtain the corresponding eigenfunction

$$u(\bar{x}) = C_1 [1 - \cos(2\pi\bar{x})] \quad (76)$$

and the associated function

$$u_1(\bar{x}) = C_2 \left(1 + \frac{C_1}{2\pi} - \cos(\pi\bar{x}) - \frac{5}{2\pi^2} \sin(\pi\bar{x}) \right) \quad (77)$$

where C_1 and C_2 are undetermined constants. For $\bar{\kappa} \neq 0$, we proceed similarly; the equations are lengthier and are not stated here.

3.2 Perturbation Analysis of the Nonconservative Problem. To analyze the stability of the gyroscopic system with dissipative and nonconservative positional forces, we use the approach of [23–25] based on the perturbation theory of Vishik and Lyusternik [31]. We will study the stability domains for the system (43), (44) depending on the parameters $\bar{\rho}$, δ , and γ . We perturb the system around a fixed speed i.e., $\bar{\rho}=\bar{\rho}+\nu$ and assume that the small parameters ν , δ , and γ are smooth functions of the

parameter ε . This corresponds to a variation along a smooth curve parameterized by ε in the parameter space. It is possible to expand $\nu(\varepsilon)$, $\delta(\varepsilon)$, and $\gamma(\varepsilon)$ around $\varepsilon=0$, assuming that $\nu(0)=\delta(0)=\gamma(0)=0$, for example,

$$\gamma(\varepsilon) = \frac{d\gamma(0)}{d\varepsilon} \varepsilon + \dots = \gamma_1 \varepsilon + \dots \quad (78)$$

$$\delta(\varepsilon) = \frac{d\delta(0)}{d\varepsilon} \varepsilon + \dots = \delta_1 \varepsilon + \dots \quad (79)$$

$$\nu(\varepsilon) = \frac{d\nu(0)}{d\varepsilon} \varepsilon + \dots = \nu_1 \varepsilon + \dots \quad (80)$$

Assuming this kind of perturbation, we write the boundary value problem perturbed up to first order in ε as

$$L(w) + \varepsilon L^{1\varepsilon}(w) = 0 \quad (81)$$

$$U_1(w) = U_1^0(w) + \varepsilon U_1^\varepsilon(w) = 0, \dots, U_8(w) = U_8^0(w) + \varepsilon U_8^\varepsilon(w) = 0 \quad (82)$$

where $L(w)$ is defined by (43):

$$L^{1\varepsilon}(w) = \nu_1(2\lambda w' + 2\tilde{\rho} w'') \quad (83)$$

$U_i^0(w)$ are defined by (46)–(48) and

$$U_i^{1\varepsilon} = \gamma_1 U_i^{1\gamma} + \delta_1 U_i^{1\delta} \quad (84)$$

with $U_i^{1\gamma, \delta}$ defined in (49) and (50).

It is known that generically only simple and double eigenvalues occur in the spectrum of a one-parameter gyroscopic system [30], as we can also see from Fig. 5. In the following, we expand the eigenvalues of the perturbed problem in a series, depending on the Jordan structure of the eigenvalue of the unperturbed problem. The leading terms of these expansions are analytical approximations to the stability boundaries of the system in the parameter space.

3.2.1 Perturbation of Simple Eigenvalues. According to [31], for a simple eigenvalue λ_0 and the corresponding eigenfunction $u(\bar{x})$, we set

$$w(\bar{x}) = u(\bar{x}) + \varepsilon w_1^\varepsilon(\bar{x}) + \dots \quad (85)$$

$$\lambda = \lambda_0 + \varepsilon \lambda_1^\varepsilon + \dots = \lambda_0 + \varepsilon (\gamma_1 \lambda_1^\gamma + \delta_1 \lambda_1^\delta + \nu_1 \lambda_1^\nu) + \dots \quad (86)$$

Taking into account the dependence of $L(w)$ on λ , we frequently use the notation

$$L(w) = [L(\lambda)](w) \quad (87)$$

Substitution into (81) yields

$$[L(\lambda_0 + \varepsilon \lambda_1^\varepsilon + \dots) + \varepsilon L^{1\varepsilon}(\lambda_0 + \varepsilon \lambda_1^\varepsilon + \dots)](u + \varepsilon w_1^\varepsilon + \dots) = 0 \quad (88)$$

where we can write

$$\begin{aligned} [L(\lambda_0 + \varepsilon \lambda_1^\varepsilon + \dots)](u) &= [L(\lambda_0)](u) + \varepsilon \lambda_1^\varepsilon \left[\frac{\partial L}{\partial \lambda}(\lambda_0) \right](u) + \dots \\ &= L(u) + \varepsilon \lambda_1^\varepsilon \frac{\partial L}{\partial \lambda}(u) + \dots \end{aligned} \quad (89)$$

Proceeding similarly with $L^{1\varepsilon}$, we arrive at

$$L(u) + \varepsilon \left[\lambda_1^\varepsilon \frac{\partial L}{\partial \lambda}(u) + L(w_1^\varepsilon) + L^{1\varepsilon}(u) \right] + \dots = 0 \quad (90)$$

Similarly, we get

$$U_i^0(u) + \varepsilon \left[\lambda_1^\varepsilon \frac{\partial U_i^0}{\partial \lambda}(u) + U_i^0(w_1^\varepsilon) + U_i^{1\varepsilon}(u) \right] + \cdots = 0 \quad (91)$$

Setting equal to zero the terms of same powers of ε and multiplying with the eigenfunction of the unperturbed adjoint problem v , we obtain

$$\lambda_1^\varepsilon \left\langle \frac{\partial L}{\partial \lambda}(u), v \right\rangle + \langle L(w_1^\varepsilon), v \rangle + \langle L^{1\varepsilon}(u), v \rangle = 0 \quad (92)$$

Using $\langle L(w_1^\varepsilon), v \rangle = \langle w_1^\varepsilon, L^*(v) \rangle + \sum_{i=1}^n U_i^0(w_1^\varepsilon) V_{n+1-i}^0(v)$, where $n=16$ for our particular problem, and

$$U_i^0(w_1^\varepsilon) = -\lambda_1^\varepsilon \frac{\partial U_i^0}{\partial \lambda}(u) - U_i^{1\varepsilon}(u) \quad (93)$$

we obtain a formula for λ_1^ε , first derived by Kirillov and Seyranian in [23–25] for general two-point non-self-adjoint boundary value problems smoothly depending on the spectral parameter and a vector of physical parameters:

$$\lambda_1^\varepsilon = - \frac{\langle L^{1\varepsilon}(u), v \rangle - \sum_{i=1}^n U_i^{1\varepsilon}(u) V_{n+1-i}^0(v)}{\left\langle \frac{\partial L}{\partial \lambda}(u), v \right\rangle - \sum_{i=1}^n \frac{\partial U_i^0}{\partial \lambda}(u) V_{n+1-i}^0(v)} \quad (94)$$

Substituting $L^{1\varepsilon} = \gamma_1 L^{1\gamma} + \delta_1 L^{1\delta}$ and $U_i^{1\varepsilon} = \gamma_1 U_i^{1\gamma} + \delta_1 U_i^{1\delta}$, using (49), (50), and (53)–(61), and taking into account that the eigenfunctions and their first derivatives are continuous (i.e., for example $u(\bar{a}^+) = u(\bar{a}^-) = u(\bar{a})$), we calculate λ_1^γ , λ_1^δ , and λ_1^ν , which for our problem, read

$$\lambda_1^\gamma = \frac{-2\bar{k}u(\bar{a})\bar{v}(\bar{a}) + (\bar{h}\bar{k}\mu u(\bar{a}) - \bar{h}\bar{N}_0\mu^2 u'(\bar{a}))\bar{v}'(\bar{a})}{\int_0^1 (2\lambda_0 u + 2\tilde{\rho}u')\bar{v}d\bar{x}} \quad (95)$$

$$\lambda_1^\delta = \frac{-2\bar{d}\lambda_0 u(\bar{a})\bar{v}(\bar{a}) + \bar{h}\bar{d}\lambda_0\mu u(\bar{a})\bar{v}'(\bar{a})}{\int_0^1 (2\lambda_0 u + 2\tilde{\rho}u')\bar{v}d\bar{x}} \quad (96)$$

and

$$\lambda_1^\nu = - \frac{\int_0^1 (2\lambda_0 u' + 2\tilde{\rho}u'')\bar{v}d\bar{x}}{\int_0^1 (2\lambda_0 u + 2\tilde{\rho}u')\bar{v}d\bar{x}} \quad (97)$$

Note that (94) can be regarded as the extension of the formulas derived in [23–25] to the important case of problems containing intermediate boundary conditions.

3.2.2 Perturbation of Double Eigenvalues. Following [31] for double eigenvalues, we set

$$w(\bar{x}) = u(\bar{x}) + \varepsilon^{\frac{1}{2}} w_1^\varepsilon(\bar{x}) + \cdots \quad (98)$$

$$\lambda = \lambda_0 + \varepsilon^{\frac{1}{2}} \lambda_1^\varepsilon + \cdots \quad (99)$$

Expanding all the terms in powers of $\varepsilon^{\frac{1}{2}}$ and ordering yields

$$\varepsilon^0: L(u) = 0 \quad (100)$$

$$\varepsilon^{\frac{1}{2}}: \lambda_1^\varepsilon \frac{\partial L}{\partial \lambda}(u) + L(w_1^\varepsilon) = 0 \quad (101)$$

$$\varepsilon: (\lambda_1^\varepsilon)^2 \frac{1}{2} \frac{\partial^2 L}{\partial \lambda^2}(u) + \lambda_2^\varepsilon \frac{\partial L}{\partial \lambda}(u) + \lambda_1^\varepsilon \frac{\partial L}{\partial \lambda}(w_1^\varepsilon) + L(w_2^\varepsilon) + L^{1\varepsilon}(u) = 0 \quad (102)$$

and similar expressions hold for the U_i , namely,

$$\varepsilon^0: U_i^0(u) = 0 \quad (103)$$

$$\varepsilon^{\frac{1}{2}}: \lambda_1^\varepsilon \frac{\partial U_i^0}{\partial \lambda}(u) + U_i^0(w_1^\varepsilon) = 0 \quad (104)$$

$$\varepsilon: (\lambda_1^\varepsilon)^2 \frac{1}{2} \frac{\partial^2 U_i^0}{\partial \lambda^2}(u) + \lambda_2^\varepsilon \frac{\partial U_i^0}{\partial \lambda}(u) + \lambda_1^\varepsilon \frac{\partial U_i^0}{\partial \lambda}(w_1^\varepsilon) + U_i^0(w_2^\varepsilon) + U_i^{1\varepsilon}(u) = 0 \quad (105)$$

From (101) follows $w_1^\varepsilon = \lambda_1^\varepsilon u_1 + Cu$, where C is a constant. Multiplying the Jordan chain by v , we get

$$\begin{aligned} \langle v, L(w_1^\varepsilon) \rangle + \lambda_1^\varepsilon \left\langle v, \frac{\partial L}{\partial \lambda}(u) \right\rangle &= \sum_{i=1}^n U_i^0(w_1^\varepsilon) V_{n+1-i}^0(v) + \lambda_1^\varepsilon \left\langle v, \frac{\partial L}{\partial \lambda}(u) \right\rangle \\ &= -\lambda_1^\varepsilon \sum_{i=1}^n \frac{\partial^2 U_i^0}{\partial \lambda^2}(u) V_{n+1-i}^0(v) \\ &\quad + \lambda_1^\varepsilon \left\langle v, \frac{\partial L}{\partial \lambda}(u) \right\rangle = 0 \end{aligned} \quad (106)$$

making use of (104). Multiplication of (102) by v and integration by parts using (105) yields

$$\begin{aligned} \langle v, L(w_2^\varepsilon) \rangle &= \sum_{i=1}^n U_i^0(w_2^\varepsilon) V_{n+1-i}^0(v) = - \sum_{i=1}^n \left[(\lambda_1^\varepsilon)^2 \frac{1}{2} \frac{\partial^2 U_i^0}{\partial \lambda^2}(u) \right. \\ &\quad \left. + \lambda_2^\varepsilon \frac{\partial U_i^0}{\partial \lambda}(u) + \lambda_1^\varepsilon \frac{\partial U_i^0}{\partial \lambda}(w_1^\varepsilon) + U_i^{1\varepsilon}(u) \right] V_{n+1-i}^0(v) \\ &= -(\lambda_1^\varepsilon)^2 \frac{1}{2} \left\langle v, \frac{\partial^2 L}{\partial \lambda^2}(u) \right\rangle - \lambda_2^\varepsilon \left\langle v, \frac{\partial L}{\partial \lambda}(u) \right\rangle \\ &\quad - \lambda_1^\varepsilon \left\langle \frac{\partial L}{\partial \lambda}(w_1^\varepsilon), v \right\rangle - \langle L^{1\varepsilon}(u), v \rangle \end{aligned}$$

With $w_1^\varepsilon = \lambda_1^\varepsilon u_1 + Cu$ and using (106), this simplifies to the formula

$$(\lambda_1^\varepsilon)^2 = - \frac{1}{\sigma_2} \left[\langle L^{1\varepsilon}(u), v \rangle - \sum_{i=1}^n U_i^{1\varepsilon}(u) V_{n+1-i}^0(v) \right] \quad (107)$$

$$\sigma_2 = \sum_{r=1}^2 \frac{1}{r!} \left[\left\langle \frac{\partial^2 L}{\partial \lambda^2}(u_{2-r}), v \right\rangle - \sum_{i=1}^n \frac{\partial^2 U_i^0}{\partial \lambda^2}(u_{2-r}) V_{n+1-i}^0(v) \right] \quad (108)$$

already derived in [23–25]. This is again the extension allowing for intermediate boundary conditions, which can obviously be done for Jordan chains of arbitrary length and several intermediate transition conditions. Using (49), (50), and (53)–(61) for our problem, it reads

$$\begin{aligned}
(\lambda_1^e)^2 = & \gamma_1 \frac{-2\bar{k}u(\bar{a})\bar{v}(\bar{a}) + [\bar{h}\bar{k}\mu u(\bar{a}) - \bar{h}\bar{N}_0\mu^2 u'(\bar{a})]\bar{v}'(\bar{a})}{\int_0^1 (2\lambda_0 u_1 + 2\bar{\rho}u_1')\bar{v}d\bar{x} + \int_0^1 2u\bar{v}d\bar{x}} \\
& + \delta_1 \frac{-2\bar{d}\lambda_0 u(\bar{a})\bar{v}(\bar{a}) + \bar{h}\bar{d}\lambda_0 \mu u(\bar{a})\bar{v}'(\bar{a})}{\int_0^1 (2\lambda_0 u_1 + 2\bar{\rho}u_1')\bar{v}d\bar{x} + \int_0^1 2u\bar{v}d\bar{x}} \\
& - \int_0^1 (2\lambda_0 u' + 2\bar{\rho}u'')\bar{v}d\bar{x} \\
& + \nu_1 \frac{\int_0^1 (2\lambda_0 u_1 + 2\bar{\rho}u_1')\bar{v}d\bar{x} + \int_0^1 2u\bar{v}d\bar{x}}{\int_0^1 (2\lambda_0 u_1 + 2\bar{\rho}u_1')\bar{v}d\bar{x} + \int_0^1 2u\bar{v}d\bar{x}} \quad (109)
\end{aligned}$$

3.3 Stability Boundaries. In Secs. 3.2.1 and 3.2.2, we derived formulas for the change of simple and double eigenvalues occurring in the spectrum of the unperturbed system, caused by small perturbations of the parameters. For a fixed velocity $\bar{\rho}$, the stability region of the system in the parameter plane γ, δ is given by those areas where all eigenvalues have a negative real part. For each simple purely imaginary eigenvalue λ_j of the unperturbed problem, there is a stable region, which in the first approximation, is a half-plane

$$\gamma \operatorname{Re}(\lambda_1^{j\gamma}) + \delta \operatorname{Re}(\lambda_1^{j\delta}) \leq 0 \quad \forall j \quad (110)$$

If, at a certain speed $\bar{\rho}$, all eigenvalues are simple, then first approximation to the stable region is given by the intersection of the half-planes (see also [17], [23–25,32]). Depending on the parameters, the intersection can be a sector limited by an angle, a line (for $\operatorname{Re}(\lambda_1^{1\delta})/\operatorname{Re}(\lambda_1^{1\gamma}) = \dots = \operatorname{Re}(\lambda_1^{n\delta})/\operatorname{Re}(\lambda_1^{n\gamma})$) or a point.

The necessary and sufficient conditions for a double purely imaginary eigenvalue λ_0 not to move to the right-hand side of the complex plane in the first approximation, i.e.,

$$\operatorname{Re}\{\sqrt{\operatorname{Re}[(\lambda_1^e)^2] + i \operatorname{Im}[(\lambda_1^e)^2]}\} \leq 0 \quad (111)$$

are $\operatorname{Im}[(\lambda_1^e)^2] = 0$ and $\operatorname{Re}[(\lambda_1^e)^2] < 0$. The condition $\operatorname{Im}[(\lambda_1^e)^2] = 0$ defines the line

$$\gamma = \frac{\operatorname{Im}[2\bar{d}\lambda_0 u(\bar{a})\bar{v}(\bar{a}) - \bar{h}\bar{d}\lambda_0 \mu u(\bar{a})\bar{v}'(\bar{a})]}{\operatorname{Im}\{2\bar{k}u(\bar{a})\bar{v}(\bar{a}) - [\bar{h}\bar{k}\mu u(\bar{a}) - \bar{h}\bar{N}_0\mu^2 u'(\bar{a})]\bar{v}'(\bar{a})\}} \delta \quad (112)$$

only half of which can be in the stable region. We are now in the position to draw pictures of the onset of the stability regions.

In Figs. 6 and 7, we see the onset of the stability boundary corresponding to the smallest two pairs of complex conjugate eigenvalues, plotted for the values of parameters

$$\mu = 0.3 \quad \bar{d} = 0.5 \quad \bar{k} = 1 \quad \bar{N}_0 = 0.1 \quad \bar{\kappa} = 3\pi \quad \bar{h} = 0.01 \quad \bar{a} = 0.51 \quad (113)$$

Below the first critical speed $\bar{\rho}_1 = 6.99$, we have a simple spectrum with purely imaginary eigenvalues. Therefore, we get a region of asymptotic stability, which for small γ and δ , is approximately a sector limited by an angle.

Therefore, the simultaneous actions of dissipative and nonconservative positional forces can cause both asymptotic stabilization and flutter instability. This is very important, since it occurs in the subcritical range in squeal problems. However, in the subcritical range, in accordance with Fig. 6, it is possible to assign a destabilizing effect to nonconservative forces, as discussed in Sec. 2.4 based on [1,16,19], and a stabilizing effect to damping forces (at least if they cause a positive semidefinite damping operator). This is no longer true in the supercritical range, since the stiffness operator becomes negative definite. Due to the angle singularity

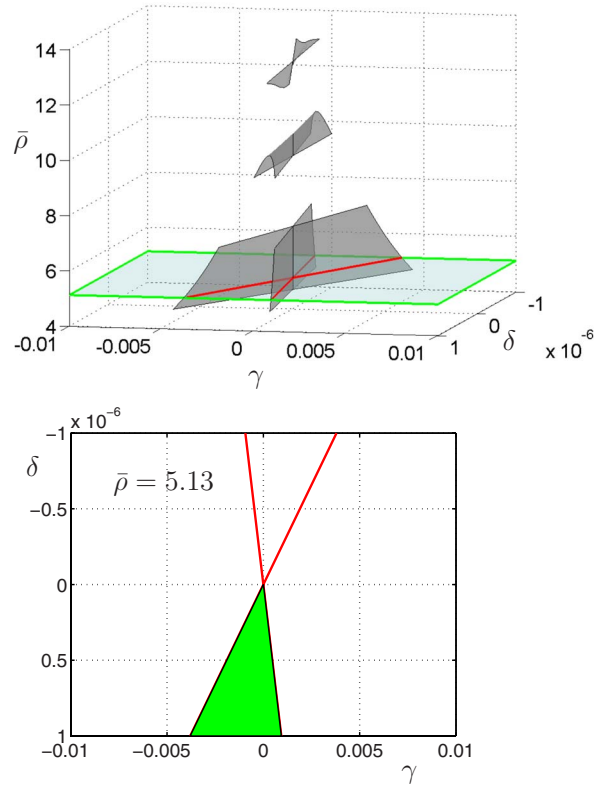


Fig. 6 Stability boundaries (subcritical range)

on the stability boundary, the choice of the stabilizing combination of the forces is nontrivial, especially in this case in agreement with [17].

At the critical speed, we get a double zero eigenvalue with a Jordan chain, characterizing the divergence boundary of the unperturbed system. At the second critical speed $\bar{\rho}_2 = 9.50$, the system stabilizes again. For the perturbed problem, the stability region is again given by a sector limited by an angle.

The next interesting point in the spectrum occurs where the first and second eigenvalues meet with nonvanishing imaginary part ($\bar{\rho}_3 = 10.30$). Here, we have again a double eigenvalue with a Jordan chain. Towards this point, the sector of the stable region shrinks to a line since the stability boundaries of first and second eigenvalue coincide at this point, as can be seen from the perturbation formulas derived for the simple and the double eigenvalue in Secs. 3.2.1 and 3.2.2. What we see around this point in Fig. 7 is, in fact, a generic singularity of the stability boundary of a three-parameter system ($\bar{\rho}, \gamma, \delta$). These singularities have been investigated by Arnold [30] and the one that we are observing is the Whitney umbrella caused by a double purely imaginary eigenvalue with a Jordan block. Above the speed corresponding to the Whitney umbrella, the unperturbed system suffers flutter instability. Increasing the speed of the beam, the unperturbed system stabilizes again at a critical speed $\bar{\rho}_4 = 12.78$, corresponding to a double eigenvalue with a Jordan block, again yielding a Whitney umbrella. Afterwards, the system again loses stability by divergence, and additionally suffers flutter at a later stage.

The Whitney umbrella also appears in other problems, e.g., on the stability boundaries of the Beck column with external and internal damping (see [23,25]). It was also found in general two degree of freedom linear gyroscopic systems with damping and circulatory forces considered in [17], as well as in circulatory systems with small velocity dependent forces [33].

We now give an expression for the Whitney umbrella. Suppose the beam without pads, i.e., $\gamma = \delta = 0$, is moving with the speed $\bar{\rho}$

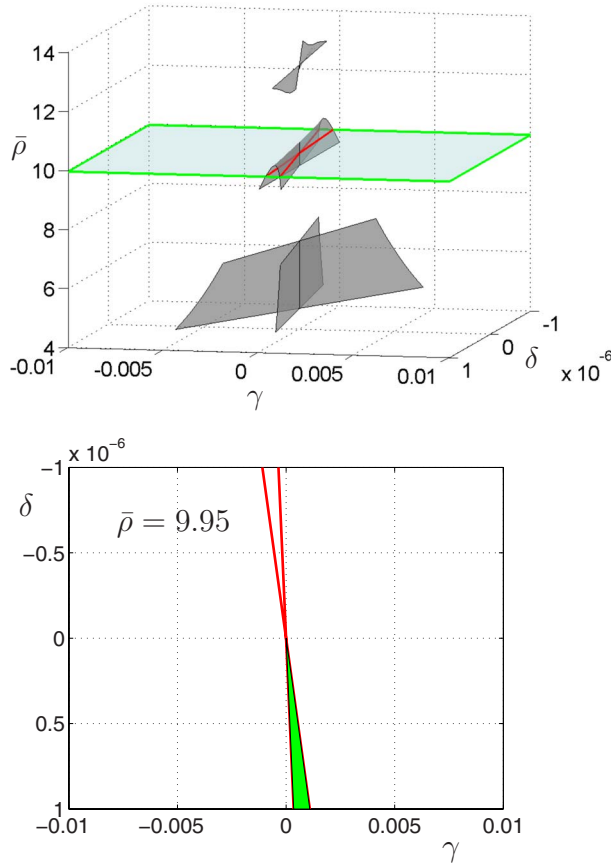


Fig. 7 Stability boundaries (supercritical range)

$=\bar{\rho}$ corresponding to the first double purely imaginary eigenvalue $\lambda_0=\tilde{\lambda}_0$, the corresponding eigenfunction $u=\tilde{u}$, and associated function $u_1=\tilde{u}_1$. Now the velocity is changed slightly by the small parameter ν . Formula (109) yields

$$(\lambda_1^\nu)^2 = (\lambda_1^\varepsilon)^2 = -\nu_1 \frac{\langle \tilde{\lambda}_0 \tilde{u}' + \tilde{\rho} \tilde{u}'', \tilde{u} \rangle}{\langle \tilde{\lambda}_0 \tilde{u}_1 + \tilde{\rho} \tilde{u}_1 + \tilde{u}, \tilde{u} \rangle} \quad (114)$$

that is seen to be a real quantity, because $\langle u_1, L(u_1) \rangle = -\langle u_1, (\partial L / \partial \lambda)(u) \rangle$ is real, which follows from the defining equations of the Jordan chain and integration by parts. Consequently, in the vicinity of the first flutter boundary $\bar{\rho}=\bar{\rho}+\nu=\bar{\rho}+\varepsilon\nu_1$, the increment $\sqrt{\nu}\lambda_1^\nu$ to the unperturbed double eigenvalue $\tilde{\lambda}_0$ is purely imaginary for $\nu<0$ and real otherwise. For negative ν , the eigenvalue $\tilde{\lambda}_0$ splits into two simple purely imaginary eigenvalues $\pm\tilde{\lambda}_0$.

Perturbation of the system for arbitrary $\bar{\rho}$ corresponding to a simple spectrum of the unperturbed problem with the forces coming from the pads yielded $\bar{\lambda}(\bar{\rho})=\bar{\lambda}_0(\bar{\rho})+\gamma\lambda_1^\gamma(\bar{\rho})+\delta\lambda_1^\delta(\bar{\rho})$ for the eigenvalues meeting at $\bar{\rho}$. The coefficients $\lambda_1^\gamma(\bar{\rho})$ and $\lambda_1^\delta(\bar{\rho})$ were given in (95) and (96), depending on $\bar{\rho}$ through the eigenvalues and eigenfunctions of the unperturbed problem ($\delta=\gamma=0$). Substituting $\bar{\lambda}_0=\tilde{\lambda}_0\pm\sqrt{\nu}\lambda_1^\nu+\dots$ and $\bar{u}=\tilde{u}\pm\sqrt{\varepsilon}\lambda_1^\varepsilon\tilde{u}_1$ into (95) and (96) and postulating that $\bar{\lambda}(\bar{\rho})=0$, we obtain an approximate equation for the critical speed for the flutter boundary for the beam with pads of the form

$$\bar{\rho}=\tilde{\rho}+\left(\frac{A\delta+B\gamma}{C\delta+D\gamma}\right)^2 \quad (115)$$

where A, B, C , and D are constants depending only on the spectral data of the unperturbed problem at $\bar{\rho}=\tilde{\rho}$, $\delta=\gamma=0$, and corresponding to the double eigenvalue $\tilde{\lambda}_0$. Equation (115) is of the canonical form for the Whitney umbrella; see [17,30].

4 Conclusion

In this paper we considered a moving beam with clamped boundary conditions in frictional contact with idealized pads. The system's equations of motion were derived, and the interactions between beam and rod equations were identified. Due to the pads, self-excited vibrations can arise in the system originating from instabilities of the trivial solution of the beam equation. The investigated mechanism not only occurs in beams, but can also be observed in rotating disks, providing an explanation for the phenomenon of squeal, and probably also in other moving continua, like shells. The problem was investigated using a perturbation approach, which enabled us to calculate analytic approximations to the stability boundaries. It was found that on the stability boundary of the system, there are generic singularities corresponding to double eigenvalues with a Jordan chain, and analytic approximations for the splitting of eigenvalues in the vicinity of these singularities were calculated. The model is an example for a system losing Hamiltonian symmetry only due to perturbations in the boundary conditions. Insights gained from the problem carry over to other problems with moving media, and are to be investigated in future research.

Acknowledgment

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On Colles' Fracture: An Experimental Study Involving Structural and Material Testing

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A two-stage experimental program was conducted, which was aimed at examining the process of initiation/propagation of fracture in human radii under the conditions simulating a fall onto an outstretched hand. It involved a number of destructive tests on dried cadaver bones. The bones were first subjected to DXA as well as spiral CT measurements to establish the density properties and the details of geometry. Subsequently, the specimens were tested under controlled boundary conditions, to induce Colles' type of fracture. Following these tests, samples of cortical bone tissue were extracted at different orientations with respect to the direction of osteons and tested in axial tension. The results of material tests were used to verify the performance of an anisotropic fracture criterion for the cortical tissue. It has been demonstrated that the proposed criterion can reproduce the basic trends in the directional dependence of the tensile strength characteristics. For the structural tests, a correlation was established between the geometric characteristics of the cortex, the strength properties and the fracture load for individual radii that were tested. It was shown that the morphological traits and/or the strength properties alone are not adequate predictors of the fracture load of intact radii. A rational assessment of the fracture load requires a mechanical analysis that incorporates the key elements of the experimental program outlined here, i.e., the information on bone geometry, material properties of the bone tissue, and the static/kinematic boundary conditions. A preliminary example of a finite element analysis, for one of the radii bones tested, has been provided. [DOI: 10.1115/1.2839902]

Introduction

An understanding of fracture mechanism in human bone is essential for predicting the fracture risk associated with age and disease. A large number of bone fractures are brittle in nature; they initiate within the cortex and are typically associated with the tensile stress regime. An example here is the low energy fracture of the distal radius, referred to as Colles' fracture. It is usually caused by a fall onto an outstretched hand and involves fracture through the distal metaphysis approximately 2–3 cm proximal to the articular surface of the radius. Colles' fracture has a dorsal displacement of the fractured fragments, which is an indicative of the tensile fracture mode (Hoynak et al. [1]). In mechanical terms, the problem is similar to that of a structural element being subjected to a compressive force of a significant eccentricity. The latter typically triggers the onset of fracture in tension regime, as the tensile strength is significantly lower than that in compression. There is strong experimental evidence indicating that the magnitude of fracture load is significantly affected by the mechanical properties of the cortical tissue as well as the geometry of the cortex (e.g., Spadaro et al. [2], Jordan et al. [3]).

Cortical bone is a composite material in which the basic unit is an osteon. It consists of cylindrically shaped lamellar bone that surrounds longitudinally oriented vascular channels (i.e., Haversian canals). In mechanical terms, this type of microstructure may be considered as transversely isotropic, with the preferred material axis coinciding with that of the osteon system. In order to investigate the fracture phenomenon, the material properties of the

bone tissue need to be identified. In this context, the basic question to be dealt with is that concerning the specification of the conditions at failure.

Despite intensive research, a reliable phenomenological framework for describing the conditions at failure, in both cortical and trabecular bones, is still lacking. Up until now, the majority of research effort has been directed at the description of the onset of fracture in trabecular bone. Examples here involve the works of Tsai and Wu [4], Cowin [5], Pietruszczak et al. [6]. The primary difficulty in formulating a general fracture criterion is the need for a specific tensorial measure of trabecular architecture and its correlation with mechanical properties. In the case of cortical bone, the most common approaches to defining the notion of failure involve implementation of strain-based criteria that are largely empirical. Such criteria identify the onset of fracture with critical values of principal strain magnitudes (Niebur et al. [7]) or the strain energy density (Ulrich et al. [8], Pistoia et al. [9], Pistoia et al. [10]). Again, there is a need for a more general approach, which is rigorous and experimentally verifiable.

In this work, a comprehensive experimental program has been carried out. It consisted of two basic stages. The first one involved testing of a number of cadaver radii under the conditions simulating Colles' fracture. The second stage was concerned with material tests on samples of cortical tissue that were extracted from the fractured radii. All tests, both structural and material, were carried out on dry bones and the limitations of this procedure are addressed in the Discussion section. The results of material tests were examined in the context of identification/verification of a simple macroscopic fracture criterion that incorporates the anisotropy of strength properties. In addition, the results for the whole radii were employed to study a correlation between the geometric characteristics of the cortex, its strength properties, and the fracture load.

It needs to be emphasized that the current experimental program was structured in such a way as to provide a benchmark for assessing the predictive abilities of numerical models incorporat-

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ing linear/nonlinear finite element formulations. The structural tests, together with CT measurements, provided comprehensive information on bone geometry, material properties of the bone tissue, and the strength characteristics of whole radii under well-defined static/kinematic constraints. This is a general methodology that needs to be followed in order to enable a reliable mechanical analysis of the fracture process. In order to illustrate this aspect, some preliminary results of a finite element analysis of one of the radii tested have been provided. The analysis is restricted to the elastic range and incorporates the actual bone geometry and the mechanical properties established from the material tests. The focus here is on examining the fracture mode, the latter being assessed based on the verification of the employed anisotropic fracture criterion. In the second part of this work, which is currently in preparation, a more extensive numerical study of the experimental results reported here will be presented. The analysis will employ a nonlinear framework that incorporates the description of localized deformation associated with initiation and propagation of fracture.

Methodology

A Tensile Fracture Criterion for Cortical Bone. Let us start with the notion of fracture criterion that is central to the main theme of this work. Cortical bone may be considered as a transversely isotropic material in which the preferred material axes coincide with the direction of the osteons (see Ashman et al. [11]). Consequently, the strength properties depend, in general, on the orientation of material axes in relation to the direction of loading. In this study, the considerations are restricted to the tensile regime and the specific form of the fracture criterion is similar to that employed by Pietruszczak et al. [12] in the context of numerical analysis of distal radius fracture. The formulation is based on the critical plane approach, whereby the conditions at failure are defined in terms of traction components acting on a physical plane, and the representation employs a set of functions specifying the spatial variation of strength parameters. The approach consists of identifying such an orientation of the critical or localization plane, for which the failure function reaches a maximum (Pietruszczak and Mroz [13]).

In the tension regime, the failure is said to occur when the normal component of the traction vector, acting on a plane with unit normal n_i , reaches a critical value. Thus, the failure function may be defined as

$$F = t^n(n_i) - c(n_i) \quad t^n = \sigma_{ij}n_j n_i \quad c = \hat{c}(1 + \Omega_{ij}n_j n_i) \quad (1)$$

Here, t^n is the normal stress along n_i and c is the strength parameter, which is assumed to be orientation dependent. The bias in the distribution of c is defined by employing a symmetric traceless tensor Ω_{ij} whose eigenvalues coincide with the principal material axes, while $\hat{c}$ is the average value of c .

The orientation of the fracture/localization plane can be determined by maximizing the function F with respect to n_i . Thus, the conditions at failure are defined as

$$\max_{n_i} F = \max_{n_i} [t^n(n_i) - c(n_i)] = 0 \quad (2)$$

Introducing a Lagrange multiplier λ , the corresponding Lagrangian function becomes

$$G = \sigma_{kj}n_k n_j - \hat{c}(1 + \Omega_{kj}n_k n_j) - \lambda(n_j n_j - 1) \quad (3)$$

The stationary conditions with respect to n_i take the form

$$\frac{\partial G}{\partial n_i} = 2(\sigma_{ij}n_j - \hat{c}\Omega_{ij}n_j) - 2\lambda\delta_{ij}n_j = 0 \quad (4)$$

from which

$$(B_{ij} - \lambda\delta_{ij})n_j = 0 \quad B_{ij} = \sigma_{ij} - \hat{c}\Omega_{ij} \quad (5)$$

Equation (5) defines an eigenvalue problem that can be solved to specify the direction cosines n_i . It is evident that for $\Omega_{ij}=0$, there

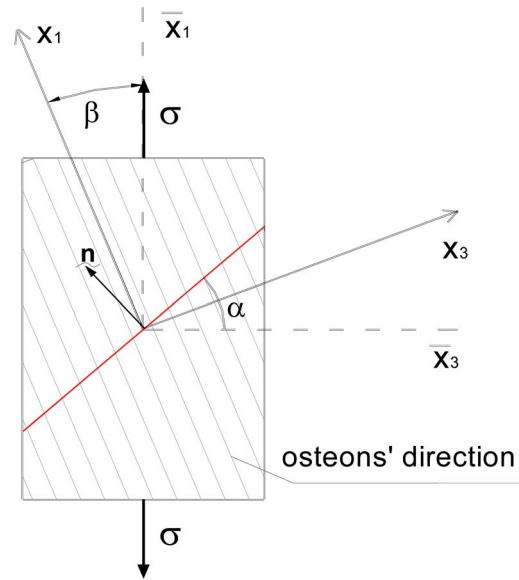


Fig. 1 Schematic representation of cortical bone sample; x_i - material coordinate system; $\bar{x}_i$ - global coordinate system. (Note: α defines the inclination of the critical plane with respect to horizontal axis, while β specifies the orientation of osteons)

is $c = \hat{c} = \text{const}$. In this case, λ 's are the eigenvalues of σ_{ij} , so that the direction of the localization plane is normal to that of the maximum tensile stress.

Note that, according to Eq. (1), the failure function F can be defined in terms of Cartesian components of both operators σ_{ij}, Ω_{ij} as

$$F = (\sigma_{ij} - \hat{c}\Omega_{ij})n_j n_i - \hat{c} \quad (6)$$

Referring the above equation to, for example, the principal stress system, $\sigma_I = \sigma_{11}, \sigma_{II} = \sigma_{22}, \sigma_{III} = \sigma_{33}$, one can express the criterion (2) as

$$\max F = (\sigma_I - \hat{c}\Omega_{11})\hat{n}_1^2 + (\sigma_{II} - \hat{c}\Omega_{22})\hat{n}_2^2 + (\sigma_{III} - \hat{c}\Omega_{33})\hat{n}_3^2 - 2\hat{c}(\Omega_{12}\hat{n}_1\hat{n}_2 - \Omega_{13}\hat{n}_1\hat{n}_3 - \Omega_{23}\hat{n}_2\hat{n}_3) = 0 \quad (7)$$

where $\hat{n}_i = \{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$ defines the components of the unit vector normal to the localization plane, Eq. (5).

Unfortunately, an analytical expression for $\hat{n}_i$ is not straightforwardly available, so that an explicit form of Eq. (7) is not viable. This is in contrast to other criteria that are expressed directly in terms of stress components referred to the principal material system. An example here is the Tsai–Wu criterion, which employs a quadratic representation in terms of stress and requires, in general, 12 independent material constants (Tsai and Wu [4]). The primary problem with such criteria, in addition to excessive number of constants and restrictive abilities in terms of describing the directional variation of strength properties, is the notion of frame indifference. Indeed, for a transversely isotropic material, such as cortical bone tissue, the value of failure function should not be affected by referring the problem to a coordinate system obtained through rotation about the preferred axis. This, however, is not enforced for the class of criteria of Tsai–Wu type.

The identification of material parameters appearing in Eq. (1) entails performing a series of direct tension tests on samples extracted at different orientations relative to the direction of the osteons. Figure 1 shows a schematic geometry of the sample. Here, β defines the orientation of osteons, while α specifies the direction of the localization/failure plane. Referring to this figure, consider a sample subjected to axial tension along the x_1 axis. For loading in the direction of osteons ($\beta=0$ deg), or in the direction

perpendicular to it ($\beta=90$ deg), the principal material directions coincide with the eigenvectors of the stress tensor. Consequently, for both these cases the localization plane is orthogonal to σ (i.e., $\alpha=0$). Noting that Ω_{ij} is a traceless operator and $\Omega_2=\Omega_3$ (transversely isotropic material), there is $\Omega_1=-2\Omega_3$. Denoting now by $c_0, c_{\pi/2}$ the ultimate strengths at $\beta=0$ deg and $\beta=90$ deg, respectively, one obtains

$$c = \hat{c}(1 + \Omega_1) = c_0 \quad c = \hat{c}(1 + \Omega_3) = \hat{c}\left(1 - \frac{1}{2}\Omega_1\right) = c_{\pi/2} \quad (8)$$

so that

$$\Omega_1 = \frac{2(c_0 - c_{\pi/2})}{c_0 + 2c_{\pi/2}} \quad \hat{c} = \frac{1}{3}(c_0 + 2c_{\pi/2}) \quad (9)$$

Given now the values of the material constants, the spatial distribution of c can be defined by invoking the representation (2). For the case as stipulated in Fig. 1, the following relations are obtained:

$$F = \sigma \cos^2 \alpha - \hat{c} \left\{ 1 + \frac{1}{2}\Omega_1[3 \cos^2(\alpha - \beta) - 1] \right\} = 0 \quad (10)$$

$$\frac{dF}{d\alpha} = -\sigma \sin 2\alpha - 3\hat{c}\Omega_1 \sin 2(\alpha - \beta) = 0$$

which, for a fixed value of β , provides a set of two equations for two unknowns, i.e., α and $\sigma=c$.

Finally, it is noted that the criterion (1) can be extended to compression regime by invoking, for example, a Coulomb-type of function on the localization plane. In this case, the problem may be formulated as

$$F = \max\{F_1, F_2\} \quad F_1 = t^s - \mu(n_i)t^n - \eta(n_i) \quad F_2 = t^n - c(n_i) \quad (11)$$

where

$$\mu(n_i) = \hat{\mu}(1 + \Omega_{ij}^\mu n_i n_j) \quad \eta(n_i) = \hat{\eta}(1 + \Omega_{ij}^\eta n_i n_j) \quad (12)$$

Here, t^s is the shear stress on the plane with the unit normal n_i and the operators $\Omega_{ij}^\mu, \Omega_{ij}^\eta$ have the definition that is analogous to that employed in Eq. (1).

Mechanical Testing. In order to identify/verify the suitability of the framework outlined above, an experimental program has been undertaken. In this program, a number of cortical bone samples were extracted from dry cadaver forearms at different orientations ($\beta=0$ deg, 45 deg, and 90 deg) with respect to the preferred material orientation (i.e., direction of osteons). The identification was restricted to the tensile regime, i.e., the samples were tested in axial tension in order to specify the function $c(n_i)$, Eq. (1).

As mentioned earlier, this specific experimental program was part of a broader investigation that was aimed at examining the process of initiation/propagation of fracture in human radii under the conditions simulating a fall onto an outstretched hand. The latter involved a number of destructive tests on whole cadaver bones. The bones were first subjected to DXA as well as spiral CT measurements. Subsequently, the specimens were tested under controlled boundary conditions, to induce Colles' type of fracture. Following these tests, small samples of cortical bone tissue were extracted and tested in direct tension in order to define the basic material parameters.

The specimens for both the structural and material testings were taken from a set of 11 dried cadaver radius bones obtained from multiple donors (gender and age were unspecified). Prior to testing, the radii were measured, weighed, and scanned using both DXA and GE CTI helical scanner. In the latter case, the digital data were acquired through a stack of images taken along longitudinal axis of radii with 1 mm spacing, Fig. 2. The results were

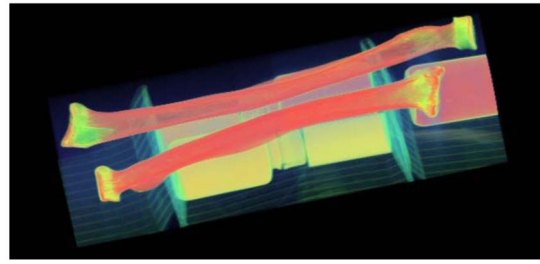


Fig. 2 CT scans of two isolated radii together with vials containing calibration solutions for the derivation of Hounsfield number

then used to obtain the details of geometry as well as the information on the cortical tissue density, which was later correlated with the tensile strength. In addition, relations between various geometric entities (i.e., minimum cortical thickness and cortical cross-section area), the clinical measurements obtained from DXA protocol and the fracture load were examined.

The first phase of experiments involved structural testing of dried radius bones in a setup simulating the condition of a fall onto an outstretched hand. The specimens were trimmed at 60–70% of the length in order to reduce the bending moment at the support. The standard experimental setup as used by other investigators (see Augat et al. [14], Pistoia et al. [9], Muller et al. [15]), which employed embedment at both trimmed and distal ends, has been modified in order to ensure a more accurate load transfer (see Fig. 3). In particular, the circular embedment at the distal end was replaced by a polymer filling applied to the articular surface of the radius. The mold had a smooth upper surface with volar tilt varying from 8 deg to 12 deg (Fig. 3(a)) and radial tilt within the range of 0–6 deg (Fig. 3(b)). The proximal (trimmed) end was embedded in the same polymer placed in a polyvinyl chloride (PVC) ring of 25 mm inner diameter. The load was applied through the steel plate placed at the top of the mold filling the articular surface. All tests were conducted by employing a constant loading rate of 0.1 mm/min. This rate corresponds to static range (McElhaney [16]) and it is substantially lower than that simulating a dynamic impact (~ 100 mm/s; Muller et al. [15]). The choice here was dictated by the overall objective of this study, i.e., to provide simple benchmark problems for validation of static

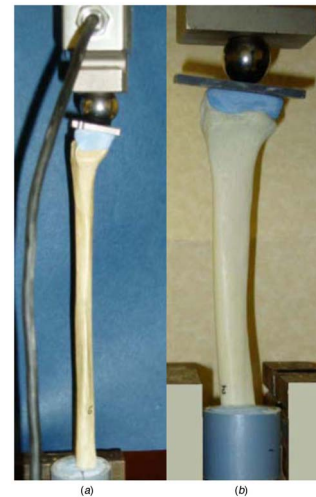


Fig. 3 Experimental setup for testing Colles' fracture; two different samples shown in (a) lateral medial view (b) palmer view. Load cell at the top of the figure.

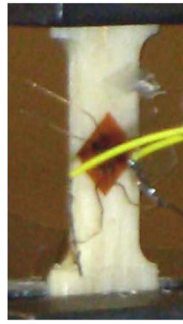


Fig. 4 Experimental setup for material tests (longitudinal sample, $\beta=0$ deg)

linear/nonlinear finite element simulations. Note that the dynamic analysis is quite ambiguous and difficult to verify, as it employs more internal variables and requires sophisticated *material* tests for assessing the sensitivity of the tissue properties to the applied loading rate.

The testing machine was equipped with internal force and deformation transducers. Both transducers were calibrated before the testing, and the force-deformation curves were recorded. In addition, the polymer that was used for embedment (PMMA, Bosworth Fastray) was itself tested in axial compression in order to determine the elastic and ultimate strength properties.

In the second phase of the experimental program, the samples of cortical bone were extracted from the middle section of the individual radii, i.e., the region where the cortical shell was the thickest. The bone tissue was carefully milled to produce rectangular specimens with the gauge length of 10–15 mm, the width of 5 mm, and the thickness of 1.5 mm. The samples were extracted at different orientations of $\beta=0$ deg, 45 deg, and 90 deg with respect to the longitudinal axis of the radius. For samples taken along the longitudinal direction, it was possible to produce specimens with enlarged ends that could be held safely by the grips in the testing machine, as shown in Fig. 4. The samples taken out at 45 deg and 90 deg (transverse direction) had a uniform width, as shown in Fig. 5(b).

Each sample was individually fabricated. For most longitudinal samples, miniature, three-element, 45 deg rectangular stacked rosette strain gauges from Micro-Measurements (gauge designation: WA-06-030WR-120) were glued to the middle of the specimens, as shown in Figs. 4 and 5(a). The longitudinal and the transverse gauges were each connected in quarter bridge configurations, while the 45 deg gauge was not employed. This configuration allowed to obtain strain measurements for identification of the elasticity moduli and the Poisson's ratios of the cortical bone in the longitudinal directions of the tissue. Note that most samples

extracted in transverse direction were too small to be outfitted with strain gauges in the same manner. Thus, in this case, the elastic moduli were estimated directly from transducer readings.

The samples were mounted in the grips of a 5 kN Instron 1011 testing machine. All specimens were subjected to uniaxial tension at a constant displacement rate of 0.5 mm/min (i.e., strain rate of approximately 0.001/s), which again is representative of static conditions (McElhaney [16]). The testing machine was equipped with internal force and deformation transducers. Both transducers were calibrated before the testing, and the force-deformation curves were simultaneously recorded on the computer screen and by a strip-chart recorder for later analyses. Additional external force and deformation transducer were employed and their signals were recorded along with the responses from the strain gauges.

Data Analysis. The first aspect of the analysis involved the identification/verification of the proposed fracture criterion (1) and (2). Extensive numerical simulations were performed, for a number of individual specimens, to specify the tensile strength distribution and the results were compared with the experimental data. In addition, a simple regression analysis was performed in order to determine which clinical measurement (i.e., spiral CT or DXA) is the best predictor of the tensile strength of cortical tissue. The accuracy of the proposed criterion was assessed by examining the relation between the values of ultimate strength predicted by the model and those obtained experimentally.

For the tests on whole radii, a regression analysis was performed in order to determine the relation between the geometric/material characteristics of individual radii and the fracture load. Variables of three different types were employed, i.e., DXA measurements of bone mineral density at distal location, geometric characteristics of cortical shell (i.e. minimum thickness and cross-sectional area) at different distal locations, and strength properties of cortical tissue. A stepwise regression was performed to establish if the combination of chosen variables predicts fracture load better than a single variable.

Experimental Results

The main results of structural tests on the whole radii are provided in Table 1. The table gives the information about the magnitude of fracture load together with classification of the type of fracture according to the *Comprehensive Classification of Fracture of Long Bones* (Fernandez et al. [17]). In addition, the clinical measurements of areal bone mineral density (BMD) (g/cm^2) and bone mineral content (BMC) (g) are also provided. The areal BMD was determined by DXA for the projection areas of 1 cm at two anatomic locations, i.e., at the midshaft area and 1/3 length from the distal end of radius, respectively. Figure 6 shows a typical example of experimentally produced Colles' fracture (Bone 5), while the corresponding load-displacement curve is given in Fig. 7.

The key information on the results of direct tension tests on cortical bone specimens is provided in Table 2. Note that the main focus in this work is on evaluation of directional distribution of tensile strength. However, some supplementary information, i.e., that on elastic properties as well as cortBMD (g/cm^3) measurements (spiral CT), has also been provided. The values of cortBMD at the midshaft are averages from the data collected over 1 cm length along the longitudinal direction, whereas the values at 1/3 distal end were derived from a single image that included only the cortical tissue. Bones marked as 6 and 10 were not subjected to spiral CT imaging.

Given the restrictions imposed by the size and the geometry of individual radii, it was not always possible to extract samples at all desired orientations. Thus, a complete set of specimens ($\beta=0$ deg, 45 deg and 90 deg) was obtained only for Bones 4, 5, 9, and 11. The tensile strength was determined for all available samples; the values given in Table 2 are the averages of two to three tests that were carried out for each specific orientation. Note

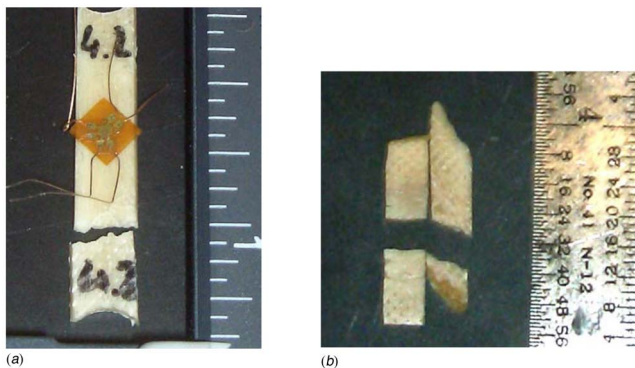


Fig. 5 Orientation of fracture plane; (a) sample tested in longitudinal direction; (b) samples tested at 90 deg and 45 deg

Table 1 Main results of structural tests on whole radii (the experimental accuracy ± 45 N)

Bone	Length (cm)	Dry weight (g)	BMC (g)	BMD midshaft (g/cm ²)	BMD 1/3 distal length (g/cm ²)	Fracture type	Fracture load (N)
1	23.3	22.01	13.63	0.55	0.39	Colles': ^a 23-A2.2	2090
2	23.9	29.79	19.48	0.62	0.49	Colles': ^a 23-A2.2	1990
3	22.8	29.55	18.73	0.62	0.41	Colles': ^a 23-A2.2	2540
4	24.6	46.24	30.36	0.82	0.55	Multifragmental shaft fracture	2240
5	24.4	39.87	25.84	0.79	0.61	Colles': ^a 23-A2.2	3720
6	21.7	22.83	14.22	0.68	0.43	Colles': ^a 23-A2.2	1180
7	25.2	41.04	26.28	0.66	0.54	Extra-articular: 23-A3.3	2990
8	23.8	40.91	32.94	0.83	0.6	Colles': ^a 23-A2.2	2580
9	23	45.19	28.77	0.83	0.58	Extra-articular: 23-A3.2	4530
10	26.8	32.52	20.53	0.57	0.36	Complete articular: 23-C2.2	2170
11	25	35.22	22.67	0.61	0.46	Complete articular: 23-C3.3	2360

^aExtra-articular fracture of the radius simple and impacted with dorsal tilt.

that the information gathered on elastic parameters is only partially complete. For specimens tested in longitudinal direction, $\beta=0$ deg, the elastic modulus E_1 and Poisson's ratio ν_{13} were determined from the strain gauge readings. For transverse samples, most values of Young's moduli were estimated based on data collected from the transducers (i.e., with accuracy within the range of $\pm 20\%$). For inclined specimens, i.e., at $\beta=45$ deg, no

values are provided as the measurements are not in the principal material system. Again, all values quoted are the averages from two to three tests.

Figure 8 shows the stress-strain curves obtained for samples extracted in the longitudinal direction ($\beta=0$ deg) from Bones 5 and 11. It is evident that the behavior of the tissue is typical of an elastic-brittle material. After reaching the critical value, the load is drastically reduced to zero due to the abrupt fracture. In samples tested in principal material directions ($\beta=0$ deg and $\beta=90$ deg), the fracture plane is orthogonal to the direction of loading. In inclined samples, the failure plane significantly deviates from the horizontal, as shown in Fig. 5(b).

Given the results of all material tests, as described above, let us focus now on one of the primary objectives of this study, i.e., the

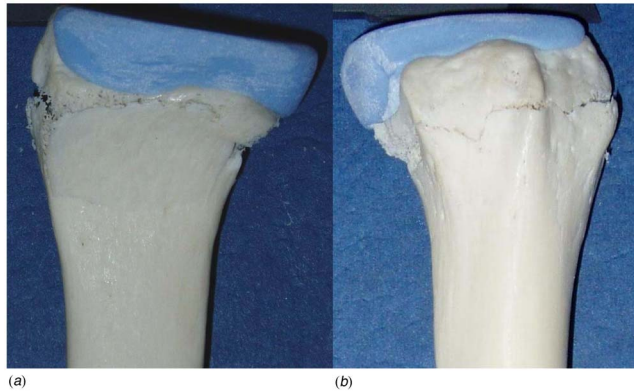


Fig. 6 Failure mode for Bone 5; (a) anterior posterior view; (b) posterior anterior view

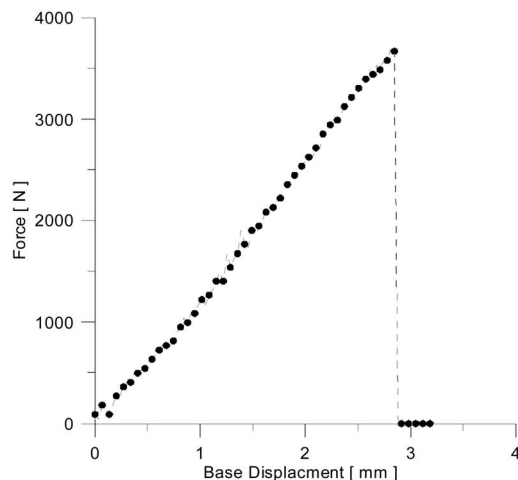


Fig. 7 Load-displacement characteristic for Bone 5

Table 2 Main results of material tests (cortical bone specimens).

Bone no.	Length (cm)	cortBMD (g/cm ³)		Sample orientation ^a (deg)	c (MPa)	$E_{1(3)}$ (GPa) ^b	$\nu_{13(31)}$ ^b
		1/3 length	Midshaft				
2	23.9	1.989	1.984	0	156.5	24.7	0.36
3	22.8	1.966	1.967	0	171.5	21.1	0.36
				90	14.4	~14	^c
4	24.6	1.962	1.941	0	129.9	32.4	0.45
				45	37.4	^d	^d
				90	25.6	~20	^c
5	24.4	1.978	1.981	0	174.9	27.5	0.29
				45	34.6	^d	^d
				90	22.4	~19	^c
6	21.7	^c	^c	0	186.7	~29	^c
9	22.9	1.932	1.941	0	134.9	27.9	0.35
				45	21.2	^d	^d
				90	15.4	19.6	0.27
11	25	1.988	1.964	0	152.9	25.9	0.37
				45	36.4	^d	^d
				90	15.6	~16	^c

Note: all results are averages from two to three samples. The term ~ means estimated based on the data from transducers.

^a0 deg and 90 deg correspond to longitudinal and transverse directions, respectively.

^bIndices 1 and 3 refer to longitudinal and transverse directions, respectively.

^cData not available.

^dNot applicable (measurements not in the principal material system).

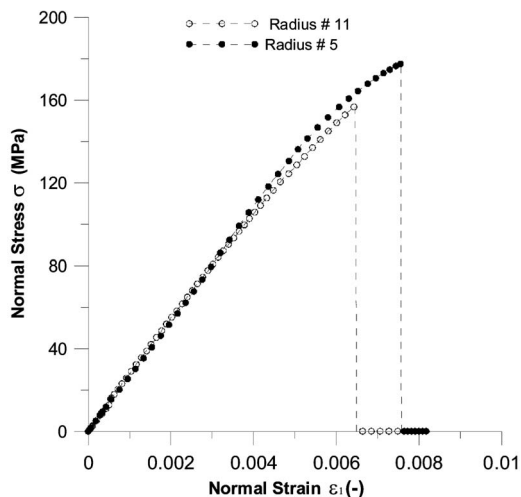


Fig. 8 Typical stress-strain curves representative of tensile tests for cortical tissue

validation of the functional form of the fracture criterion, Eqs. (1) and (2). Figures 9(a), 10(a), and 11(a) present the spatial distribution of the tensile strength, c , as a function of sample orientation, β . The variation of strength is supplemented by the evolution of the orientation of fracture plane, as shown in Figs. 9(b), 10(b), and 11(b). The results correspond to specimens extracted from a number of different radii (in this case, 5, 9, and 11). For the numerical simulations, the material parameters $\hat{c}$, Ω_1 were evaluated from Eq. (7) by employing the values of tensile strengths in longitudinal and transverse directions, as provided in Table 2. Given both $\hat{c}$ and Ω_1 , the spatial distribution of strength and the corresponding orientation of failure plane were determined by solving the set of simultaneous equations (8) for a number of discrete orientations β . The graphs in Figs. 9–11 give both the numerical solution (solid line) as well as experimental data (black dots). The experimental values of strength are the ones obtained from tests on samples extracted at $\beta=0$ deg, 45 deg, and 90 deg. Concerning the orientation of the fracture plane, for all longitudinal and transverse samples tested, the direction of tension was along one of the principal material axes and, in this case, the fracture plane always remained perpendicular to the loading direction, i.e., $\alpha=0$. For samples extracted at $\beta=45$ deg, the fracture

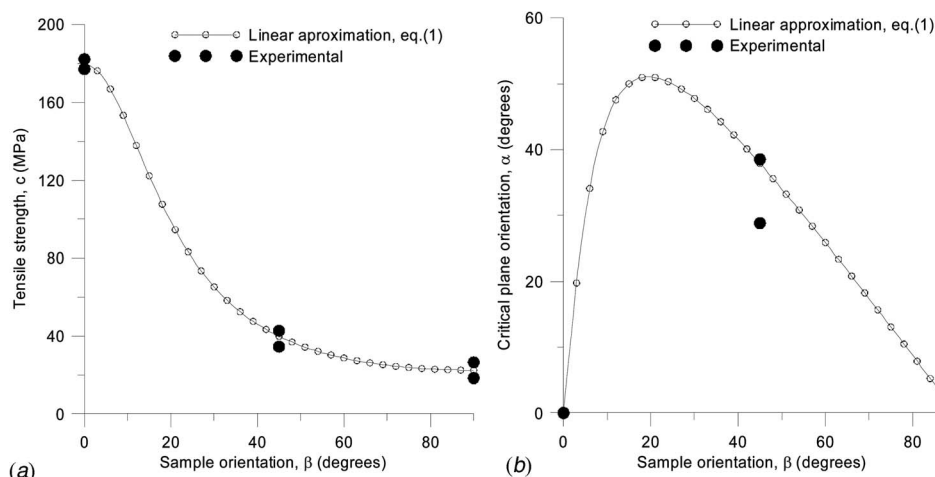


Fig. 9 Numerical simulation of direct tension test; Bone 5, $\hat{c}=74.86$ MPa, $\Omega_1=1.400$. (a) Distribution of axial tensile strength; (b) orientation of fracture plane.

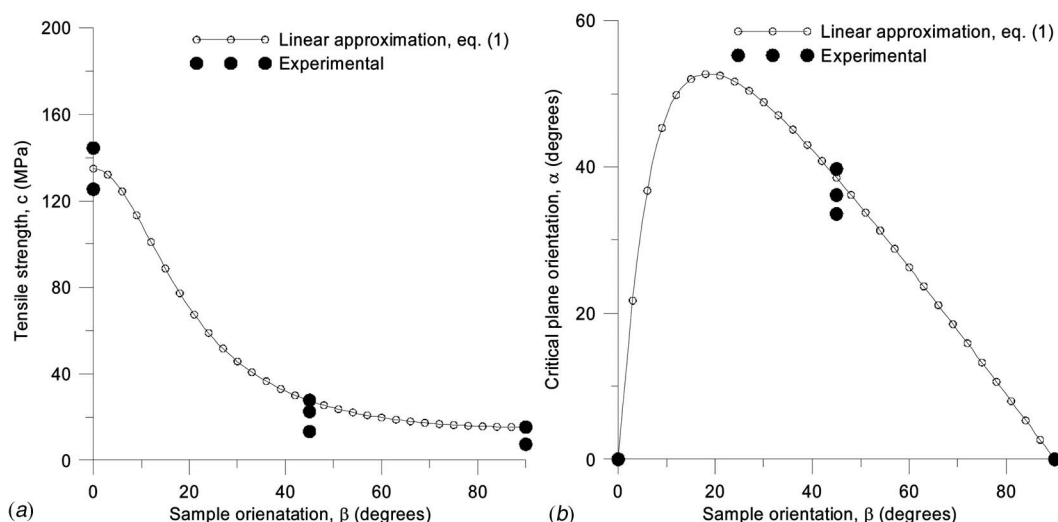


Fig. 10 Numerical simulation of direct tension test; Bone 9, $\hat{c}=55.21$ MPa, $\Omega_1=1.444$. (a) Distribution of axial tensile strength; (b) orientation of fracture plane

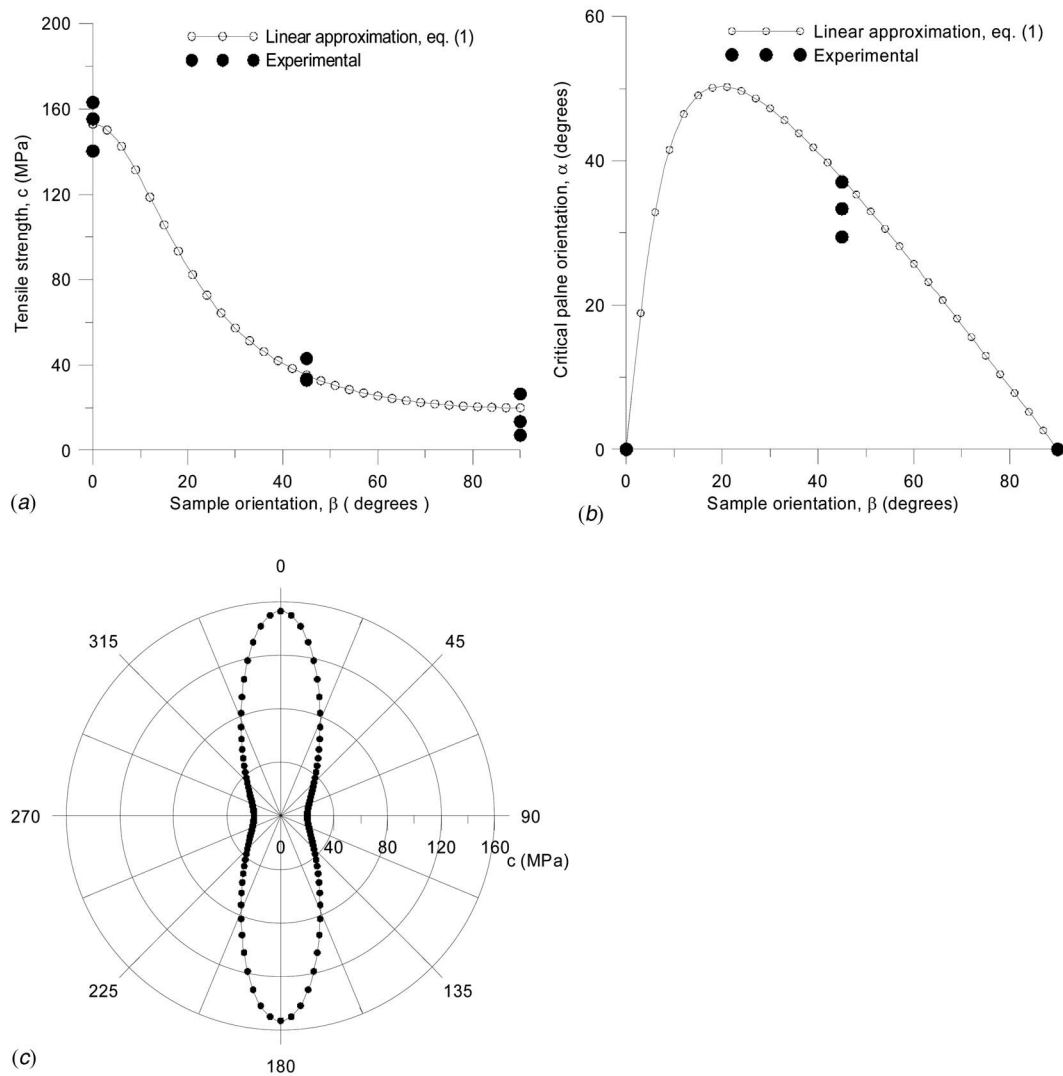


Fig. 11 Numerical simulation of direct tension test; Bone 11, $c_0=64.28$ MPa, $\Omega_1=1.379$. (a) Distribution of axial tensile strength, (b) orientation of fracture plane, and (c) distribution of axial tensile strength - polar coordinate representation

plane deviated substantially from the direction of the normal to the applied load. All the measured orientations are shown as black dots in Figs. 9(b), 10(b), and 11(b).

Figure 12 shows the relation between the tensile strength predicted by the model versus the available experimental data. In this case, all data acquired from specimens extracted at 45 deg have been pooled, since the remaining values, i.e., those for vertical and horizontal samples, were used to identify the parameters employed in the fracture criterion. The dashed line corresponds to a perfect agreement, i.e., the slope equal to 1. The solid line represents the fitted regression line; the regression coefficient is 0.912 ($R^2=0.68$). Note that the regression line shown here was forced through zero; the natural regression produced insignificantly small intercept ($y=0.9111x+0.0417$; $R^2=0.68$). Finally, Fig. 13 shows the correlation between the tensile strength in longitudinal direction, as predicted by the model, and the volumetric BMD of the cortical tissue. The correlation coefficient is 0.73.

The results of structural tests, as given in Table 1, were examined in the context of establishing a correlation between the geometric characteristics of the cortex, the tensile strength of the cortical tissue and the fracture load for individual radii that were tested. For this purpose, two distinct geometric measures were introduced, viz., minimum thickness of cortical shell ($cortT_{min}$

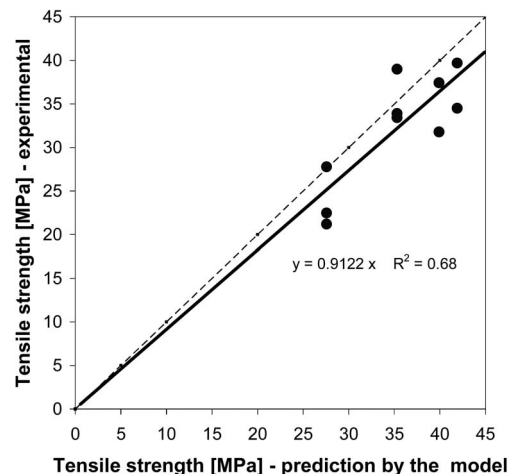


Fig. 12 Correlation between the tensile strength predicted by the model and the experimental results (excludes samples that were used for the calibration of the model). Note: the dashed line is a reference line illustrating perfect agreement; the solid line is the actual regression line forced through zero.

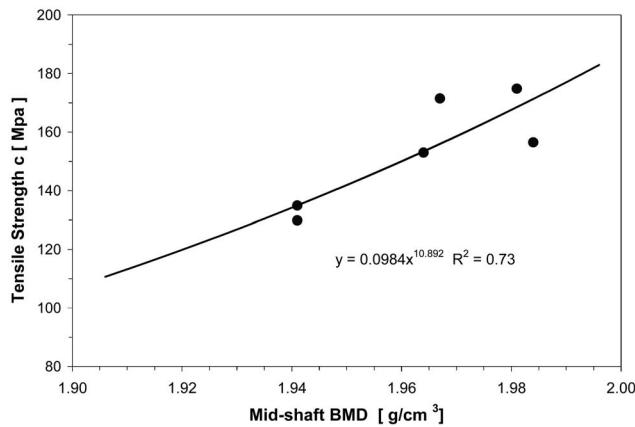


Fig. 13 Correlation between the tensile strength in longitudinal direction, as predicted by the model, and volumetric BMD of cortical tissue

(mm)) and cross-section area of cortical shell (cortCSA (mm²)), both obtained from spiral CT data. These new measures, in addition to standard DXA parameters (BMD and BMC), were included as independent variables.

Figure 14 shows the correlations between the fracture load and BMD (at 1/3 distal end) as well as cort $T_{\min}$ (at 6% distal end, i.e., just below the articular surface). The P value for both cases is less than 0.000001; BMD model explains 40% ($R^2=0.40$) of the variability of the data, whereas for cort $T_{\min}$ there is $R^2=0.39$. Note that in both cases the regression line was again forced through zero in order to preserve the physical meaning of the investigated correlation. The complete results of linear regression employing several different measures are given in Table 3. In general, the clinical measurements (BMD and BMC) are weak predictors of fracture load ($R^2 < 0.50$). Also, the tensile strength in dominant direction is a very weak predictor ($R^2=0.18$).

The geometric measures of cortex (cort $T_{\min}$, cortCSA) at ultradistal 6% and distal 15% ends do not approximate the failure load adequately. Combination of two independent geometric and DXA variables results in an improvement in coefficient of determination. At 15% distal length, the combination of BMD 1/3 with cort $T_{\min}$ and BMD with cortCSA gives $R^2=0.49$ and $R^2=0.59$, respectively. At 6% ultradistal location, combining BMD 1/3 with cort $T_{\min}$ allows to explain 53% of variability of the fracture load.

Table 3 Results of linear regression analysis

Location (% of distal length)	Parameters	R^2	P
	BMC	0.32	<0.00001
	BMD 1/3	0.40	<0.00001
	c (max. tensile strength)	0.18	<0.001
6 %	Cort $T_{\min}$	0.39	<0.00001
	Cort $T_{\min} \pm$ BMD 1/3	0.53	<0.00001
15%	Cort $T_{\min}$	0.22	<0.00001
	Cort $T_{\min} \pm$ BMD 1/3	0.49	<0.00001
15%	cortCSA	0.21	0
	cortCSA $\pm$ BMD 1/3	0.59	<0.00001

(Note: significance level $\alpha=0.05$; for $p < \alpha$ regression is statistically significant)

Preliminary Numerical Simulations

A reliable mechanical analysis of the fracture process requires the information on the bone geometry, the kinematic/static boundary conditions as well as the key material properties. Thus, all stages of the experimental program, as described in this work, are interrelated and are essential to provide the required input parameters. In order to illustrate this aspect, preliminary finite element (FE) simulations have been conducted for the radius Bone 5. The actual geometry was retrieved from CT scans and incorporated in a FE model, whereby the entire analyzed domain was discretized using eight-noded solid elements. The preliminary study carried out here involved an elastic analysis for a transversely isotropic material, which was followed by a verification of the anisotropic fracture criterion, viz., Eq. (1). The key material parameters employed, i.e., elastic constants and the strength parameters, are listed in Table 2.

The preliminary results, which focus on the prediction of the failure mode, are shown in Fig. 15. The figure on the right presents the experimental failure mechanism for the radius Bone 5. At the same time, the figure on the left shows the distribution of the failure function F , Eq. (1), obtained for the loading environment analogous to that in the full scale test. The darkened zone shows the region that experiences $F \rightarrow 0$; the latter being indicative of the onset of tensile fracture. Note that the location of fracture is consistent with experimental observation and the results clearly confirm that the fracture commences in the tensile regime.

As mentioned earlier, a more comprehensive numerical analysis

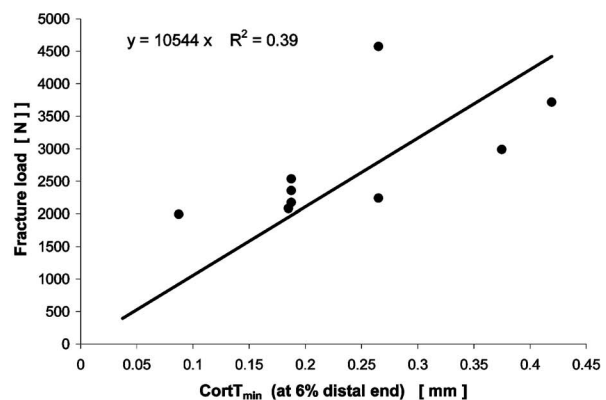
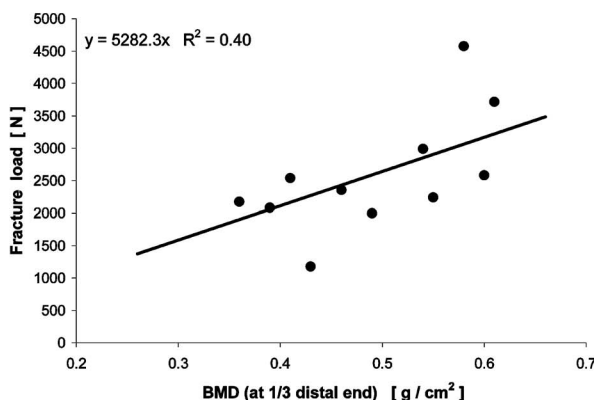


Fig. 14 Correlation between the ultimate load and clinical/geometric measurements

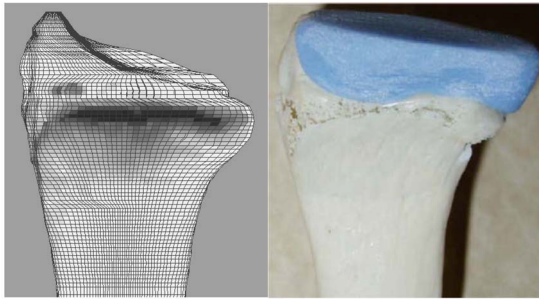


Fig. 15 Failure mode for the radius Bone 5: right: experimental result; left: FE simulation; distribution of failure function (darkened zone shows the region where the onset of tensile fracture occurs)

of the experimental work reported here will be presented separately. The analysis will employ a nonlinear framework that incorporates the description of localized deformation associated with initiation and propagation of fracture.

Discussion

In this paper, the results of a two-stage experimental program have been presented. The program involved a series of structural tests on the whole radii accompanied by material tests on samples of cortical tissue, the latter extracted from the fractured bones. The program was designed in such a way as to supply the information on bone geometry, material properties, and the kinematic/static boundary conditions. Such a methodology is essential for providing a comprehensive set of data that is required for conducting a quantitative numerical analysis of the bone fracture process. At this stage, some preliminary simulations were carried out in order to illustrate the methodology itself. In addition, the experimental results presented here were analyzed in two different contexts. The results of material tests were employed to examine the predictive abilities of a simple macroscopic criterion for describing the onset of brittle fracture within the human cortical bone tissue. For the structural tests, a correlation between the magnitude of fracture load, associated with Colles' fracture, and the geometric/strength properties of the cortex was investigated.

One of the main limitations of this work is the fact that the tests were all conducted on dry bone tissues. The experimental evidence indicates that the presence of fluid results in increased ductility and the reduction in ultimate tensile strength (Evans [18], Currey [19]). In general, the difference in mechanical properties of dry and wet tissues stems from the chemical interaction, which affects the mineral content, as well as from generation of the excess of pore pressure that affects the response of solid matrix. Given that the effect of drying and rewetting on the mechanical properties of cortical bone is relatively small (Currey [19]), it appears that the latter, i.e., solid-fluid interaction, may be a dominant factor. In this context, the properties of the dry tissue are of a significant importance in assessing the overall macroscopic response in the presence of fluid.

Another related aspect is that of the influence of the loading rate. As mentioned earlier, the tests reported here were conducted in static range, as the primary objective was to provide simple benchmark problems for the purpose of numerical validation. Thus, both rates employed (i.e., 0.5 mm/min for material testing and 0.1 mm/min for structural testing) were significantly lower than the impact rate (~ 100 mm/s; Muller et al. [15]). It is well known that at high strain rates, which are representative of an impact, both the stiffness and strength increase (Currey [20]). It is interesting to note though that under high loading rates the mechanical characteristics of the wet cortical bone become similar to those of dry bones tested under quasistatic regime (McElhaney [16]). Namely, the wet tissue becomes more brittle and the in-

crease in strength at amplified strain rates compensates for the loss associated with the drying process. In this context, the static tests on dry bones may be quite useful in assessing the basic trends under the clinically relevant state.

In what follows, the main observations regarding the behavior at both the material and the structural level are addressed in detail.

Mechanical Properties of Cortical Tissue. In cortical bone, the orientation of principal material axes can be estimated a priori based on the geometry of the cortical shell. Compact bone is composed of osteons, which are the main structural units. Every osteon has a number of lamellae, which contain collagen fibers laid in an ordered array. The resistance to tension is primarily due to the presence of those fibers. The average orientation of osteons is parallel to the axis of long bones (Martin et al. [21]). Thus, the preferred material orientation is largely defined by the geometry of the whole bone. The microstructure of the cortical tissue may be idealized as transversely isotropic, so that the remaining material directions are confined to the plane orthogonal to the direction of osteons. This well organized structure is quite different from that of the trabecular tissue. In the latter case, the specification of the local material triad entails the use of some specific measures of material microstructure (e.g., Odgaard [22], Smit et al. [23], Inglis and Pietruszczak [24]), which significantly complicates the mathematical formulation of the problem.

The typical trends in the mechanical response, as presented in Figs. 9–11, are consistent with those reported by other investigators (see Guo et al. [25], Reilly and Burstein [26]). In the tension regime, the dry cortical tissue is an elastic-brittle material. There is no evidence of ductility and the onset of fracture occurs at the axial strain of less than 1% (Jepsen et al. [27]). The values of Poisson's ratio, as recorded in the experiments here, are within the range of 0.29–0.45. This is consistent with the results reported by Reilly and Burstein [26], which quote the values within a similar interval (i.e., 0.34–0.47 in transverse plane and 0.29–0.5 in the longitudinal direction). Young's moduli along the preferred orientation are within the range of 20–32 GPa, while the transverse modulus reaches 19.6 GPa. The values reported in the literature are mostly for a wet cortical tissue and therefore are generally lower. For example, Reilly and Burstein [26] quote the values to be around 10–14 and 16–18 GPa in transversal and longitudinal directions, respectively. Turner et al. [28] determined the elastic properties from a series of nanoindentation tests and obtained values that are substantially higher, i.e., 16.6 ± 0.3 and 23.5 ± 0.2 GPa. Guo et al. [25] provided an overview of the estimates of elastic moduli based on nanoindentation experiments conducted on wet, dry, and embedded samples. As quoted by the authors, for a dry cortical tissue the values are in the range of 25–34 GPa with a high standard deviation. The latter interval is consistent with the range of values reported in the present work.

The tensile strength, as reported here, was within the range of 135–187 MPa for testing along the preferred orientation, and 14–26 MPa in transverse direction. Guo et al. [25] quote the values for the wet cortical tissue to be 135 ± 16 and 53 ± 11 MPa; a similar range of tensile strengths is recorded in the work of Reilly and Burstein [26]. Thus, the values obtained here are, in general, higher for testing in longitudinal direction and lower in transverse direction. This again stems from the fact that a dry tissue was being tested, in which case the bonding between the fibers is reduced leading to a lower strength in transverse direction.

As mentioned earlier, the main focus of the material testing was the validation of a simple fracture criterion (1). The key results are those presented in Figs. 9–11 and summarized in Figs. 12 and 13. Figure 12 gives the plot of experimental values of strength against those predicted by the model. Note, again, that only the values corresponding to samples extracted at 45 deg have been included here, as these were not directly employed in the calibration of the model parameters. The slope of the regression line is close to unity, indicating a fairly good agreement. It is recognized that the

number of tests performed is limited. At the same time, however, the level of accuracy is very encouraging and provides a considerable degree of confidence in the predictive abilities of the model. The question on whether the distribution of strength can be described sufficiently accurately by employing a linear dependence on the dyadic product $\Omega_{ij}n_i n_j$ is still open. As noted above, the predictions for samples tested at 45 deg are quite accurate. Also, the overall trend in the distribution of strength as reported by Reilly and Burstein [26], who took measurements along four different orientations, are fairly similar to those depicted in Figs. 9–11. In general, however, one or two additional testing angles seem to be necessary to assess the accuracy.

Finally, it is evident from Fig. 13 that the value of tensile strength is very sensitive to the variations in tissue density. In this work, a relatively good correlation has been obtained between volumetric BMD of the cortical tissue and the predicted tensile strength ($R^2=0.73$, $p<0.005$), which is significant in terms of the possibility of assessing the values of strength parameters by means of noninvasive measurements. In general, the existing evidence is not conclusive in this respect. While some authors report a poor correlation between the mechanical properties and CT density measurements (e.g., Snyder et al. [29]), others give the evidence of a moderate to good correlation. The examples of the latter include the work of Rho et al. [30], who found a good correlation between the apparent density and the elastic modulus ($R^2=0.79$), and Wachter et al. [31], who found a moderate correlation between the yield stress in compression and BMD ($R^2=0.72$).

Structural Tests on Whole Radii. The range of magnitudes of the fracture load, as reported in Table 1, remains consistent with the existing experimental evidence for both the dynamic (Augat et al. [14], Muller et al. [15]) and quasistatic (Spadaro et al. [2]) loading conditions. The load-displacement characteristics (Fig. 7) confirm the brittle nature of Colles' fracture. The dominating linear part is followed by an unstable branch associated with localized deformation.

It is evident that the clinical DXA measurements and/or the strength properties alone cannot be adequately correlated with the fracture load. Also, a weak correlation is found by employing various geometric measures of cortex at locations close to the distal end. By combining the geometric and strength properties, the predictive abilities improve; in general, however, they still remain unsatisfactory. It is therefore evident that the assessment of fracture load is a strictly mechanical problem and as such requires a proper numerical analysis. The value of ultimate load is affected by mechanical properties of the bone tissue, the geometry of the whole radius, and the boundary conditions.

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Exact Response of a Translating String With Arbitrarily Varying Length Under General Excitation

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The exact response of a translating string with constant tension and arbitrarily varying length is determined under general initial conditions and external excitation. The governing equation is transformed to a standard hyperbolic equation using characteristic transformation. The domain of interest for the transformed equation is divided into groups of subdomains according to the properties of wave propagation. d'Alembert's solution for any point in the zeroth subdomain group is obtained by using the initial conditions. The solution is extended to the whole domain of interest by using the boundary conditions, and a recursive mapping is found for the solution in the second and higher groups of subdomains. The least upper bound of the displacement of the freely vibrating string is obtained for an arbitrary movement profile. The forced response of the string with nonhomogeneous boundary conditions is obtained using a transformation method and the direct wave method. A new method is used to derive the rate of change of the vibratory energy of the translating string from the system viewpoint. Three different approaches are used to derive and interpret the rate of change of the vibratory energy of the string within a control volume, and the energy growth mechanism of the string during retraction is elucidated. The solution methods are applied to a moving elevator cable with variable length. An interesting parametric instability phenomenon in a translating string with sinusoidally varying length is discovered. [DOI: 10.1115/1.2839903]

Keywords: wave propagation, moving string with variable length, characteristic transformation, boundary excitation, rate of change of energy, unstable shortening string behavior, parametric instability, exact solution

1 Introduction

A translating string with variable length is a prototypical model of translating media with variable length. Translating media with variable length can model such systems as elevator cables, crane and mine hoists, satellite tethers, robotic arms through prismatic joints, flexible appendages, and paper sheets through copiers. The dynamics of translating media has typically been studied for cases with constant span length and transport velocity [1,2]. Some classes of time-varying translating media, including translating media with variable length and/or velocity, have been studied recently [3]. A recent review of research on the transverse vibration of translating strings is given in Chen [4].

The vibration of translating strings is governed by hyperbolic partial differential equations. Using Hamilton's principle, Miranker [5] derived the linear equation for the transverse vibration of a translating string with constant length and tension and an arbitrary velocity profile. Hamilton's principle can be applied to a control volume with variable mass when the virtual displacement on an open control surface vanishes [6].

Swope and Ames [7] obtained the response of a translating string with constant length and velocity and homogenous boundary conditions using the method of characteristics and discussed the properties of wave propagation in the string. Tan and Ying [8] presented an exact solution for the response of a translating string with constant length and velocity and various boundary conditions by applying the transfer function formulation. Besides exact solutions, spatial discretization methods, such as Galerkin and as-

sumed modes methods, have been commonly used for a translating medium. Instead of using eigenfunctions of a stationary string, Wickert and Mote [9] showed that the use of complex eigenfunctions of a translating string as the basis functions can significantly improve the eigenvalue predictions. Chen [10] used the translating string eigenfunctions to discretize a translating string coupled to a mass-spring-damper system. Jha and Parker [11] compared the spatial discretization methods using the stationary string and translating string eigenfunctions and examined the effectiveness of each method.

Compared to time-invariant translating strings, methods of solution for time-varying translating strings are far more underdeveloped. Carrier [12] first studied the dynamics of a translating string with parabolically varying length. Schaffers [13] studied the longitudinal vibration of a slowly moving mine hoist, and obtained the response of the hoist by using Riemann's method of characteristic curves. Using the method of distorted images, Ram and Caldwell [14] derived a wave solution for a stationary string with both boundaries moving with the same time-varying velocity; the problem can be related to a translating string with constant length and variable velocity through a coordinate transformation. Zhu and Guo [15] determined the free and forced responses of a translating string with an arbitrary velocity profile using the method of characteristic transformation. Zhu and Chen [16] determined the free and forced responses of translating strings and beams with variable length using three spatial discretization methods and showed that the three methods are mathematically equivalent.

Miranker [5] first analyzed the rate of change of the total mechanical energy of a translating string with constant length and showed that there is a periodic energy transfer between the portions of the string within and outside the two fixed boundaries. Wickert and Mote [17] added energy flux terms to the rate of change of the energy in Ref. [5] and explained the rates of change

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of the vibratory energies by the work-energy relation. Lee and Mote [18] studied the energy transfer mechanisms of a translating string at various boundaries for propagating harmonic waves. Renshaw et al. [19] defined Lagrangian and Eulerian energy functionals for translating media with constant length and velocity. Zhu [20] introduced control volume and system viewpoints for differentiating energies of translating media with constant and variable lengths. The rate of change of energy in Ref. [5] corresponds to the Eulerian energy functional in Ref. [19] and the control volume viewpoint in Ref. [20]. The rates of change of energies in Ref. [17] correspond to the Lagrangian energy functionals in Ref. [19] and the system viewpoint in Ref. [20].

Some unstable phenomena have been revealed in studies of strings with time-varying length. The spaghetti problem [12] exhibits an instability similar to that of a simple pendulum with variable length, i.e., the vibration amplitude increases as the length of the string decreases. Cooper [21] analyzed the long-time behavior and the energy growth of a stationary string with a sinusoidally moving boundary. The unstable behavior of the string is characterized by the bounded displacement and exponentially growing energy. Using the rates of change of vibratory energies from the control volume viewpoint, Zhu and Ni [22] investigated the general stability characteristics of horizontally and vertically translating strings and beams with variable length. The vibratory energy was found to decrease and increase in general during extension and retraction, respectively.

The dynamics of a translating string with variable length is an interesting research topic due to its important application, especially to the elevator industry [23]. While approximate methods, such as perturbation methods [24,25], modal methods [16], finite element methods [26], and multibody dynamics methods [27], have been used to find the free and forced responses of translating media with variable length, the exact solutions for the free and forced responses of the systems with an arbitrary movement profile have not been obtained.

In this work, the characteristic transformation in Ref. [15] is used to determine the exact response of a translating string with constant tension and arbitrarily varying length under general initial conditions and external excitation. The least upper bound of the displacement of the freely vibrating string is obtained. The forced response of the string with nonhomogeneous boundary conditions is obtained using two methods. The rate of change of the vibratory energy of the string is derived and interpreted from system and control viewpoints. A new approach is presented to derive the rate of change of the vibratory energy from the system viewpoint. The rate of change of the vibratory energy from the control volume viewpoint is derived using three different approaches: direct differentiation, employment of a new energy flux function describing the rate of energy transfer along the translating string, and use of the wave propagation properties. The energy transfer mechanism of the translating string at a boundary to which the string has a relative motion is provided and the unstable shortening string behavior is explained. The solution methods are applied to a moving elevator cable and the free response of the cable is compared to that from the modal methods in Ref. [16]. An interesting parametric instability phenomenon in a translating string with sinusoidally varying length is found numerically. It is discovered that when the length variation frequency of the translating string equals a natural frequency of the string with the averaged length and velocity, a parametric instability characterized by the bounded displacement and unbounded energy occurs, and a shock wave forms.

2 Equation of Motion

Consider a uniform string of mass per unit length ρ and constant tension P , translating at a subcritical speed $|V(T)| < \sqrt{P/\rho}$, where T is time, between two boundaries (Fig. 1); the tension change due to acceleration of the string is assumed to be negligible. When the tension change due to acceleration of the string is

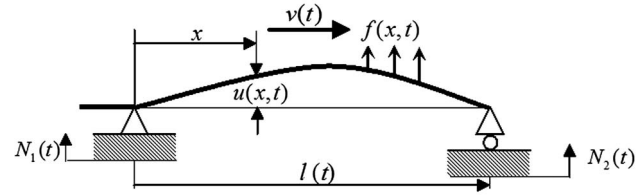


Fig. 1 Schematic of a translating string with variable length; the variables labeled are dimensionless

considered, finite difference and modal [16] methods can be used to find the dynamic response. The string has a relative motion to the left boundary and is fixed to the right boundary. The span length of the string is $L(T)$, satisfying $dL(T)/dT = V(T)$; the longitudinal vibration of the string is not considered. The transverse displacement of a material particle, instantaneously located at a spatial position $X \in [0, L(T)]$ at time T , is described by $U(X, T)$. The string is subjected to a distributed external force $F(X, T)$. A concentrated external force can be represented by $F(X, T)$ using a spatial Dirac delta function.

The variables $L(0)$, P , and ρ are used to nondimensionalize the system parameters, and the corresponding dimensionless variables are defined by

$$x = \frac{X}{L(0)}, \quad t = T \sqrt{\frac{P}{\rho L^2(0)}}, \quad l(t) = \frac{L(T)}{L(0)} \quad (1)$$

$$u(x, t) = \frac{U(X, T)}{L(0)}, \quad f(x, t) = \frac{F(X, T)L(0)}{P}, \quad v(t) = \frac{V(T)}{\sqrt{P/\rho}}$$

where $v(t) = \dot{l}(t) \in (-1, 1)$, with the overdot denoting differentiation with respect to t , is the dimensionless subcritical transport velocity, and $l(0) = 1$. Note that a positive velocity $v(t)$ indicates that the length of the string $l(t)$ increases instantaneously with time and a negative velocity indicates that the length decreases instantaneously.

The nondimensional governing equation for the linear transverse vibration of the translating string is [5]

$$u_{tt}(x, t) + 2v(t)u_{xt}(x, t) - [1 - v^2(t)]u_{xx}(x, t) + \dot{v}(t)u_x = f(x, t), \quad 0 < x < l(t) \quad (2)$$

where a subscript denotes partial differentiation. The boundary conditions are of the general form

$$u(0, t) = N_1(t), \quad u(l(t), t) = N_2(t) \quad (3)$$

where $N_1(t)$ and $N_2(t)$ are the prescribed, dimensionless displacements of the string at the two boundaries. The initial conditions are

$$u(x, 0) = a(x), \quad u_t(x, 0) = b(x), \quad 0 < x < 1 \quad (4)$$

where $a(x)$ and $b(x)$ are the dimensionless initial displacement and velocity of the string, respectively. The following displacement continuity conditions are assumed at the two boundaries at $t=0$:

$$N_1(0) = a(0), \quad N_2(0) = a(1) \quad (5)$$

3 Method of Solution

3.1 Characteristic Transformation. The characteristic transformation in Ref. [15] is employed here because the governing equation in Eq. (2) is of the same form as that in Ref. [15]:

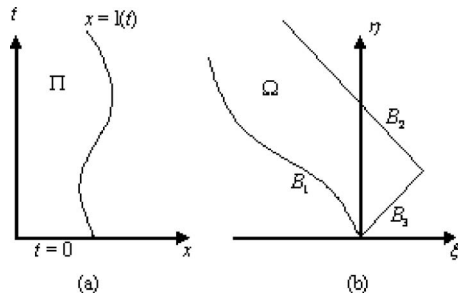


Fig. 2 Domains of the original (a) and transformed (b) equations and their boundaries

$$\begin{cases} \xi = \phi(x, t) = x - \int_0^t v(\tau) d\tau - t \\ \eta = \psi(x, t) = x - \int_0^t v(\tau) d\tau + t \end{cases} \quad (6)$$

where (ξ, η) are the characteristic coordinates, which are curvilinear when $\dot{v}(t) \neq 0$. This represents the characteristics of a wave that propagates downstream with a time-varying velocity $1+v(t)$ and a wave that propagates upstream with a time-varying velocity $1-v(t)$. The two coordinate systems, (ξ, η) and (x, t) , have a one-to-one correspondence because the Jacobian does not vanish. By use of the chain rule of differentiation, Eq. (2) becomes [15]

$$z_{\xi\eta}(\xi, \eta) = -\frac{1}{4}Q(\xi, \eta) \quad (7)$$

where

$$z(\xi, \eta) = z(\phi(x, t), \psi(x, t)) = u(x, t) \quad (8)$$

is the dependent variable in the (ξ, η) coordinate system, and

$$Q(\xi, \eta) = f\left(\frac{\xi + \eta}{2} + \int_0^{(\eta-\xi)/2} v(\tau) d\tau, \frac{\eta - \xi}{2}\right) \quad (9)$$

is the forcing function in the transformed equation.

The boundary and initial conditions associated with the governing equation in Eq. (7) are determined next. Substituting the boundary equation $x=0$ into the first equation in Eq. (6) yields

$$\xi = -\int_0^t v(\tau) d\tau - t, \quad t \in [0, \infty) \quad (10)$$

Since $\dot{\xi}(t) = -v(t) - 1 < 0$, ξ is a strictly monotonically decreasing function of t . Hence, the inverse function for Eq. (10) exists, and it is denoted by $t=w(\xi)$. It is assumed that $|v(t)| \leq r < 1$, where $r \in R$ is a constant. Consequently,

$$\xi = -\int_0^t v(\tau) d\tau - t \leq \int_0^t |v(\tau)| d\tau - t \leq \int_0^t r d\tau - t = (r-1)t \quad (11)$$

and $\xi \rightarrow -\infty$ as $t \rightarrow \infty$. Therefore, $t=w(\xi)$ is a strictly monotonically decreasing function of $\xi \in (-\infty, 0]$. Substituting $t=w(\xi)$ into the second equation in Eq. (6) gives the equation for the boundary B_1 in the $\xi-\eta$ plane (Fig. 2(b)), which corresponds to the boundary $x=0$ in the $x-t$ plane (Fig. 2(a)):

$$\eta = -\int_0^t v(\tau) d\tau + t = \xi + 2w(\xi), \quad \xi \in (-\infty, 0] \quad (12)$$

Note that B_1 is a strictly monotonically decreasing curve in the $\xi-\eta$ plane since

$$\eta'(\xi) = 1 + 2w'(\xi) = 1 - 2\frac{1}{\dot{\xi}(t)} = -\frac{1-v(t)}{1+v(t)} < 0 \quad (13)$$

where a prime denotes differentiation with respect to the argument. Because $|v(t)| \leq r < 1$, one has by use of Eq. (13),

$$-\frac{1+r}{1-r} \leq \eta'(\xi) \leq -\frac{1-r}{1+r} \quad (14)$$

Hence, B_1 is not asymptotic to any vertical line $\xi=\text{const}$ in the left half of the $\xi-\eta$ plane. Using $t=w(\xi)$, the first equation in Eq. (3), and Eq. (8) yields the boundary condition on B_1 in the $\xi-\eta$ plane:

$$z(\xi, \eta) = N_1(w(\xi)), \quad (\xi, \eta) \in B_1 \quad (15)$$

Substituting $x=l(t)=1+\int_0^t v(\tau) d\tau$ into Eq. (6) yields the equation for the boundary B_2 in the $\xi-\eta$ plane (Fig. 2(b)), which corresponds to the boundary $x=l(t)$ in the $x-t$ plane (Fig. 2(a)):

$$\eta = 2 - \xi, \quad \xi \in (-\infty, 1] \quad (16)$$

By use of $t=1-\xi$, the second equation in Eq. (3), and Eq. (8), the boundary condition on B_2 in the $\xi-\eta$ plane is

$$z(\xi, \eta) = N_2(1-\xi), \quad (\xi, \eta) \in B_2 \quad (17)$$

Finally, by use of $t=0$ in Eq. (6), the equation for the line B_3 in the $\xi-\eta$ plane (Fig. 2(b)), corresponding to $t=0$ in the $x-t$ plane (Fig. 2(a)), is

$$\xi = \eta, \quad \xi \in (0, 1) \quad (18)$$

Using the chain rule of differentiation and Eq. (6) yields

$$\frac{\partial}{\partial t} = [-v(t) - 1] \frac{\partial}{\partial \xi} + [-v(t) + 1] \frac{\partial}{\partial \eta} \quad (19)$$

By use of $t=0$, $x=\xi$, and Eqs. (19), (4), and (8), the initial conditions on B_3 in the $\xi-\eta$ plane are

$$z(\xi, \eta) = a(\xi), \quad (\xi, \eta) \in B_3 \quad (20)$$

$$[-v(0) - 1] \frac{\partial z}{\partial \xi} + [-v(0) + 1] \frac{\partial z}{\partial \eta} = b(\xi), \quad (\xi, \eta) \in B_3 \quad (21)$$

Let Ω be the domain enclosed by B_i ($i=1, 2, 3$) in the $\xi-\eta$ plane (Fig. 2(b)); it corresponds to the domain Π enclosed by $x=0$, $x=l(t)$, and $t=0$ in the $x-t$ plane in Fig. 2(a). The partial differential equation and its boundary and initial conditions in the (x, t) coordinate system (Eqs. (2)–(4)) have been transformed to a standard hyperbolic equation in the (ξ, η) coordinate system (Eqs. (7), (15), (17), (20), and (21)). Hence, by Eq. (6), $\xi=\text{const}$ and $\eta=\text{const}$ are the characteristic curves in the $\xi-\eta$ plane. Using these characteristic curves, the domain of interest Ω in the $\xi-\eta$ plane is partitioned into subdomains L_i , R_i , and C_i shown in Fig. 3, where L_i ($i=1, 2, \dots$) is the i th left subdomain, R_i ($i=1, 2, \dots$) is the i th right subdomain, and C_i ($i=0, 1, 2, \dots$) is the i th central subdomain. The zeroth subdomain group consists of only C_0 , and the i th ($i=1, 2, 3, \dots$) group of subdomains consists of L_i , R_i , and C_i .

3.2 d'Alembert Solution in the Subdomain C_0 . A classical solution for a stationary string with an infinite length and arbitrary initial conditions is given by d'Alembert [28]. The d'Alembert solution for the translating string governed by Eq. (7), in the subdomain C_0 in the $\xi-\eta$ plane, is of the general form

$$z(\xi, \eta) = h_0(\xi) + g_0(\eta) - \frac{1}{4} \int_0^\xi \int_0^\eta Q(s, \tau) d\tau ds, \quad (\xi, \eta) \in C_0 \quad (22)$$

where $h_0(\cdot)$ and $g_0(\cdot)$ are arbitrary functions, representing the waves that originate from B_3 propagating along the characteristic curves, $\xi=\text{const}$ and $\eta=\text{const}$, in the $\xi-\eta$ plane, respectively;

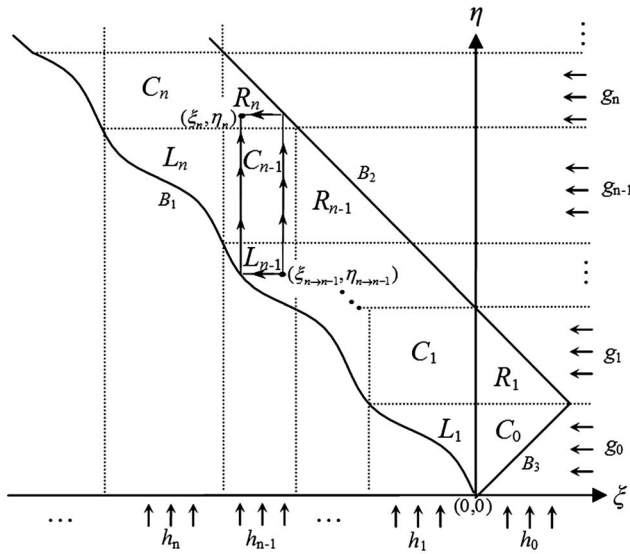


Fig. 3 Partition of the domain of interest and wave propagation in the ξ - η coordinate system. The wave functions h_n and g_n ($n=0,1,2,\dots$) that propagate along $\xi=\text{const}$ and $\eta=\text{const}$, respectively, are labeled.

$h_0(\cdot)$ and $g_0(\cdot)$ also represent the rightward and leftward propagating waves in the x - t plane, respectively. The third term on the right-hand side of Eq. (22) shows the effect of the distributed external force on the response. It should be mentioned that when one carries out the integration for this term, $Q(\xi, \eta)$ has the form of Eq. (9) if $(\xi, \eta) \in \Omega$ and is set to zero otherwise.

Using Eq. (22) in the initial condition in Eq. (20) and using Eq. (18) yield for $(\xi, \eta) \in B_3$,

$$h_0(\xi) + g_0(\xi) - \frac{1}{4} \int_0^\xi \int_0^\xi Q(s, \tau) d\tau ds = a(\xi), \quad \xi \in (0, 1) \quad (23)$$

Differentiating Eq. (23) with respect to ξ and using Leibnitz's rule yield

$$h'_0(\xi) + g'_0(\xi) - \frac{1}{4} \int_0^\xi Q(\xi, \tau) d\tau - \frac{1}{4} \int_0^\xi Q(s, \xi) ds = a'(\xi), \quad \xi \in (0, 1) \quad (24)$$

Using Eq. (22) in the initial condition in Eq. (21) and using Eq. (18) yield for $(\xi, \eta) \in B_3$,

$$[-v(0) - 1]h'_0(\xi) + [-v(0) + 1]g'_0(\xi) - \frac{1}{4}[-v(0) - 1] \int_0^\xi Q(\xi, \tau) d\tau - \frac{1}{4}[-v(0) + 1] \int_0^\xi Q(s, \xi) ds = b(\xi), \quad \xi \in (0, 1) \quad (25)$$

Solving Eqs. (24) and (25) for $h'_0(\xi)$ and $g'_0(\xi)$ yields

$$h'_0(\xi) = \frac{1-v(0)}{2} a'(\xi) - \frac{1}{2} b(\xi) + \frac{1}{4} \int_0^\xi Q(\xi, \tau) d\tau \quad (26)$$

$$g'_0(\xi) = \frac{1+v(0)}{2} a'(\xi) + \frac{1}{2} b(\xi) + \frac{1}{4} \int_0^\xi Q(s, \xi) ds \quad (27)$$

Integrating the above equations over $\xi \in (0, 1)$ yields

$$h_0(\xi) = h_0(0) + \frac{1-v(0)}{2} a(\xi) - \frac{1}{2} \int_0^\xi b(s) ds + \frac{1}{4} \int_0^\xi \int_0^\xi Q(\xi, \tau) d\tau d\xi \quad (28)$$

$$g_0(\xi) = g_0(0) + \frac{1+v(0)}{2} a(\xi) + \frac{1}{2} \int_0^\xi b(s) ds + \frac{1}{4} \int_0^\xi \int_0^\xi Q(s, \xi) ds d\xi \quad (29)$$

Substituting Eqs. (28) and (29) into Eq. (22) and using the first equation in Eqs. (3) and (5) yield

$$h_0(0) + g_0(0) = 0 \quad (30)$$

By use of Eq. (30), Eqs. (28) and (29) can be written as

$$h_0(\xi) = \frac{1-v(0)}{2} a(\xi) - \frac{1}{2} \int_{\xi_0}^\xi b(s) ds + \frac{1}{4} \int_0^\xi \int_0^\xi Q(\xi, \tau) d\tau d\xi, \quad \xi \in (0, 1) \quad (31)$$

$$g_0(\xi) = \frac{1+v(0)}{2} a(\xi) + \frac{1}{2} \int_{\xi_0}^\xi b(s) ds + \frac{1}{4} \int_0^\xi \int_0^\xi Q(s, \xi) ds d\xi, \quad \xi \in (0, 1) \quad (32)$$

where $\xi_0 \in R$ satisfies $g_0(0) = -(1/2) \int_0^{\xi_0} b(s) ds$. The d'Alembert solution of Eq. (7) in the subdomain C_0 , with the initial conditions in Eqs. (20) and (21), is of the form in Eq. (22), where $h_0(\xi)$ is given by Eq. (31) and $g_0(\cdot)$ is given by Eq. (32). Note that ξ_0 in Eqs. (31) and (32) can be set to any value because its effects cancel when one adds $h_0(\xi)$ and $g_0(\eta)$ in Eq. (22); ξ_0 can be conveniently set to zero.

3.3 Solution in the i th Group of Subdomains ($i \geq 1$) for the Case With Homogenous Boundary Conditions ($N_1(t) = N_2(t) = 0$). The solution of the governing equation in Eq. (7) in the subdomain L_1 , as shown in Fig. 3, is of the general form

$$z(\xi, \eta) = h_1(\xi) + g_0(\eta) - \frac{1}{4} \int_0^\xi \int_0^\eta Q(s, \tau) d\tau ds, \quad (\xi, \eta) \in L_1 \quad (33)$$

where $g_0(\cdot)$ is given by Eq. (32), representing a wave that originates from B_3 propagating along a characteristic curve $\eta=\text{const}$, and $h_1(\cdot)$ is a wave function, representing a wave that originates from the portion of B_1 in contact with L_1 , denoted here by $B_1 \cap L_1$ in which the notation " $\cap$ " is used in what follows to convey the same meaning, propagating along a characteristic curve $\xi=\text{const}$. Using Eq. (33) in the homogeneous boundary condition $z(\xi, \xi + 2w(\xi)) = 0$ on B_1 yields

$$h_1(\xi) = -g_0(\xi + 2w(\xi)) + \frac{1}{4} \int_0^\xi \int_0^{\xi+2w(\xi)} Q(s, \tau) d\tau ds, \quad \xi \in [w'(1), 0] \quad (34)$$

Note that due to the continuity of the displacement, the solution for an intersection point, such as the point corresponding to $\xi = w'(1)$ in Eq. (34), of two neighboring curves can be calculated from the formula for each curve. This is also true for an intersection curve of two neighboring subdomains. Substituting Eq. (34) into Eq. (33) yields the solution of Eq. (7) in the subdomain L_1 :

$$z(\xi, \eta) = g_0(\eta) - g_0(\xi + 2w(\xi)) + \frac{1}{4} \int_{\xi}^0 \int_{\xi+2w(\xi)}^{\eta} Q(s, \tau) d\tau ds, \quad (\xi, \eta) \in L_1 \quad (35)$$

Define a wave function $g_1(\eta)$ on $\{\eta | (\xi, \eta) \in B_2 \cap R_1\}$, representing a wave that originates from the portion of B_2 in contact with R_1 , propagating along a characteristic curve $\eta = \text{const}$. The solution of Eq. (7) in the subdomain R_1 is of the general form

$$z(\xi, \eta) = h_0(\xi) + g_1(\eta) - \frac{1}{4} \int_0^{\xi} \int_0^{\eta} Q(s, \tau) d\tau ds, \quad (\xi, \eta) \in R_1 \quad (36)$$

Using Eq. (36) in the homogeneous boundary condition $z(2 - \eta, \eta) = 0$ on B_2 yields

$$g_1(\eta) = -h_0(2 - \eta) + \frac{1}{4} \int_0^{2-\eta} \int_0^{\eta} Q(s, \tau) d\tau ds, \quad \eta \in [1, 2] \quad (37)$$

Substituting Eq. (37) into Eq. (36) yields the solution of Eq. (7) in the subdomain R_1 :

$$z(\xi, \eta) = h_0(\xi) - h_0(2 - \eta) + \frac{1}{4} \int_{\xi}^{2-\eta} \int_0^{\eta} Q(s, \tau) d\tau ds, \quad (\xi, \eta) \in R_1 \quad (38)$$

The solution of Eq. (7) in the subdomain C_1 is of the general form

$$z(\xi, \eta) = h_1(\xi) + g_1(\eta) - \frac{1}{4} \int_0^{\xi} \int_0^{\eta} Q(s, \tau) d\tau ds, \quad (\xi, \eta) \in C_1 \quad (39)$$

Substituting Eqs. (34) and (37) into Eq. (39) yields the solution of Eq. (7) in the subdomain C_1 :

$$z(\xi, \eta) = -g_0(\xi + 2w(\xi)) - h_0(2 - \eta) + \frac{1}{4} \int_0^{\xi} \int_0^{\xi+2w(\xi)} Q(s, \tau) d\tau ds + \frac{1}{4} \int_0^{2-\eta} \int_0^{\eta} Q(s, \tau) d\tau ds, \quad (\xi, \eta) \in C_1 \quad (40)$$

The solution of Eq. (7) in the i th ($i \geq 2$) group of subdomains L_i , R_i , and C_i is obtained by finding a recursive relation so that the solutions in these subdomains can eventually be related to those in the zeroth and first groups of subdomains C_0 , L_1 , R_1 , and C_1 . As shown in Fig. 3, the coordinates (ξ_n, η_n) of a point in the n th group of subdomains L_n , R_n , and C_n , with $i \geq n \geq 2$, correspond to the coordinates (ξ_{n-1}, η_{n-1}) of a point in the $(n-1)$ th group of subdomains L_{n-1} , R_{n-1} , and C_{n-1} , and the correspondence depends on the boundary curves B_1 and B_2 . More specifically, a point $(\xi_n, \eta_n) \in C_n$ corresponds to a point $(\xi_{n-1}, \eta_{n-1}) \in C_{n-1}$, a point $(\xi_n, \eta_n) \in L_n$ corresponds to a point $(\xi_{n-1}, \eta_{n-1}) \in R_{n-1}$, and a point $(\xi_n, \eta_n) \in R_n$ corresponds to a point $(\xi_{n-1}, \eta_{n-1}) \in L_{n-1}$. Furthermore, the correspondence above provides an insight on wave propagation, as shown below.

Define a wave function $h_n(\cdot)$ on $\{\xi | (\xi, \eta) \in B_1 \cap L_n\}$, representing a wave that originates from the portion of B_1 in contact with L_n , propagating along a characteristic curve $\xi = \text{const}$, and a wave function $g_n(\cdot)$ on $\{\eta | (\xi, \eta) \in B_2 \cap R_n\}$, representing a wave that originates from the portion of B_2 in contact with R_n , propagating along a characteristic curve $\eta = \text{const}$. An infinitesimal portion of a wave in a small region of the point (ξ_{n-1}, η_{n-1}) , propagating along $\xi = \xi_{n-1}$, is denoted by $h_{n-1}(\xi_{n-1})$. When it arrives at the boundary B_2 , it becomes a new infinitesimal wave $g_n(\eta_n)$ that

propagates along $\eta = \eta_n$. Similarly, an infinitesimal portion of a wave in a small region of the point (ξ_{n-1}, η_{n-1}) , propagating along $\eta = \eta_{n-1}$, is denoted by $g_{n-1}(\eta_{n-1})$. When it arrives at the boundary B_1 , it becomes a new infinitesimal wave $h_n(\xi_n)$ that propagates along $\xi = \xi_n$. The two new waves $g_n(\eta_n)$ and $h_n(\xi_n)$ meet subsequently at the point (ξ_n, η_n) in the $\xi - \eta$ plane.

More generally, a point (ξ_{i-m}, η_{i-m}) in the m th ($m < i$) group of subdomains corresponds to a point (ξ_i, η_i) in the i th group of subdomains by recursively applying the correspondence established above for the two points in the neighboring groups of subdomains. For instance, if (ξ_{i-1}, η_{i-1}) corresponds to (ξ_i, η_i) and (ξ_{i-1}, η_{i-1}) corresponds to (ξ_i, η_i) , then (ξ_{i-1}, η_{i-1}) corresponds to (ξ_i, η_i) and (ξ_{i-1}, η_{i-1}) is denoted by (ξ_{i-2}, η_{i-2}) .

The general forms of solutions of Eq. (7) in the subdomains L_n , C_n , and R_n are

$$z(\xi, \eta) = h_n(\xi) + g_{n-1}(\eta) - \frac{1}{4} \int_0^{\xi} \int_0^{\eta} Q(s, \tau) d\tau ds, \quad (\xi, \eta) \in L_n \quad (41)$$

$$z(\xi, \eta) = h_n(\xi) + g_n(\eta) - \frac{1}{4} \int_0^{\xi} \int_0^{\eta} Q(s, \tau) d\tau ds, \quad (\xi, \eta) \in C_n \quad (42)$$

$$z(\xi, \eta) = h_{n-1}(\xi) + g_n(\eta) - \frac{1}{4} \int_0^{\xi} \int_0^{\eta} Q(s, \tau) d\tau ds, \quad (\xi, \eta) \in R_n \quad (43)$$

respectively. Note that the general solutions in the subdomains L_{n-1} , C_{n-1} , and R_{n-1} can be obtained by changing the index n to $n-1$ in Eqs. (41)–(43). If $(\xi_n, \eta_n) \in L_n$, then $(\xi_{n-1}, \eta_{n-1}) \in R_{n-1}$. Using Eqs. (41) and (43) yields

$$z(\xi_n, \eta_n) = h_n(\xi_n) + g_{n-1}(\eta_n) - \frac{1}{4} \int_0^{\xi_n} \int_0^{\eta_n} Q(s, \tau) d\tau ds, \quad (\xi_n, \eta_n) \in L_n \quad (44)$$

$$z(\xi_{n-1}, \eta_{n-1}) = h_{n-2}(\xi_{n-1}) + g_{n-1}(\eta_{n-1}) - \frac{1}{4} \int_0^{\xi_{n-1}} \int_0^{\eta_{n-1}} Q(s, \tau) d\tau ds, \quad (\xi_{n-1}, \eta_{n-1}) \in R_{n-1} \quad (45)$$

respectively. Applying the homogeneous boundary conditions, $z(\xi_n, \eta_{n-1}) = 0$ along $B_1 \cap L_n$ and $z(\xi_{n-1}, \eta_n) = 0$ along $B_2 \cap R_{n-1}$, to Eqs. (41) and (43) yields

$$h_n(\xi_n) = -g_{n-1}(\eta_{n-1}) + \frac{1}{4} \int_0^{\xi_n} \int_0^{\eta_{n-1}} Q(s, \tau) d\tau ds \quad (46)$$

$$g_{n-1}(\eta_n) = -h_{n-2}(\xi_{n-1}) + \frac{1}{4} \int_0^{\xi_{n-1}} \int_0^{\eta_n} Q(s, \tau) d\tau ds \quad (47)$$

respectively. Substituting Eqs. (46) and (47) into Eq. (44) and using Eq. (45) yield

$$z(\xi_n, \eta_n) = -z(\xi_{n \rightarrow n-1}, \eta_{n \rightarrow n-1}) + \frac{1}{4} \int_{\xi_n}^{\xi_{n \rightarrow n-1}} \int_{\eta_{n \rightarrow n-1}}^{\eta_n} Q(s, \tau) d\tau ds, \quad (\xi_n, \eta_n) \in L_n \quad (48)$$

If $(\xi_n, \eta_n) \in C_n$ or R_n , using the similar procedure to that from Eqs. (44)–(47), one finds that Eq. (48) still holds. Hence, the solution at any point in the n th ($n \geq 2$) group of subdomains is related to that at the corresponding point in the $(n-1)$ th group of subdomains through

$$z(\xi_n, \eta_n) = -z(\xi_{n \rightarrow n-1}, \eta_{n \rightarrow n-1}) + \frac{1}{4} \int_{\xi_n}^{\xi_{n \rightarrow n-1}} \int_{\eta_{n \rightarrow n-1}}^{\eta_n} Q(s, \tau) d\tau ds, \quad (\xi_n, \eta_n) \in L_n \cup R_n \cup C_n \quad (49)$$

To obtain the solution at any point in the i th ($i \geq 2$) group of subdomains, one uses Eq. (49) recursively for the corresponding points from the i th group to the second group of subdomains:

$$z(\xi_i, \eta_i) = -z(\xi_{i \rightarrow i-1}, \eta_{i \rightarrow i-1}) + \frac{1}{4} \int_{\xi_i}^{\xi_{i \rightarrow i-1}} \int_{\eta_{i \rightarrow i-1}}^{\eta_i} Q(s, \tau) d\tau ds$$

$$z(\xi_{i \rightarrow i-1}, \eta_{i \rightarrow i-1}) = -z(\xi_{i \rightarrow i-2}, \eta_{i \rightarrow i-2}) + \frac{1}{4} \int_{\xi_{i \rightarrow i-1}}^{\xi_{i \rightarrow i-2}} \int_{\eta_{i \rightarrow i-2}}^{\eta_{i \rightarrow i-1}} Q(s, \tau) d\tau ds$$

$$\dots$$

$$z(\xi_{i \rightarrow 2}, \eta_{i \rightarrow 2}) = -z(\xi_{i \rightarrow 1}, \eta_{i \rightarrow 1}) + \frac{1}{4} \int_{\xi_{i \rightarrow 2}}^{\xi_{i \rightarrow 1}} \int_{\eta_{i \rightarrow 1}}^{\eta_{i \rightarrow 2}} Q(s, \tau) d\tau ds$$

Using the equations in Eq. (50) and replacing (ξ_i, η_i) by (ξ, η) yield

$$z(\xi, \eta) = (-1)^{i-1} z(\xi_{i \rightarrow 1}, \eta_{i \rightarrow 1}) + (-1)^{i-2} \frac{1}{4} \int_{\xi_{i \rightarrow 2}}^{\xi_{i \rightarrow 1}} \int_{\eta_{i \rightarrow 1}}^{\eta_{i \rightarrow 2}} Q(s, \tau) d\tau ds$$

$$+ \dots + \frac{1}{4} \int_{\xi}^{\xi_{i \rightarrow i-1}} \int_{\eta_{i \rightarrow i-1}}^{\eta} Q(s, \tau) d\tau ds, \quad (\xi, \eta) \in L_i \cup R_i \cup C_i \quad (51)$$

where $z(\xi_{i \rightarrow 1}, \eta_{i \rightarrow 1})$ can be obtained from Eq. (35), (38), or (40) depending on which subdomain the point $(\xi_{i \rightarrow 1}, \eta_{i \rightarrow 1})$ belongs to.

In summary, the solution of Eq. (7), with homogenous boundary conditions as given by Eqs. (15) and (17) with $N_1(w(\xi)) = N_2(1-\xi) = 0$, and the initial conditions in Eqs. (20) and (21), is given by Eqs. (22), (35), (38), and (40) if $(\xi, \eta) \in C_0 \cup L_1 \cup R_1 \cup C_1$, and by Eq. (51) if $(\xi, \eta) \in L_i \cup R_i \cup C_i$ ($i \geq 2$).

3.4 Boundedness of the Displacement of a Freely $f(x, t) = N_1(t) = N_2(t) = 0$ Vibrating String. When $f(x, t) = 0$, one has $Q(\xi, \eta) = 0$. Setting $Q(\xi, \eta) = 0$ in Eq. (51) and taking the absolute value on both sides of the equation yield

$$|z(\xi, \eta)| = |z(\xi_{i \rightarrow 1}, \eta_{i \rightarrow 1})|, \quad (\xi, \eta) \in L_i \cup R_i \cup C_i \quad (52)$$

for $i \geq 2$. Hence, the least upper bound of the solution for the whole domain of interest Ω equals to that for the domain $C_0 \cup L_1 \cup R_1 \cup C_1$. By Eqs. (22), (35), (38), and (40), the solutions for the zeroth and first groups of subdomains satisfy the following inequalities:

$$|z(\xi, \eta)| \leq \max_{0 \leq \xi_{1,2} \leq 1} |h_0(\xi_1) + g_0(\xi_2)|, \quad (\xi, \eta) \in C_0 \quad (53)$$

$$|z(\xi, \eta)| \leq \max_{0 \leq \xi_{1,2} \leq 1} |g_0(\xi_1) - g_0(\xi_2)|, \quad (\xi, \eta) \in L_1 \quad (54)$$

$$|z(\xi, \eta)| \leq \max_{0 \leq \xi_{1,2} \leq 1} |h_0(\xi_1) - h_0(\xi_2)|, \quad (\xi, \eta) \in R_1 \quad (55)$$

$$|z(\xi, \eta)| \leq \max_{0 \leq \xi_{1,2} \leq 1} |h_0(\xi_1) + g_0(\xi_2)|, \quad (\xi, \eta) \in C_1 \quad (56)$$

With the above inequalities, the displacement in the domain $C_0 \cup L_1 \cup R_1 \cup C_1$ is bounded by

$$|z(\xi, \eta)| \leq \max \left\{ \max_{0 \leq \xi_{1,2} \leq 1} |h_0(\xi_1) + g_0(\xi_2)|, \max_{0 \leq \xi_{1,2} \leq 1} |g_0(\xi_1) - g_0(\xi_2)|, \max_{0 \leq \xi_{1,2} \leq 1} |h_0(\xi_1) - h_0(\xi_2)| \right\} \quad (57)$$

Substituting Eqs. (31) and (32) into the right-hand side of Eq. (57) yields the least upper bound of the displacement expressed in terms of the initial conditions:

$$\max \left\{ \max_{0 \leq \xi_{1,2} \leq 1} \left| \frac{1-v(0)}{2} a(\xi_1) + \frac{1+v(0)}{2} a(\xi_2) + \frac{1}{2} \int_{\xi_1}^{\xi_2} b(s) ds \right|, \max_{0 \leq \xi_{1,2} \leq 1} \left| \frac{1+v(0)}{2} [a(\xi_1) - a(\xi_2)] + \frac{1}{2} \int_{\xi_2}^{\xi_1} b(s) ds \right|, \max_{0 \leq \xi_{1,2} \leq 1} \left| \frac{1-v(0)}{2} [a(\xi_1) - a(\xi_2)] - \frac{1}{2} \int_{\xi_2}^{\xi_1} b(s) ds \right| \right\} \quad (58)$$

When the initial velocity $b(x)$ is zero, the least upper bound of the displacement in Eq. (58) becomes

$$\max \left\{ \max_{0 \leq \xi_{1,2} \leq 1} \left| \frac{1-v(0)}{2} a(\xi_1) + \frac{1+v(0)}{2} a(\xi_2) \right|, \max_{0 \leq \xi_{1,2} \leq 1} \left| \frac{1+v(0)}{2} [a(\xi_1) - a(\xi_2)] \right|, \max_{0 \leq \xi_{1,2} \leq 1} \left| \frac{1-v(0)}{2} [a(\xi_1) - a(\xi_2)] \right| \right\} \quad (59)$$

Without loss of generality, it is assumed that the initial transport velocity $v(0) \geq 0$, and one has

$$\max_{0 \leq \xi_{1,2} \leq 1} \left| \frac{1-v(0)}{2} [a(\xi_1) - a(\xi_2)] \right| \leq \max_{0 \leq \xi_{1,2} \leq 1} \left| \frac{1+v(0)}{2} [a(\xi_1) - a(\xi_2)] \right| \quad (60)$$

The least upper bound of the displacement in Eq. (59) can be reduced to

$$\max \left\{ \max_{0 \leq \xi_{1,2} \leq 1} \left| \frac{1-v(0)}{2} a(\xi_1) + \frac{1+v(0)}{2} a(\xi_2) \right|, \max_{0 \leq \xi_{1,2} \leq 1} \left| \frac{1+v(0)}{2} [a(\xi_1) - a(\xi_2)] \right| \right\} \quad (61)$$

In addition, when the initial displacement $a(x)$ is non-negative or nonpositive for all $0 < x < 1$, the following inequalities hold:

$$\begin{aligned} & \max_{0 \leq \xi_{1,2} \leq 1} \left| \frac{1+v(0)}{2} [a(\xi_1) - a(\xi_2)] \right| \\ & \leq \max_{0 \leq \xi_{1,2} \leq 1} \left| \frac{1-v(0)}{2} a(\xi_1) + \frac{1+v(0)}{2} a(\xi_2) \right| \\ & \leq \max_{0 \leq \xi_1 \leq 1} \left| \frac{1-v(0)}{2} a(\xi_1) \right| + \max_{0 \leq \xi_2 \leq 1} \left| \frac{1+v(0)}{2} a(\xi_2) \right| \\ & = \max_{0 \leq x \leq 1} |a(x)| \end{aligned} \quad (62)$$

Using Eq. (62) in Eq. (61), one finds that the least upper bound of the displacement is $\max_{0 \leq x \leq 1} |a(x)|$, which is, as expected, the maximum initial displacement of the string along its spatial domain.

3.5 Solution for the Case With Nonhomogeneous Boundary Conditions

3.5.1 Transformation Method. A standard transformation [29] can be used to convert the linear partial differential equation with nonhomogeneous boundary conditions to one with homogeneous boundary conditions. For instance, for the case of $N_1(t) \neq 0$ and $N_2(t)=0$, the following transformation can be used:

$$u(x,t) = u^*(x,t) + \frac{l(t)-x}{l(t)} N_1(t) \quad (63)$$

where $u^*(x,t)$ satisfies the partial differential equation

$$\begin{aligned} & u_{tt}^* + 2v(t)u_{xt}^* - [1-v^2(t)]u_{xx}^* + \dot{v}(t)u_x^* \\ & = f(x,t) - \ddot{N}_1(t) + \ddot{l}(t) \frac{N_1(t)}{l(t)} + 2 \left[\frac{\dot{l}(t)\dot{N}_1(t)}{l(t)} - \frac{\dot{l}^2(t)N_1(t)}{l^2(t)} \right] \\ & + x \left[\frac{\ddot{N}_1(t)}{l(t)} - \frac{\ddot{l}(t)N_1(t)}{l^2(t)} - \frac{2\dot{l}(t)\dot{N}_1(t)}{l^2(t)} + \frac{2\dot{l}^2(t)N_1(t)}{l^3(t)} \right] \end{aligned} \quad (64)$$

with the homogenous boundary conditions

$$u^*(0,t) = 0, \quad u^*(l(t),t) = 0 \quad (65)$$

and the initial conditions

$$u^*(x,0) = a(x) - (1-x)N_1(0), \quad (66)$$

The method for finding the solution for u^* that satisfies Eqs. (64)–(66) has been presented in Secs. 3.1–3.3. The solution for u is obtained by substituting the solution for u^* into Eq. (63).

3.5.2 Direct Wave Method. Since the domain of dependence for any point in the subdomain C_0 is a subset of C_0 itself, the boundary conditions do not affect the solution in Eq. (22) for the subdomain C_0 . The solution procedure for the other groups of subdomains is essentially the same as that in Sec. 3.3, as demonstrated below for the first group of subdomains.

The solutions of Eq. (7) in the subdomains L_1 and R_1 are given by Eqs. (33) and (36), respectively. Applying the nonhomogeneous boundary conditions, $z(\xi, \xi+2w(\xi)) = N_1(w(\xi))$ and $z(2-\eta, \eta) = N_2(\eta-1)$, to Eqs. (33) and (36) yields

$$\begin{aligned} h_1(\xi) &= -g_0(\xi+2w(\xi)) + \frac{1}{4} \int_0^\xi \int_0^{\xi+2w(\xi)} Q(s,\tau) d\tau ds \\ &+ N_1(w(\xi)), \quad \xi \in [w'(1), 0] \end{aligned} \quad (67)$$

$$g_1(\eta) = -h_0(2-\eta) + \frac{1}{4} \int_0^{2-\eta} \int_0^\eta Q(s,\tau) d\tau ds + N_2(\eta-1), \quad (68)$$

$$\eta \in [1, 2]$$

respectively. Substituting Eqs. (67) and (68) into Eqs. (33) and (36) yields the solutions in the subdomains L_1 and R_1 , respectively:

$$\begin{aligned} z(\xi, \eta) &= g_0(\eta) - g_0(\xi+2w(\xi)) + \frac{1}{4} \int_\xi^0 \int_{\xi+2w(\xi)}^\eta Q(s,\tau) d\tau ds \\ &+ N_1(w(\xi)), \quad (\xi, \eta) \in L_1 \end{aligned} \quad (69)$$

$$\begin{aligned} z(\xi, \eta) &= h_0(\xi) - h_0(2-\eta) + \frac{1}{4} \int_\xi^{2-\eta} \int_0^\eta Q(s,\tau) d\tau ds \\ &+ N_2(\eta-1), \quad (\xi, \eta) \in R_1 \end{aligned} \quad (70)$$

Similarly, the solution of Eq. (7) in the subdomain C_1 is given by Eq. (39). Substituting Eqs. (67) and (68) into Eq. (39) yields the solution in the subdomain C_1 :

$$\begin{aligned} z(\xi, \eta) &= -g_0(\xi+2w(\xi)) - h_0(2-\eta) + \frac{1}{4} \int_0^\xi \int_0^{\xi+2w(\xi)} Q(s,\tau) d\tau ds \\ &+ \frac{1}{4} \int_0^{2-\eta} \int_0^\eta Q(s,\tau) d\tau ds + N_1(w(\xi)) + N_2(\eta-1), \\ &(\xi, \eta) \in C_1 \end{aligned} \quad (71)$$

To obtain the solution of Eq. (7) in the i th ($i \geq 2$) group of subdomains, one first establishes the correspondence between a point (ξ_n, η_n) in the n th ($i \geq n \geq 2$) group of subdomains and a point $(\xi_{n-n-1}, \eta_{n-n-1})$ in the $(n-1)$ th group of subdomains by following the procedure in Sec. 3.3 and using the nonhomogeneous boundary conditions $z(\xi_n, \eta_{n-n-1}) = N_1(w(\xi_n))$ and $z(\xi_{n-n-1}, \eta_n) = N_2(\eta_n-1)$:

$$\begin{aligned} z(\xi_n, \eta_n) &= -z_{n-1}(\xi_{n-n-1}, \eta_{n-n-1}) + \frac{1}{4} \int_{\xi_n}^{\xi_{n-n-1}} \int_{\eta_{n-n-1}}^{\eta_n} Q(s,\tau) d\tau ds \\ &+ N_1(w(\xi_n)) + N_2(\eta_n-1), \quad (\xi_n, \eta_n) \in L_n \cup R_n \cup C_n \end{aligned} \quad (72)$$

The solution at any point in the i th ($i \geq 2$) group of subdomains is obtained by using Eq. (72) recursively, as shown in Sec. 3.3:

$$\begin{aligned} z(\xi, \eta) &= (-1)^{i-1} z_1(\xi_{i-1}, \eta_{i-1}) + (-1)^{i-2} \int_{\xi_{i-2}}^{\xi_{i-1}} \int_{\eta_{i-1}}^{\eta_{i-2}} Q(s,\tau) d\tau ds \\ &+ \cdots + \int_\xi^{\xi_{i-i-1}} \int_{\eta_{i-i-1}}^\eta Q(s,\tau) d\tau ds + (-1)^{i-2} N_1(w(\xi_{i-2})) \\ &+ (-1)^{i-3} N_1(w(\xi_{i-3})) \cdots - N_1(w(\xi_{i-i-1})) \\ &+ N_1(w(\xi)) + (-1)^{i-2} N_2(\eta_{i-2}-1) + (-1)^{i-3} N_2(\eta_{i-3} \\ &- 1) \cdots - N_2(\eta_{i-i-1}-1) + N_1(\eta-1) \end{aligned} \quad (73)$$

The boundary excitation shows up in the last four lines of Eq. (73).

4 Energy Consideration

4.1 Rate of Change of Energy From System and Control Volume Viewpoints

4.1.1 System Viewpoint. The dimensionless energy density (energy per unit length) associated with the transverse vibration of the translating string, at a fixed spatial position x at time t , is [22]

$$\varepsilon(x,t) = \varepsilon_p(x,t) + \varepsilon_k(x,t) \quad (74)$$

where

$$\varepsilon_p(x, t) = \frac{1}{2} u_x^2(x, t) \quad (75)$$

$$\varepsilon_k(x, t) = \frac{1}{2} (u_t + v u_x)^2 \quad (76)$$

are the dimensionless potential and kinetic energy densities that result from the transverse vibration of the string, respectively.

A system is defined here as the collection of material particles occupying the spatial domain $\{x|x \in [0, l(t_0)]\}$ at a time t_0 . At time t , the system occupies the spatial domain $\{x|x \in [\int_{t_0}^t v(\tau) d\tau, l(t_0) + \int_{t_0}^t v(\tau) d\tau]\}$ and the vibratory energy of the system is

$$E_s(t) = \int_{\int_{t_0}^t v(\tau) d\tau}^{l(t_0) + \int_{t_0}^t v(\tau) d\tau} \varepsilon(x, t) dx \quad (77)$$

The rate of change of $E_s(t)$ is

$$\frac{dE_s(t)}{dt} = \frac{d}{dt} \left[\int_{\int_{t_0}^t v(\tau) d\tau}^{l(t_0) + \int_{t_0}^t v(\tau) d\tau} \varepsilon(x, t) dx \right] \quad (78)$$

To move the time derivative inside the integral in Eq. (78), a moving coordinate $\tilde{x}$ that moves with the same velocity as the string and coincides with the coordinate x at time t_0 is introduced:

$$x = \tilde{x} + \int_{t_0}^t v(\tau) d\tau \equiv \chi(\tilde{x}, t) \quad (79)$$

Using Eq. (79) in Eq. (78) yields

$$\frac{dE_s(t)}{dt} = \frac{d}{dt} \left[\int_0^{l(t_0)} \varepsilon(\chi(\tilde{x}, t), t) d\tilde{x} \right] \quad (80)$$

Since the limits of the integral in Eq. (80) are fixed, the time derivative can be moved inside the integral in Eq. (80):

$$\begin{aligned} \frac{dE_s(t)}{dt} &= \int_0^{l(t_0)} \frac{d}{dt} \varepsilon(\chi(\tilde{x}, t), t) d\tilde{x} = \int_0^{l(t_0)} [\varepsilon_x(\chi(\tilde{x}, t), t) \chi_t \\ &+ \varepsilon_t(\chi(\tilde{x}, t), t)] d\tilde{x} = \int_0^{l(t_0)} \left[\frac{\partial}{\partial t} + v(t) \frac{\partial}{\partial x} \right] \varepsilon(\chi(\tilde{x}, t), t) d\tilde{x} \end{aligned} \quad (81)$$

where Eq. (79) has been used. Using Eq. (79) in Eq. (81) yields

$$\frac{dE_s(t)}{dt} = \int_{\int_{t_0}^t v(\tau) d\tau}^{l(t_0) + \int_{t_0}^t v(\tau) d\tau} \left[\frac{\partial}{\partial t} + v(t) \frac{\partial}{\partial x} \right] \varepsilon(x, t) dx \quad (82)$$

The rate of change of the vibratory energy of the system at time t_0 is obtained from Eq. (82):

$$\frac{dE_s(t_0)}{dt} = \int_0^{l(t_0)} \left[\frac{\partial}{\partial t} + v(t_0) \frac{\partial}{\partial x} \right] \varepsilon(x, t_0) dx \quad (83)$$

Substituting Eqs. (74)–(76) into Eq. (83) and using Eqs. (2) and (3) at $t=t_0$ yield

$$\begin{aligned} \frac{dE_s(t_0)}{dt} &= -v(t_0) u_x^2(0, t_0) + \int_0^{l(t_0)} f(x, t_0) [u_t(x, t_0) + v(t_0) u_x(x, t_0)] dx \\ &- u_x(0, t_0) \dot{N}_1(t_0) + u_x(l(t_0), t_0) \dot{N}_2(t_0) \end{aligned} \quad (84)$$

Note that the time derivatives of Eq. (3) at $t=t_0$, given by $u_t(0, t_0) = \dot{N}_1(t_0)$ and $u_t(l(t_0), t_0) + v(t_0) u_x(l(t_0), t_0) = \dot{N}_2(t_0)$, are also used in Eq. (84).

Equation (84) establishes the work-energy relation of the system along the transverse direction at time t_0 . The first and third terms on the right-hand side of Eq. (84) can be written as

$-u_x(0, t_0)[v u_x(0, t_0) + \dot{N}_1(t_0)]$, which is the product of the transverse component of the tension and the transverse velocity of the string at $x=0$ at $t=t_0$, representing the rate of work done by the transverse component of the tension at $x=0$. The second term on the right-hand side of Eq. (84) represents the rate of work done by the distributed external force. The fourth term on the right-hand side of Eq. (84) represents the rate of work done by the transverse component of the tension at $x=l(t_0)$.

4.1.2 Control Volume Viewpoint. A control volume at time t is defined as the spatial domain $\{x|x \in [0, l(t)]\}$. The vibratory energy of the string within the control volume is

$$E_{cv}(t) = \int_0^{l(t)} \varepsilon(x, t) dx \quad (85)$$

The rate of change of $E_{cv}(t)$ is obtained by differentiating Eq. (85) using Leibnitz's rule and Eqs. (74)–(76):

$$\begin{aligned} \frac{dE_{cv}(t)}{dt} &= \int_0^{l(t)} \{u_x(x, t) u_{xt}(x, t) + [u_t(x, t) + v(t) u_x(x, t)] [u_{tt}(x, t) \\ &+ v u_{xt}(x, t) + \dot{v}(t) u_x(x, t)]\} dx + \frac{1}{2} v(t) \{u_x^2(l(t), t) \\ &+ [u_t(l(t), t) + v(t) u_x(l(t), t)]^2\} \end{aligned} \quad (86)$$

Using Eqs. (2) and (3) and the time derivatives of Eq. (3) in Eq. (86) yields

$$\begin{aligned} \frac{dE_{cv}(t)}{dt} &= -\frac{1}{2} v(t) [1 - v^2(t)] u_x^2(0, t) + \int_0^{l(t)} f(x, t) [u_t(x, t) \\ &+ v(t) u_x(x, t)] dx - u_x(0, t) \dot{N}_1(t) + u_x(l(t), t) \dot{N}_2(t) \end{aligned} \quad (87)$$

Evaluating Eq. (87) at $t=t_0$ yields the rate of change of the vibratory energy of the string within the control volume at time t_0 . Comparing the resulting expression with Eq. (84) yields

$$\frac{dE_{cv}(t_0)}{dt} = \frac{dE_s(t_0)}{dt} + v(t_0) \varepsilon(0, t_0) \quad (88)$$

The same equation is obtained in Ref. [22] using a different approach. The first term on the right-hand side of Eq. (88) represents the rates of work done by nonconservative domain and boundary forces, as explained in Sec. 4.1.1. The second term on the right-hand side of Eq. (88) represents the energy flux due to the mass transfer across the boundary $x=0$. A more detailed description of the energy flux in the translating string is given in Sec. 4.2.

4.2 Energy Flux for a Translating String. The energy flux describes the rate of energy transfer at a spatial point in a structure. The energy flux function for a stationary string is defined by [30]

$$S(x, t) = -u_x(x, t) u_t(x, t) \quad (89)$$

It is the product of the nondimensional transverse component of the tension $-u_x(x, t)$, exerted by the neighboring portion of the string whose spatial coordinate is less than x on the string particle at the spatial position x , and the corresponding velocity of the string particle $u_t(x, t)$. Hence, it is the rate of work done by the portion of the string with the spatial coordinate less than x on the string particle at the spatial position x . When this rate of work done is positive, the energy flux is along the x direction. The energy flux for a stationary string is the transfer of the rate of work done by the transverse component of the tension at a fixed spatial point along the string.

The energy flux function for a translating string is defined here by

$$S(x,t) = -u_x \frac{Du(x,t)}{Dt} + v(t)\varepsilon(x,t), \quad x \in (0, l(t)) \quad (90)$$

where $D/Dt = \partial/\partial t + v(t)(\partial/\partial x)$ is the material derivative and $\varepsilon(x,t)$ is the vibratory energy density in Eq. (74). The energy flux function for a translating string differs from that for a stationary string in that it is the sum of the transfer of the rate of work done by the transverse component of the tension at a fixed spatial point x along the string, $-u_x(Du(x,t)/Dt)$, and a transfer of the vibratory energy per unit time, $v\varepsilon(x,t)$, that results from the translational motion of the string. While the analysis below can be extended to the forced vibration problem, only the free vibration problem ($f(x,t)=N_1(t)=N_2(t)=0$) is considered here to explain the energy transfer mechanism within the control volume.

Substituting Eqs. (74)–(76) into Eq. (90) yields

$$S(x,t) = -[1 - v^2(t)]u_x u_t - \frac{1}{2}v(t)[1 - v^2(t)]u_x^2 + \frac{1}{2}v(t)u_t^2, \quad x \in (0, l(t)) \quad (91)$$

For instance, by use of $u_t(0,t)=0$, the energy flux at $x=0^+$ is

$$S(0^+,t) = -\frac{1}{2}v(t)[1 - v^2(t)]u_x^2(0,t) \quad (92)$$

Note that the reaction force at the boundary $x=0$ produces an energy source at $x=0$, and $S(0^+,t)$ represents the energy flux from the source $x=0$ to $x>0$. Similarly, by use of $u_t(l(t),t) + v(t)u_x(l(t),t)=0$, the energy flux at $x=l(t)^-$ is

$$S(l(t)^-,t) = \frac{1}{2}v(t)u_x^2(l(t),t) \quad (93)$$

and $S(l(t)^-,t)$ represents the energy flux from the source $x=l(t)$, due to the reaction force at the boundary, to $x<l(t)$.

Differentiating Eq. (91) with respect to x and Eq. (74) with respect to t , adding the two resulting expressions, and using the homogeneous governing equation corresponding to Eq. (2) yield the differential relation between the energy flux function and the energy density:

$$\frac{\partial \varepsilon(x,t)}{\partial t} + \frac{\partial S(x,t)}{\partial x} = 0, \quad x \in (0, l(t)) \quad (94)$$

which is of the same form as that for the stationary string in Ref. [30]. Integrating Eq. (94) over $x \in (x_1, x_2)$, where $x_1, x_2 \in (0, l(t))$, yields

$$-\int_{x_1}^{x_2} \frac{\partial \varepsilon(x,t)}{\partial t} dx = S(x,t)|_{x_1}^{x_2} \quad (95)$$

Exchanging the order of integration and partial differentiation on the left-hand side of Eq. (95) yields the integral relation between the energy flux function and the energy density:

$$-\frac{d}{dt} \left(\int_{x_1}^{x_2} \varepsilon(x,t) dx \right) = S(x,t)|_{x_1}^{x_2} \quad (96)$$

where $x_1, x_2 \in (0, l(t))$ must be fixed. Equations (94) and (96) are the differential and integral forms of energy conservation that describe the energy balance for a particular point and section of the translating string, respectively. Equation (96) states that the rate of change of the vibratory energy is equal to the net energy flux across the boundaries. When x_1 or x_2 is a function of time, a modified integral form needs to be derived. If x_1 is fixed and x_2 is a function of time, while Eq. (95) still holds, the order of integration and partial differentiation in Eq. (95) cannot be interchanged. Using Leibnitz's rule yields

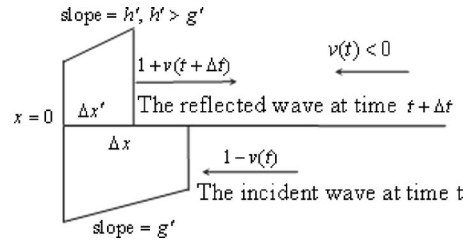


Fig. 4 Compression of an infinitesimal wave at the boundary $x=0$

$$\int_{x_1}^{x_2} \frac{\partial \varepsilon(x,t)}{\partial t} dx + \frac{dx_2}{dt} \varepsilon(x_2,t) = \frac{d}{dt} \left(\int_{x_1}^{x_2} \varepsilon(x,t) dx \right) \quad (97)$$

Substituting Eq. (97) into Eq. (95) yields the modified form of the integral relation between the energy flux function and the energy density:

$$-\frac{d}{dt} \left(\int_{x_1}^{x_2} \varepsilon(x,t) dx \right) = S(x,t)|_{x_1}^{x_2} + \frac{dx_2}{dt} \varepsilon(x_2,t) \quad (98)$$

Equation (87) for the free vibration problem can be obtained here by setting $x_1=0$ and $x_2=l(t)$ in Eq. (98) and using Eqs. (92) and (93) in Eq. (98):

$$\frac{dE_{cv}(t)}{dt} = \frac{d}{dt} \left(\int_0^{l(t)} \varepsilon(x,t) dx \right) = -\frac{1}{2}v(t)[1 - v^2(t)]u_x^2(0,t) = S(0^+,t) \quad (99)$$

Equation (99) states that the rate of change of the vibratory energy within the control volume is equal to the energy flux into the control volume across the boundary $x=0$, which includes the energy flux due to the mass transfer, $v(t)\varepsilon(0,t)=(1/2)v(t)[1 + v^2(t)]u_x^2(0,t)$, as shown in Eq. (88), and the energy flux due to the rate of work done by the transverse component of the tension at the boundary $x=0$, $-v(t)u_x^2(0,t)$, which is $dE_s(t)/dt$ in Eq. (88).

4.3 Wave Propagation in the Translating String. While the wave analysis below can be extended to the forced vibration problem, only the free vibration problem ($f(x,t)=N_1(t)=N_2(t)=0$) is considered here to explain the energy growth mechanism at the boundary $x=0$. The homogeneous solution of Eq. (7) is of the general form in the $x-t$ coordinate system:

$$u(x,t) = h \left(x - \int_0^t v(\tau) d\tau - t \right) + g \left(x - \int_0^t v(\tau) d\tau + t \right), \quad x \in [0, l(t)] \quad (100)$$

where $h(\cdot)$ and $g(\cdot)$ are arbitrary functions, representing rightward and leftward propagating waves, respectively. Consider an infinitesimal portion of a leftward propagating wave of width Δx , occupying the domain $[0, \Delta x]$ at time t and traveling with a velocity $1-v(t)$ (see Fig. 4). At time $t+\Delta t$, the infinitesimal wave reflects from the left boundary and occupies the domain $[0, \Delta x']$; this reflected infinitesimal wave of width $\Delta x'$ propagates rightward. Using the boundary condition $u(0,r)=0$, where $r \in [t, t+\Delta t]$, in Eq. (100) yields

$$g \left(-\int_0^t v(\tau) d\tau + \tau \right) = -h \left(-\int_0^t v(\tau) d\tau - \tau \right), \quad r \in [t, t+\Delta t] \quad (101)$$

which means that the reflected wave has the same amplitude as the incident wave; this explains the bounded displacement of the freely vibrating string, as shown in Sec. 3.4. Since the propagating

wave is nondispersive here, the amplitude of the wave front of the reflected wave at time t , located at $x=0$, is the same as that of the reflected wave at time $t+\Delta t$, located at $x=\Delta x'$:

$$h\left(-\int_0^t v(\tau)d\tau-t\right)=h\left(\Delta x'-\int_0^{t+\Delta t} v(\tau)d\tau-(t+\Delta t)\right) \quad (102)$$

Combining Eq. (101) at $r=t$ and Eq. (102) yields

$$g\left(-\int_0^t v(\tau)d\tau+t\right)=-h\left(\Delta x'-\int_0^{t+\Delta t} v(\tau)d\tau-(t+\Delta t)\right) \quad (103)$$

Similarly, the amplitude of the waveback of the incident wave at time t , located at $x=\Delta x$, is the same as that of the incident wave at time $t+\Delta t$, located at $x=0$:

$$g\left(\Delta x-\int_0^t v(\tau)d\tau+t\right)=g\left(-\int_0^{t+\Delta t} v(\tau)d\tau-(t+\Delta t)\right) \quad (104)$$

Combining Eq. (101) at $r=t+\Delta t$ and Eq. (104) yields

$$g\left(\Delta x-\int_0^t v(\tau)d\tau+t\right)=-h\left(-\int_0^{t+\Delta t} v(\tau)d\tau-(t+\Delta t)\right) \quad (105)$$

Subtracting Eq. (103) from Eq. (105) yields

$$\begin{aligned} &g\left(\Delta x-\int_0^t v(\tau)d\tau+t\right)-g\left(-\int_0^t v(\tau)d\tau+t\right) \\ &=h\left(\Delta x'-\int_0^{t+\Delta t} v(\tau)d\tau-(t+\Delta t)\right) \\ &-h\left(-\int_0^{t+\Delta t} v(\tau)d\tau-(t+\Delta t)\right) \end{aligned} \quad (106)$$

The slope of the incident wave at $x=0$ at time t is

$$\begin{aligned} &g'\left(-\int_0^t v(\tau)d\tau+t\right) \\ &=\lim_{\Delta x \rightarrow 0} \frac{g(\Delta x-\int_0^t v(\tau)d\tau+t)-g(-\int_0^t v(\tau)d\tau+t)}{\Delta x} \end{aligned} \quad (107)$$

which is related to the left-hand side of Eq. (106); this is also the slope of the incident wave with an infinitesimal width. Similarly, the slope of the reflected wave at $x=0$ at time $t+\Delta t$ is

$$\begin{aligned} &h'\left(-\int_0^{t+\Delta t} v(\tau)d\tau-(t+\Delta t)\right)=\lim_{\Delta x' \rightarrow 0} \\ &\times \frac{h(\Delta x'-\int_0^{t+\Delta t} v(\tau)d\tau-(t+\Delta t))-h(-\int_0^{t+\Delta t} v(\tau)d\tau-(t+\Delta t))}{\Delta x'} \end{aligned} \quad (108)$$

which is related to the right-hand side of Eq. (106); this is also the slope of the reflected wave with an infinitesimal width. When Δt approaches zero, Eq. (108) becomes

$$\begin{aligned} &h'\left(-\int_0^t v(\tau)d\tau-t\right) \\ &=\lim_{\Delta x' \rightarrow 0} \frac{h(\Delta x'-\int_0^t v(\tau)d\tau-t)-h(-\int_0^t v(\tau)d\tau-t)}{\Delta x'} \end{aligned} \quad (109)$$

Dividing Eq. (106) by Δx yields

$$\begin{aligned} &\frac{g(\Delta x-\int_0^t v(\tau)d\tau+t)-g(-\int_0^t v(\tau)d\tau+t)}{\Delta x} \\ &=\frac{h(\Delta x'-\int_0^{t+\Delta t} v(\tau)d\tau-(t+\Delta t))-h(-\int_0^{t+\Delta t} v(\tau)d\tau-(t+\Delta t))}{\Delta x'} \\ &\times \frac{\Delta x'}{\Delta x} \end{aligned} \quad (110)$$

When Δt approaches zero, Δx and $\Delta x'$ approach zero and Eq. (110) becomes, after using Eqs. (107) and (109),

$$\lim_{\Delta x, \Delta x' \rightarrow 0} \frac{\Delta x'}{\Delta x} = \frac{g'(-\int_0^t v(\tau)d\tau+t)}{h'(-\int_0^t v(\tau)d\tau-t)} \quad (111)$$

The left-hand side of Eq. (111) represents the ratio of the width of the reflected wave to that of the incident wave, and the right-hand side of Eq. (111) represents the ratio of the slope of the incident wave to that of the reflected wave. Equation (111) shows that the width ratio of the reflected wave to the incident wave is almost inversely proportional to their slope ratio.

Define the ratio of compression α at the boundary $x=0$ as

$$\alpha = \frac{h'(-\int_0^t v(\tau)d\tau-t)}{g'(-\int_0^t v(\tau)d\tau+t)} \quad (112)$$

Consequently, by Eq. (111),

$$\frac{\Delta x'}{\Delta x} \approx \frac{1}{\alpha} \quad (113)$$

when Δx and $\Delta x'$ are small enough. Using $u_i(0,t)=0$ in Eq. (100) and comparing the resulting expression to Eq. (112) yield

$$\alpha = \frac{1-v(t)}{1+v(t)} \quad (114)$$

Using Eqs. (74)–(76) with $x=0$ yields the vibratory energy of the incident wave $g(x-\int_0^t v(\tau)d\tau+t)$ within the spatial domain $[0, \Delta x]$:

$$\begin{aligned} E_i(t) &= \frac{1}{2} \Delta x \left\{ [1-v(t)] g'\left(-\int_0^t v(\tau)d\tau+t\right) \right. \\ &\quad \left. + v(t) g'\left(-\int_0^t v(\tau)d\tau+t\right) \right\}^2 + \frac{1}{2} \Delta x g'^2 \left(-\int_0^t v(\tau)d\tau+t\right) \\ &= \Delta x g'^2 \left(-\int_0^t v(\tau)d\tau+t\right) \end{aligned} \quad (115)$$

Similarly, the vibratory energy of the reflected wave within the domain $[0, \Delta x']$ is

$$E_r(t+\Delta t) = \Delta x' h'^2 \left(-\int_0^{t+\Delta t} v(\tau)d\tau-(t+\Delta t)\right) \quad (116)$$

The average rate of change of the vibratory energy of the infinitesimal wave during reflection from the boundary $x=0$ in the time duration $[t, t+\Delta t]$ is obtained by using Eqs. (115), (116), (112), and (113):

$$\frac{E_r(t + \Delta t) - E_i(t)}{\Delta t} = \frac{(\alpha - 1)\Delta x g'^2(-\int_0^t v(\tau)d\tau + t)}{\Delta t} \quad (117)$$

Using Eq. (114) in Eq. (117) and letting $\Delta t \rightarrow 0$ yield the exact rate of change of the vibratory energy of the infinitesimal wave during reflection from the left boundary at time t :

$$\frac{dE_{\text{left}}(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{E_r(t + \Delta t) - E_i(t)}{\Delta t} = \frac{v(t)[v(t) - 1]}{v(t) + 1} \times g'^2(-\int_0^t v(\tau)d\tau + t) \quad (118)$$

The relation between the slope of the incident wave at $x=0$, $g'(-\int_0^t v(\tau)d\tau + t)$, and the slope of the string at $x=0$, $u_x(0, t)$, is established by first differentiating Eq. (100) with respect to x at $x=0$:

$$u_x(0, t) = h' \left(-\int_0^t v(\tau)d\tau - t \right) + g' \left(-\int_0^t v(\tau)d\tau + t \right) \quad (119)$$

Using Eqs. (112) and (114) in Eq. (119) yields

$$g' \left(-\int_0^t v(\tau)d\tau + t \right) = \frac{1 + v(t)}{2} u_x(0, t) \quad (120)$$

Substituting Eq. (120) into Eq. (118) yields

$$\frac{dE_{\text{left}}(t)}{dt} = -\frac{1}{2} v(t) [1 - v^2(t)] u_x^2(0, t) \quad (121)$$

When $v(t) < 0$, the velocity of the incident wave towards the left boundary is along the direction of the relative velocity of the string to the boundary, the ratio of compression is greater than 1 by Eq. (114), the width of the reflected wave is smaller than that of the incident wave by Eq. (113), and the slope of the reflected wave is greater than that of the incident wave by Eq. (112). Hence, the wave is compressed during reflection from the left boundary, and the vibratory energy increases by Eq. (117). Conversely, when $v(t) > 0$, the ratio of compression is less than 1, the width of the reflected wave increases, and the slope of the reflected wave is smaller than that of the incident wave. Hence, the wave is flattened during reflection from the left boundary and the vibratory energy decreases.

Following the similar procedure, the rate of change of the vibratory energy of an infinitesimal wave during reflection from the right boundary $x=l(t)$ is

$$\frac{dE_{\text{right}}(t)}{dt} = 0 \quad (122)$$

Since the translating string has no relative motion to the right boundary, the ratio of compression is 1, and the width and slope of the reflected wave from the right boundary are the same as those of the incident wave. The vibratory energy of the wave remains unchanged during reflection from the right boundary.

The displacement of the translating string $u(x, t)$ comprises two trains of infinitesimal waves, one propagating rightward with a velocity $1+v(t)$ and one leftward with a velocity $1-v(t)$. When there is a relative motion between the string and a boundary, and the velocity of an incident infinitesimal wave at the boundary is along (opposite to) the direction of the relative velocity of the string to the boundary, the ratio of compression is greater (less) than 1, the reflected wave from the boundary is compressed (flattened), and the vibratory energy of the wave increases (decreases) during reflection from the boundary. The vibratory energies of the infinitesimal waves in the internal spatial domain of the string remain unchanged. Consequently, the rate of change of the vibratory energy of the string within the control volume, as shown in Eq. (87) with $f(x, t) = N_1(t) = N_2(t) = 0$, is the sum of the rates of

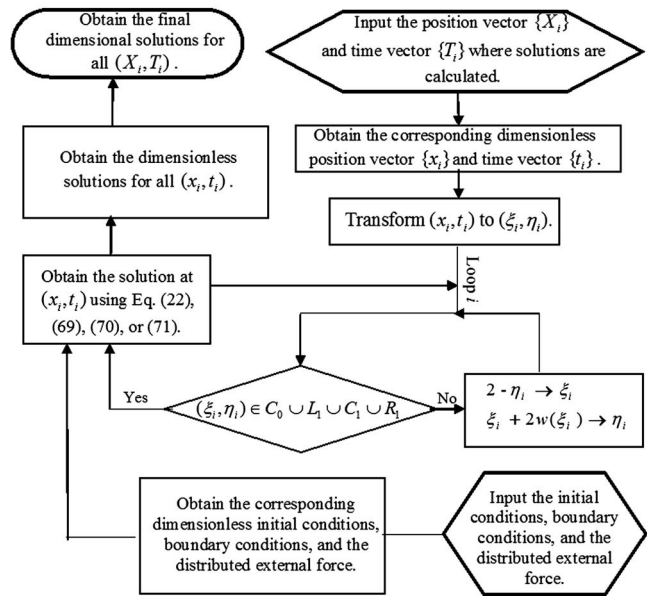


Fig. 5 Flowchart for calculating the free and forced responses of a translating string with arbitrarily varying length

change of the vibratory energies at the two boundaries in Eqs. (121) and (122).

When $v(t) = \dot{l}(t) \in (-1, 0)$, i.e., the length of the translating string gets shorter, by Eq. (87) with $f(x, t) = N_1(t) = N_2(t) = 0$, and Eqs. (92), (121), and (122), one has

$$\frac{dE_{\text{cv}}(t)}{dt} = S(0^+, t) = \frac{dE_{\text{left}}(t)}{dt} = -\frac{1}{2} v(t) [1 - v^2(t)] u_x^2(0, t) > 0 \quad (123)$$

The vibratory energy of the string within the control volume increases because the propagating waves are compressed at the left boundary $x=0$ and there is a continuous energy flux into the control volume across the boundary $x=0$. This unstable shortening string behavior is characterized by the bounded displacement and monotonically increasing vibratory energy. Furthermore, as the length of the string gets shorter, the vibratory energy density increases at a higher rate than the vibratory energy. It is noted that the positive energy flux across the boundary $x=0$ attributes to (1) a positive rate of work $-vu_x^2(0, t)$ done by the transverse component of the tension at $x=0$ and (2) a modification coefficient $(1 - v^2)/2$ due to the mass transfer across $x=0$. As the modification coefficient is greater than 0 and less than 1 for a subcritical speed, the effect of mass transfer tends to prevent the vibratory energy of the string from increasing and decreasing during retraction and extension, respectively.

5 Examples and Discussion

A computer program is written using MATLAB following the procedures described in Sec. 3. It provides the exact free and forced responses of the translating string with an arbitrary movement profile, and general initial conditions and external excitation. A flowchart for the program using the direct wave method is shown in Fig. 5. The inverse function $w(\xi)$ in Eqs. (69), (71), and (73) is obtained using a bisection root-finding algorithm.

5.1 Free Response of an Elevator Cable During Upward Movement. Consider first the free response of a hoist cable in an elevator, traveling from the first floor to the 46th floor in a 54-story building [16]. The bending stiffness of the cable and the tension change due to the weight and acceleration of the cable are

Table 1 Upward movement profile regions of an elevator cable and their polynomial coefficients

Region i	T_i (s)	$C_0^{(i)}$ (m)	$C_1^{(i)}$ (m/s)	$C_2^{(i)}$ (m/s ²)	$C_3^{(i)}$ (m/s ³)	$C_4^{(i)}$ (m/s ⁴)	$C_5^{(i)}$ (m/s ⁵)
1	1.33	162.00	0	0	0	-0.106	0.0316
2	6.67	157.38	-2.252	-0.33	0	0	0
3	8	155.51	-2.748	-0.33	0	0.106	-0.0316
4	30	125.99	-5	0	0	0	0
5	31.33	59.99	-5	0	0	0.106	-0.0316
6	36.67	30.48	-2.748	0.33	0	0	0
7	42	28.60	-2.252	0.33	0	-0.106	0.0316

neglected here. The lateral vibration of the cable is hence governed by Eq. (2). The tension in the cable is $P=9388.2$ N and the mass per unit length of the cable is $\rho=1.005$ kg/m. The upward movement profile is divided into seven regions. In each region i ($i=1, 2, \dots, 7$), the length of the cable is given by a polynomial:

$$L(T) = C_0^{(i)} + C_1^{(i)}(T - T_{(i-1)}) + C_2^{(i)}(T - T_{(i-1)})^2 + C_3^{(i)}(T - T_{(i-1)})^3 + C_4^{(i)}(T - T_{(i-1)})^4 + C_5^{(i)}(T - T_{(i-1)})^5 \quad (124)$$

where $T_{i-1} \leq T \leq T_i$, and T_i and $C_m^{(i)}$ ($m=0, 1, \dots, 5$) are given in Table 1. The initial and final lengths of the cable are 162 m and 21 m, respectively, and the total travel time is $T_7=42$ s. The initial displacement of the cable is

$$A(X) = \begin{cases} 0.25X/36 \text{ m} & 0 < X \leq 36 \text{ m} \\ -0.25(X - 162)/126 \text{ m} & 36 \text{ m} \leq X < 162 \text{ m} \end{cases} \quad (125)$$

and the initial velocity is zero. There are no distributed external force and boundary excitation.

The displacement of the cable at a fixed spatial point, $X=10$ m, from 0 s to 42 s is shown in Fig. 6. The exact response here is compared to those obtained using the spatial discretization methods in Ref. [16] with 10-term and 50-term approximations. It is seen that the response from the 50-term approximation is essentially the same as the exact response for the total time duration considered, and a large number of terms are needed in the spatial discretization methods to capture the shortening cable behavior. The results also show that the time duration between two consecutive peaks, as they pass through $X=10$ m, becomes smaller as time goes on, which results from the increase of the instantaneous fundamental frequency of the cable, $\pi\sqrt{P/\rho L^2(t)}$, during retrac-

tion. The time average of the local velocity of the cable at the spatial point $X=10$ m, reflected as the average slope in Fig. 6, representing the vibration intensity at $X=10$ m, tends to increase with time since the peak gets sharper. However, the time average of the magnitude of the displacement of the cable at the spatial point $X=10$ m increases first and then decreases. The displacement of the cable in Fig. 6 is bounded by the maximum initial displacement 0.25 m, as predicted by Eq. (62). As the elevator approaches the top of its hoistway, the vibratory energy density increases dramatically, as indicated in Sec. 4.3, and the vibration problem becomes most severe.

5.2 Response of an Elevator Cable Under Boundary Excitation During Upward Movement. Consider a moving elevator cable with the same initial length, tension, and mass per unit length as in Sec. 5.1. The transport velocity is $V(T)=-8.1$ m/s, and the total travel time is 30 s. The initial displacement and velocity are zero. The boundary displacement at $x=0$ is $U(0, T)=0.005 \sin(9.97T)$ m, and there are no distributed external force and boundary excitation at $x=l(t)$. The forced responses at a fixed spatial point $X=10$ m from 0 s to 30 s, using both the transformation method and the direct wave method in Sec. 3.5, are shown in Fig. 7. Since the transformation method is computationally expensive as it involves numerical evaluation of a double integral of $Q(\xi, \eta)$ introduced by the transformation, only 201 points are used for the time duration $T \in [0, 30]$ s in calculating and plotting the response with the transformation method, and 1001 points are used for the same time duration with the direct wave method. The data points from the transformation method coincide with the corresponding points from the direct wave method, and the two methods yield the same results.

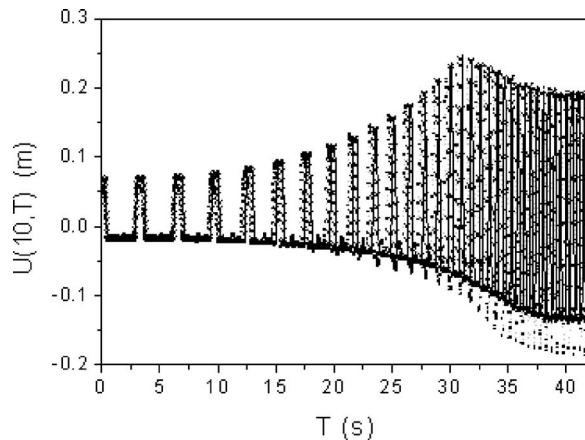


Fig. 6 Free response of a moving elevator cable at $X=10$ m from 0 s to 42 s: exact solution (solid line), 10-term approximation (dot), and 50-term approximation (cross)

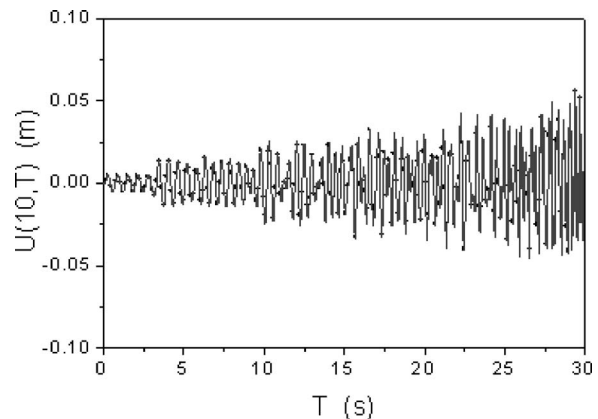


Fig. 7 Response of a moving elevator cable under boundary excitation at $X=10$ m from 0 s to 30 s: transformation method (dot) and direct wave method (solid line)

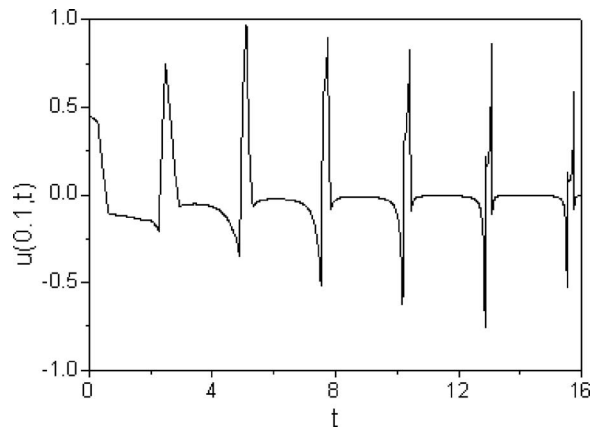


Fig. 8 Free response of a translating ($v(t)=0.2 \sin(\pi t)$) string with sinusoidally varying length at $x=0.1$

5.3 Parametric Instabilities in a Translating String With Sinusoidally Varying Length. The long-time behavior and the energy growth of a stationary string with a sinusoidally moving boundary were studied in Ref. [21]. The long-time behavior and parametric instabilities of a translating string with sinusoidally varying length are studied here.

Consider a translating string with the dimensionless velocity $v(t)=0.2 \sin(\omega t)$, where $\omega=\pi$ is the dimensionless fundamental frequency of the translating string with the average length $l(0)=1$ and zero average velocity. The dimensionless initial displacement is

$$a(x) = \begin{cases} x/0.22 & 0 < x \leq 0.22 \\ -(x-1)/0.78 & 0.22 \leq x < 1 \end{cases} \quad (126)$$

and the initial velocity is zero. There are no distributed external force and boundary excitation.

The dimensionless displacement of the translating string at a fixed spatial point $x=0.1$ from $t=0$ to 16 is shown in Fig. 8. The vibration observed at $x=0.1$ evolves into a shock as time goes on, passing through the spatial point $x=0.1$. Unlike the case in Fig. 6, the time duration between two consecutive peaks remains almost unchanged here, since the length of the string changes sinusoidally in a small region around $l(0)$ and the corresponding instantaneous, dimensionless fundamental frequency of the string $\pi/l(t)$ varies only slightly. In contrast, the slopes of these peaks increase dramatically with time.

The dimensionless displacements of the string at the times $t=4, 10$, and 16 are shown as dashed, dotted, and solid lines, respectively, in Fig. 9. It is observed that all the waves are attracted to a small region around a point at the final time, and the attracting

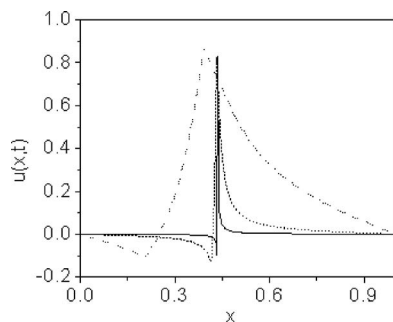


Fig. 9 Free response of a translating ($v(t)=0.2 \sin(\pi t)$) string with sinusoidally varying length at $t=4$ (dashed line), 10 (dotted line), and 16 (solid line)

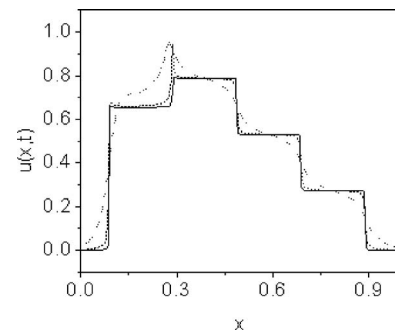


Fig. 10 Free response of a translating ($v(t)=0.2 \sin(5\pi t)$) string with sinusoidally varying length at $t=4$ (dashed line), 10 (dotted line), and 16 (solid line)

point moves along the string with a downstream velocity $1+v(t)$ and upstream velocity $1-v(t)$. The vibratory energy of the string accumulates continuously as the shock reflects from the boundary $x=0$, where the string has a relative motion to the boundary, as illustrated in Sec. 4.3.

In fact, it is found numerically that when $\omega=n\pi$ ($n=1, 2, 3, \dots$), which is the n th dimensionless natural frequency of the string with the averaged length and velocity, the similar behavior occurs. For instance, when $n=5$, the dimensionless displacements of the string at the times $t=4, 10$, and 16 are shown as dashed, dotted, and solid lines, respectively, in Fig. 10. The vibration evolves into square waves traveling along the string. It is observed that when $\omega=n\pi$, there are n abrupt changes of the displacement along the string, where shocks are developed and the vibratory energy is concentrated and accumulated.

The unstable phenomenon observed here is considered as the parametric resonance for the distributed-parameter system considered here, because some coefficients of the governing equation in Eq. (2) vary periodically with time and so does its spatial domain in the present case.

6 Conclusions

The exact response of a translating string with constant tension and arbitrarily varying length is obtained for general initial conditions and external excitation using the method of characteristics. Compared to the spatial discretization methods, the new wave method presented here is more accurate and efficient, especially for a shortening or unstable string, and it provides physical insight into wave propagation. The method is, however, limited to the case where the standard form of the governing equation does not contain the zeroth and first order partial derivatives, which means that the propagating wave is nondispersive.

The least upper bound of the displacement of the freely vibrating string depends only on the initial conditions. The direct wave method is more efficient than the transformation method in handling the nonhomogeneous boundary conditions. The energy flux function for a translating string is introduced; the relation between the energy flux function and the vibratory energy density is presented in the differential, integral, and modified integral forms, and the energy growth mechanism of a shortening string is explained using the modified integral form. A new wave method is developed to calculate the rate of change of the vibratory energy within the control volume and to explain the energy growth mechanism associated with wave compression at a boundary. A new parametric instability phenomenon is discovered for the translating string with periodically varying length. When the length of the string varies at the n th natural frequency of the string with the averaged length and velocity, n shocks characterized by the bounded displacement and unbounded energy are developed.

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Analytical Solution of the Perturbed Magnetic Fields of Plates Under Tensile Stress

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Development of magnetism based nondestructive testing technology and the Microelectronic mechanical system require accurate computation of perturbed magnetic fields generated by mechanical stress. In this paper, based on the linearized magnetoelastic theory, the governing equations and continuity conditions to determine the perturbed magnetic fields were formulated for the case of weak external magnetic fields such as the earth's magnetic field. Under those weak magnetic fields, the effect of the magnetic fields on mechanical deformation was neglected. As a result, the interaction between the deformation and the magnetic field was simplified. The effect of deformation on the perturbed magnetic field was taken into account by introducing the displacement gradient into the boundary conditions that the perturbed field should satisfy. As examples, analytic solutions of the perturbed magnetic field of infinite plates with and without a round hole, which are subjected to tensile stresses and weak external magnetic fields, were obtained by the approach presented. The results show that the perturbed magnetic fields induced by stress are three orders less in magnitude of intensity than that of magnetic fields without stress, and some prominent local features such as that has more peaks and decays more rapidly in the radial direction than the case of stress free that are predicted by the solutions. [DOI: 10.1115/1.2870266]

Keywords: stress, magnetic fields, perturbed, magnetoelasticity

1 Introduction

In magnetism based nondestructive testing technologies [1] and noncontact measurement of stress distribution [2], the effect of mechanical stress on the magnetic fields is an important topic to the development of those technologies. Analysis of the perturbed magnetic fields generated by a tension fault was regarded as a possible way to understand what happen in the deep earth before strong earthquakes [3]. In the microelectronic mechanical system (MEMS), the effect of the strain and stress on magnetic thin films, which are extensively used in MEMS and other devices, becomes crucial problem to ensure the device is working functionally [4].

The interaction between magnetic fields and elastic solids was particularly treated by Brown [5]. Some theoretical models [6–8] are available to describe the macrobehaviors of ferromagnetic materials to some extent. Among them, Pao and Yeh's model [6] provides a linearized magnetoelasticity theory and is widely used to attack the interaction problems. More recently, new progress has been made. The theory for the equilibrium response of magnetoelastic membranes [9] and the equilibrium equations for large magnetoelastic deformations [10] have been formulated for the applications and analysis of elastomers endowed with magnetic properties by the embedding of distributions of ferrous particles, which are used as smart materials in devices such as that for controlling the damping characteristics of vibration absorbers.

This paper focuses on the perturbed magnetic field induced by mechanical stress. Based on Pao and Yeh's linearized theory, Yeh [11] obtained the perturbed magnetic field generated by a line mechanical singularity in a magnetized elastic half plane. Huang and Wang [12] derived a complete solution of the magnetoelasticity in the magnetic half space. Under weak magnetic fields, the stress induced perturbed magnetic field was investigated [13], and

the perturbed magnetic field of a half plane subjected to a point force has been obtained analytically. In this paper, an approach to obtain the perturbed magnetic fields induced by the deformation due to the mechanical loads is presented, and the analytic solution of the perturbed magnetic field of an infinite plate with a round hole is derived by the approach.

Section 2 presents the equations to solve the rigid magnetic fields and the perturbed fields. Particular attention is paid to the case of weak internal magnetic fields. The continuity conditions the perturbed fields should satisfy are derived in this section, which are necessary to solve the boundary-value problem. In Sec. 3, the perturbed magnetic fields of an infinite plate subjected to tensile stress and external magnetic field are solved as an example to show how the proposed approach is implemented, and to indicate that the displacement gradient plays an important role to the initiation of the perturbed magnetic fields. The perturbed magnetic fields of an infinite plate with a round hole are solved in Sec. 4, the results are discussed in Sec. 5, and some conclusions are drawn in Sec. 6.

2 Interaction Between Deformation and Magnetic Fields

2.1 Rigid Magnetic Fields. Without deformation, the rigid stationary magnetic fields of an isotropic magnetoelastic solid occupying a spatial domain Ω and subjected to an external magnetic field can be described by the magnetic flux density $\mathbf{B}^-$, the field intensity $\mathbf{H}^-$, and the magnetized intensity $\mathbf{M}^-$ in Ω , and $\mathbf{B}^+$, $\mathbf{H}^+$, and $\mathbf{M}^+=0$ in the free space where might be air or vacuum. These magnetic quantities are governed by the Ampere law and the Gauss law as

$$e_{ijk}H_{k,j}=0, \quad B_{i,i}=0 \quad (1)$$

where e_{ijk} is the permutation symbol and $i, j, k=1,2,3$ for the three-dimensional problems. The “,” in subscripts denotes the partial derivative with respect to the spatial coordinates, for example, $H_{k,j}=\partial H_k/\partial x_j$. Whether in Ω or in the free space, Eq. (1) must be

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held; therefore, the superscripts “−” and “+” of the quantities are intentionally omitted for brevity. Hence, the quantities without superscripts “−” and “+” in Eq. (1) and in the following part of this paper denote ones both in Ω and in the free space. The constitutive laws for the magnetic fields are

$$M_i = \chi H_i, \quad B_i = \mu_0(H_i + M_i) = \mu_0\mu_r H_i, \quad \mu_r \equiv 1 + \chi \quad (2)$$

where χ is the magnetic susceptibility of the material, $\mu_0 4\pi \times 10^{-7}$ H/m is the universal constant, and μ_r the relative magnetic permeability. On the boundary of Ω , the fields satisfy the following continuity conditions:

$$N_i(B_i^+ - B_i^-) = 0 \quad (3a)$$

$$e_{ijk}N_j(H_k^+ - H_k^-) = 0 \quad (3b)$$

where N_i are the three components of the normal vector of the boundary.

Introducing magnetic scalar potential Φ and defining $\mathbf{H} = -\nabla\Phi$, where ∇ is the Nabla operator, then Eq. (1) can be transferred into two Laplace equations for the free space and the material space, respectively,

$$\nabla^2\Phi^+ = 0, \quad \nabla^2\Phi^- = 0 \quad (4)$$

2.2 Effect of Magnetic Field on Mechanical Deformation.

The deformation of the solid considered in Sec. 2.1 consists of two parts. One is caused by the applied mechanical loads and the other by the Maxwell forces from the external magnetic field.

More complicated, the deformation of the solid generates perturbed magnetic fields in the material space and in the free space. Similar to the rigid situation discussed in Sec. 2.1, the perturbed fields can be described by the magnetic flux density $\mathbf{b}^-$, the magnetic field intensity $\mathbf{h}^-$, and the magnetized intensity $\mathbf{m}^-$ in the material space Ω , and $\mathbf{b}^+$, $\mathbf{h}^+$, and $\mathbf{m}^+$ in the free space. As the gradient of displacements is assumed small, according to the linearized theory in Ref. 6, the total magnetic fields are superposition of the rigid magnetic fields and the perturbed fields, i.e.,

$$B_i^{\text{total}} = B_i + b_i, \quad H_i^{\text{total}} = H_i + h_i, \quad M_i^{\text{total}} = M_i + m_i \quad (5)$$

The total magnetic quantities should satisfy Eq. (1); this leads to the governing equations of the perturbed fields

$$e_{ijk}h_{k,j} = 0, \quad b_{i,i} = 0 \quad (6)$$

The equilibrium equation under mechanical loads and magnetic fields is [11]

$$t_{ij,i} + \mu_0(M_i H_{j,i} + M_j h_{i,i} + m_i H_{j,i}) + f_j = 0 \quad (7)$$

where t_{ij} is the magnetomechanical stress tensor and f_j the mechanical body force per unit volume. Neglecting the effect of magnetostriction, the constitutive equations are [6]

$$t_{ij} = \frac{\mu_0}{\chi} M_i M_j + \tau_{ij} + \mu_0(H_j m_i + H_i m_j) \quad (8)$$

$$m_i = \chi h_i, \quad b_i = \mu_0(h_i + m_i) = \mu_0\mu_r h_i \quad (9)$$

where

$$\tau_{ij} = \lambda u_{k,k} \delta_{ij} + G(u_{i,j} + u_{j,i}) \quad (10)$$

is the Cauchy stress tensor. λ and G are the Lamé constants, and δ_{ij} is the Kronecker delta symbol. Substituting Eqs. (8)–(10) into Eq. (7) and using Eqs. (1) and (6), omitting the mechanical body force, it yields

$$G u_{j,ii} + (\lambda + G) u_{i,ji} + 2\mu_0\chi(H_i H_{j,i} + h_i H_{j,i} + H_i h_{j,i}) = 0$$

Considering the condition $|b_i|/|B_i| \ll 1$ and $|h_i|/|H_i| \ll 1$, it can be further simplified as

$$u_{j,ii} + \frac{1}{1-2\nu} u_{i,ji} + \frac{2\chi B_i}{G\mu_r} h_{j,i} = 0 \quad (11)$$

Here, the third term in the left side of the equation represents the effect of the Maxwell forces. It has been shown that for non-soft ferromagnetic material when the external magnetic field $\mathbf{B}$ is of magnitude of the earth's magnetic field (about 40 A/m), the effect of the Maxwell forces on displacement can be neglected [11,13]. Therefore, the third item in Eq. (11) can be neglected and thus Eq. (11) is reduced to the Lamé–Navier's equation, which is commonly seen in textbooks on the theory of elasticity.

2.3 Perturbed Magnetic Fields Induced by Deformation.

Considering a particle in a magnetoelastic solid originally located at $a_i (i=1,2,3)$, it displaces u_i to x_i , then its position after the displacement is

$$x_i = a_i + u_i \quad (12)$$

The geometry equations of the boundary of the material space Ω before and after the deformation can be written as

$$F(a_1, a_2, a_3) = 0, \quad f(x_1, x_2, x_3) = 0 \quad (13)$$

Let $\mathbf{N}$ and $\mathbf{n}$ be the normal vectors of the boundary surfaces before and after the deformation, respectively, their components are calculated by

$$N_i = \frac{\partial F}{\partial a_i} \left(\frac{\partial F}{\partial a_k} \frac{\partial F}{\partial a_k} \right)^{-1/2} \equiv K \frac{\partial F}{\partial a_i}, \quad n_i = \frac{\partial f}{\partial x_i} \left(\frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_k} \right)^{-1/2} \equiv K' \frac{\partial f}{\partial x_i} \quad (14)$$

Note that

$$f(x_1, x_2, x_3) = F(a_1(x_i), a_2(x_i), a_3(x_i)) = 0$$

and assuming that the magnitude of the outward normal vectors keeps unchanged before and after the deformation, i.e., $K' = K$, from Eq. (14), the normal vector of the boundary surface after the deformation is

$$n_i = K' \frac{\partial f}{\partial x_i} = K' \frac{\partial F}{\partial a_m} \frac{\partial a_m}{\partial x_i} = N_m \left(\delta_{mi} - \frac{\partial u_m}{\partial x_i} \right) = N_i - N_m u_{m,i} \quad (15)$$

Replacing the quantities N_i, B_i , and H_i in Eqs. (3a) and (3b) by n_i, B_i^{total} , and H_i^{total} , respectively, and substituting Eqs. (5) and (15) into Eq. (3a), we have

$$N_i(b_i^+ - b_i^-) + N_i(B_i^+ - B_i^-) - N_m u_{m,i}(B_i^+ - B_i^-) - N_m u_{m,i}(b_i^+ - b_i^-) = 0$$

Note that the second item in the left side of the equation is the boundary condition, Eq. (3a), in the rigid body state; it should be zero. The fourth term is a second order infinitesimal quantity, which can be neglected, thus have

$$N_i(b_i^+ - b_i^-) = N_m u_{m,i}(B_i^+ - B_i^-) \quad (16a)$$

Substituting Eqs. (5) and (15) into Eq. (3b), we have

$$e_{ijk}N_j(h_k^+ - h_k^-) = e_{ijk}N_m u_{m,j}(H_k^+ - H_k^-) \quad (16b)$$

Equations (16a) and (16b) are the boundary conditions the perturbed magnetic field should satisfy. Equations (16a) and (16b) indicate that the displacement gradient, $u_{m,i}$, plays a key role to initiate the perturbed magnetic field. In other words, the perturbed magnetic field does not arise when the displacement gradient on the boundary is zero.

For plane problems, the continuity conditions (16a) and (16b) can be expressed in the planar polar coordinate system as

$$n_r(h_\theta^+ - h_\theta^-) - n_\theta(h_r^+ - h_r^-) = \varepsilon_r(H_\theta^+ - H_\theta^-) - \varepsilon_\theta(H_r^+ - H_r^-) \quad (17a)$$

$$n_r(b_r^+ - b_r^-) - n_\theta(b_\theta^+ - b_\theta^-) = \varepsilon_r(B_r^+ - B_r^-) + \varepsilon_\theta(B_\theta^+ - B_\theta^-) \quad (17b)$$

where

$$\varepsilon_r = n_r \frac{\partial u_r}{\partial r} + n_\theta \frac{\partial u_\theta}{\partial r}, \quad \varepsilon_\theta = \frac{1}{r} \left(n_r \frac{\partial u_r}{\partial \theta} + n_\theta \frac{\partial u_\theta}{\partial \theta} + n_\theta u_r - n_r u_\theta \right)$$

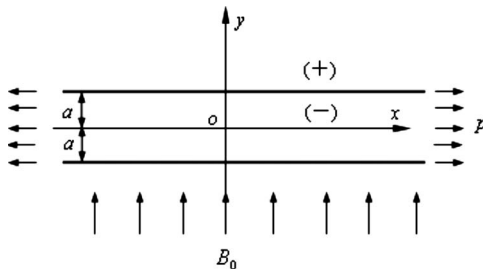


Fig. 1 Infinite plate subjected to tensile stress p and magnetic field B_0

Similar to the rigid magnetic fields in Sec. 2.1, we introduce the perturbed magnetic scalar potentials φ^+ and φ^- for the free space and the material space, respectively. By defining $\mathbf{h}^+ = -\nabla\varphi^+$ and $\mathbf{h}^- = -\nabla\varphi^-$, Eq. (6) is reduced to

$$\nabla^2\varphi^+ = 0, \quad \nabla^2\varphi^- = 0 \quad (18)$$

Equations (16a), (16b), and (18) are the equations used to determine the perturbed fields.

In summary, there are three main steps to obtain the perturbed magnetic field.

Step 1. Solving Eq. (4) to obtain the rigid magnetic field $\mathbf{B}$ and $\mathbf{H}$.

Step 2. Solving the Lamè–Navier equation to obtain the displacement field $\mathbf{u}$ and the displacement gradient on the boundary.

Step 3. Solving Eq. (18) with Eqs. (16a) and (16b) to obtain the perturbed magnetic field $\mathbf{b}$ and $\mathbf{h}$.

3 Perturbed Magnetic Fields of a Plate Subjected to Tensile Stress

Figure 1 shows an infinite plate subjected to tensile stress p and to an external magnetic field specified by magnetic flux density B_0 . The thickness of the plate is $2a$.

The rigid state magnetic field of the problem is

$$B_y^+ = B_0, \quad H_y^+ = B_0/\mu_0, \quad M_y^+ = 0 \quad (\text{in the free space})$$

$$B_y^- = B_0, \quad H_y^- = \frac{B_0}{\mu_0\mu_r}, \quad M_y^- = \frac{\chi B_0}{\mu_0\mu_r} \quad (\text{in the plate})$$

All the other magnetic quantities are zero.

The displacement components of the problem are $u_x = px/E$ and $u_y = -\nu py/E$, respectively. Here, E and ν are Young's modulus and Poisson's ratio of the plate. The perturbed magnetic field satisfies the governing equation (18) and the boundary conditions (16a) and (16b). On the boundary $y = \pm a$, it gives

$$h_x^+ - h_x^- = -\frac{\chi B_0}{\mu_0\mu_r} \frac{\partial u_y}{\partial x}, \quad b_y^+ - b_y^- = 0$$

Because there is no displacement gradient ($\partial u_y/\partial x = 0$), the right hand term of the first equation is zero and subsequently there is no perturbed magnetic field induced by the stress p .

4 Perturbed Magnetic Fields of an Infinite Plate With a Round Hole

4.1 Displacement Solutions. Figure 2 shows an infinite plate with a centered round hole. The radius of the hole is a . The plate is subjected to tensile stress p and external magnetic field B_0 .

For convenience, the planar polar coordinates (θ, r) are used. The stress solution of this problem can be found in many textbooks of elasticity. From the stress solution and Hooke's law, the displacement components can be integrated from the geometry equations as

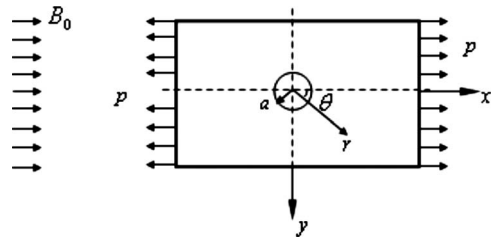


Fig. 2 Infinite plate with a hole subjected to stress p and magnetic field B_0

$$u_r = \frac{p}{E} \left[\left(\frac{1-\nu}{2} + \frac{1+\nu}{2} \cos 2\theta \right) r + \left(\frac{1+\nu}{2} + 2 \cos 2\theta \right) \frac{a^2}{r} + \frac{1+\nu}{2} \cos 2\theta \frac{a^4}{r^3} \right] + A \sin \theta + B \cos \theta$$

$$u_\theta = \frac{p}{E} \left[- (1-\nu) \frac{a^2}{r} - \frac{1+\nu}{2} r - (1+\nu) \frac{a^4}{r^3} \right] \sin 2\theta + A \cos \theta - B \sin \theta + Cr$$

where $r = \sqrt{x^2 + y^2}$, E and ν are Young's modulus and Poisson's ratio of the plate, respectively. In order to find the integration constants A , B , and C , use the symmetry conditions of the problem

$$u_r|_{\theta=0} = u_r|_{\theta=\pi}, \quad u_r|_{\theta=\pi/2} = u_r|_{\theta=3\pi/2}$$

Then have $B=0$ and $A=0$. Besides, on the line of $\theta=0$, the vertical displacement component $u_\theta=0$, this leads to $C=0$. Thus, the displacement solution is

$$u_r = \frac{p}{E} \left[\left(\frac{1-\nu}{2} + \frac{1+\nu}{2} \cos 2\theta \right) r + \left(\frac{1+\nu}{2} + 2 \cos 2\theta \right) \frac{a^2}{r} + \frac{1+\nu}{2} \cos 2\theta \frac{a^4}{r^3} \right]$$

$$u_\theta = \frac{p}{E} \left[- (1-\nu) \frac{a^2}{r} - \frac{1+\nu}{2} r - (1+\nu) \frac{a^4}{r^3} \right] \sin 2\theta$$

4.2 Rigid Magnetic Field Solutions. In order to compute the perturbed magnetic fields, the rigid magnetic fields must be computed first. The general solution of Eq. (4) is

$$\Phi(r, \theta) = (A_0 + B_0 \ln r)(C_0 + D_0 \theta) + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \times (C_n \cos n\theta + D_n \sin n\theta)$$

When $r \rightarrow \infty$, the magnetic scalar potential has a limited value $\Phi^- = -(B_0/\mu)r \cos \theta$, where $\mu = \mu_0\mu_r$. This requires that $A_0 = B_0 = D_n = 0$. When $n=1$, then

$$\Phi^- = A_1 C_1 r \cos \theta + \frac{B_1 C_1}{r} \cos \theta = -\frac{B_0}{\mu} r \cos \theta + \frac{F_1}{r} \cos \theta \quad (19)$$

On the boundary of the hole ($r=a$), the continuity conditions require $\Phi^+ = \Phi^-$. This implies that Φ^+ has a form of

$$\Phi^+ = -F_2 r \cos \theta + \frac{F_3}{r} \cos \theta \quad (20)$$

Here, F_1 , F_2 , and F_3 are constants need to be found. If the point at $r=0$ is picked as the reference potential point, that is, $\Phi^+=0$ at

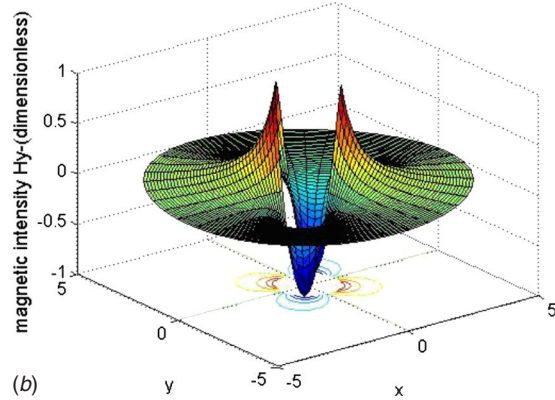
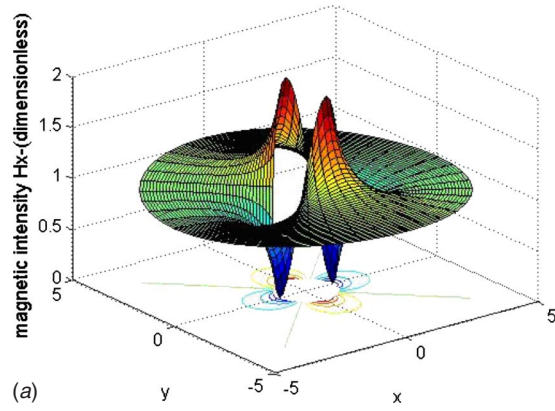


Fig. 3 The rigid magnetic field intensities: (a) H_x and (b) H_y

$r=0$, this leads to $F_3=0$.

The continuity conditions, Eqs. (3a) and (3b), suggest that as $r=a$, $\Phi^+=\Phi^-$ and $\mu_0(\partial\Phi^+/\partial r)=\mu(\partial\Phi^-/\partial r)$, it results that

$$F_1 = -\frac{\mu - \mu_0}{\mu + \mu_0} a^2 \frac{B_0}{\mu}, \quad F_2 = \frac{2B_0}{\mu + \mu_0}$$

Then the potentials are

$$\begin{cases} \Phi^+(r, \theta) = -\frac{2B_0}{\mu_0(\mu_r + 1)} r \cos \theta & (r \leq a) \\ \Phi^-(r, \theta) = -\frac{B_0}{\mu_0\mu_r} r \cos \theta - \frac{\mu_r - 1}{\mu_r + 1} \cdot \frac{B_0}{\mu_0\mu_r} \frac{a^2}{r} \cos \theta & (r \geq a) \end{cases}$$

and the field intensities are

$$\begin{aligned} \mathbf{H}^+(r, \theta) &= \frac{2B_0}{\mu_0(\mu_r + 1)} (\mathbf{e}_r \cos \theta - \mathbf{e}_\theta \sin \theta) \\ \mathbf{H}^-(r, \theta) &= \mathbf{e}_r \left(1 - \frac{\mu_r - 1}{\mu_r + 1} \cdot \frac{a^2}{r^2} \right) \frac{B_0}{\mu_0\mu_r} \cos \theta \\ &\quad + \mathbf{e}_\theta \left(-1 - \frac{\mu_r - 1}{\mu_r + 1} \cdot \frac{a^2}{r^2} \right) \frac{B_0}{\mu_0\mu_r} \sin \theta \end{aligned}$$

In the Descartes coordinate system, the potentials and intensities have forms of

$$\begin{cases} \Phi^+(x, y) = -\frac{2B_0}{\mu_0(\mu_r + 1)} x & (r \leq a) \\ \Phi^-(x, y) = -\frac{B_0}{\mu_0\mu_r} x - \frac{\mu_r - 1}{\mu_r + 1} \cdot \frac{B_0}{\mu_0\mu_r} \frac{a^2 x}{r^2} & (r \geq a) \end{cases}$$

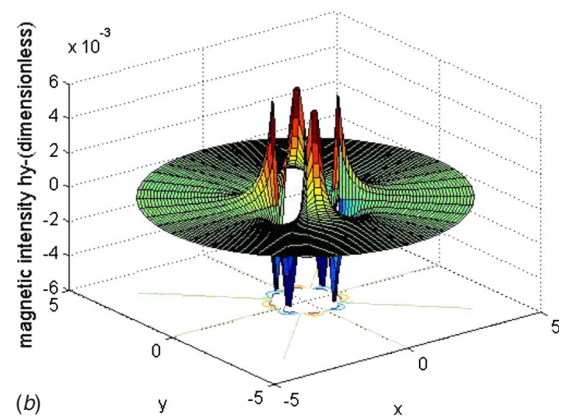
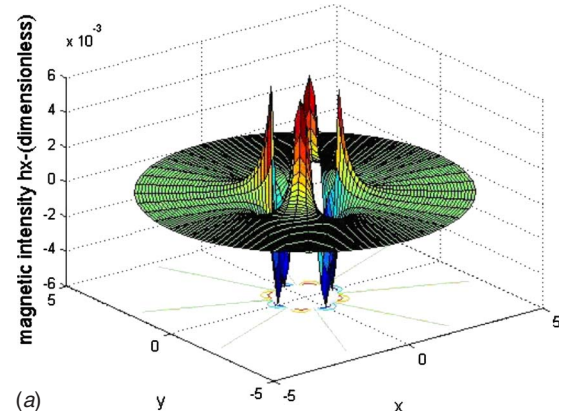


Fig. 4 The perturbed magnetic field intensities: (a) h_x and (b) h_y

$$\begin{cases} H_x^+ = -\frac{\partial\Phi^+}{\partial x} = \frac{2B_0}{\mu_0(\mu_r + 1)} & (r \leq a) \\ H_y^+ = -\frac{\partial\Phi^+}{\partial y} = 0 \end{cases} \quad (21)$$

$$\begin{cases} H_x^- = -\frac{\partial\Phi^-}{\partial x} = \frac{B_0}{\mu_0\mu_r} - \frac{\mu_r - 1}{\mu_r + 1} \cdot \frac{B_0}{\mu_0\mu_r} \cdot \frac{a^2(x^2 - y^2)}{(x^2 + y^2)^2} \\ H_y^- = -\frac{\partial\Phi^-}{\partial y} = -\frac{\mu_r - 1}{\mu_r + 1} \cdot \frac{B_0}{\mu_0\mu_r} \cdot \frac{2a^2xy}{(x^2 + y^2)^2} \end{cases} \quad (r \geq a) \quad (22)$$

Equation (21) indicates that in the hole there is a uniform magnetic field.

4.3 Perturbed Magnetic Fields Induced by the Deformation.

The perturbed magnetic scalar potentials φ^+ and φ^- are governed by Eq. (18). For this problem, the continuity conditions, Eqs. (17a) and (17b), were used for convenience. For the boundary $r=a$, the components of the normal vector are $n_r=-1$ and $n_\theta=0$, thus Eqs. (17a) and (17b) become

$$\begin{cases} h_\theta^+ - h_\theta^- = \frac{\partial u_r}{\partial r} (H_\theta^+ - H_\theta^-) - \frac{1}{r} \left(\frac{\partial u_r}{\partial \theta} - u_\theta \right) (H_r^+ - H_r^-) \\ b_r^+ - b_r^- = \frac{\partial u_r}{\partial r} (B_r^+ - B_r^-) + \frac{1}{r} \left(\frac{\partial u_r}{\partial \theta} - u_\theta \right) (B_\theta^+ - B_\theta^-) \end{cases} \quad (r=a) \quad (23)$$

Note that $B^+=\mu_0 H^+$, $B^-=\mu_0\mu_r H^-$, $b^+=\mu_0 h^+$, $b^-=\mu_0\mu_r h^-$, and substituting the solved u_r , H_θ^+ , H_θ^- , H_r^+ , H_r^- into Eq. (23), then have

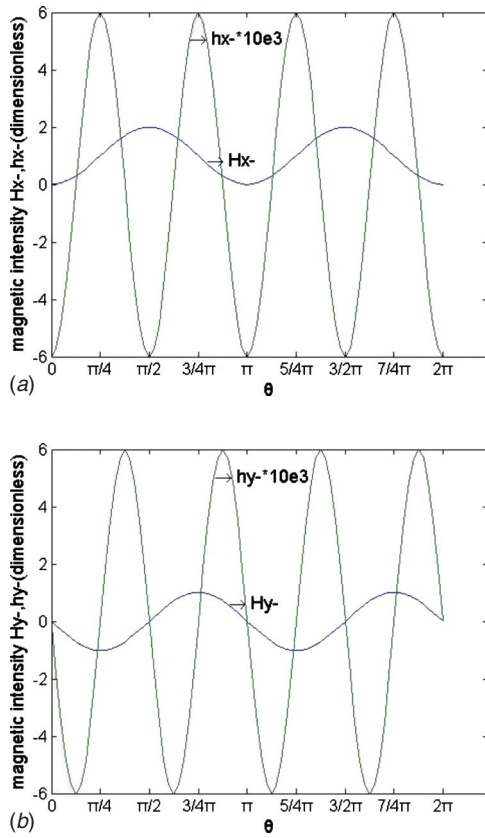


Fig. 5 Comparison of intensities along the hole boundary ($r=a$): (a) H_x and h_x ; (b) H_y and h_y

$$\begin{cases} \frac{\partial \varphi^+}{\partial \theta} - \frac{\partial \varphi^-}{\partial \theta} = -\frac{1}{E}(7+3\nu) \cdot \left(\frac{\mu_r-1}{\mu_r+1}\right) pa \cdot \frac{B_0}{\mu_0 \mu_r} \cdot \sin 2\theta \cdot \cos \theta \\ \frac{\partial \varphi^+}{\partial r} - \mu_r \frac{\partial \varphi^-}{\partial r} = \frac{1}{E}(7+3\nu) \cdot \left(\frac{\mu_r-1}{\mu_r+1}\right) p \cdot \frac{B_0}{\mu_0} \cdot \sin 2\theta \cdot \sin \theta \end{cases} \quad (r=a) \quad (24)$$

The general solutions of Eq. (18) are

$$\begin{aligned} \varphi^+(r, \theta) &= (a_0^+ + b_0^+ \ln r)(c_0^+ + d_0^+ \theta) + \sum_{n=1}^{\infty} (a_n^+ r^n + b_n^+ r^{-n}) \\ &\quad \times (c_n^+ \cos n\theta + d_n^+ \sin n\theta) \\ \varphi^-(r, \theta) &= (a_0^- + b_0^- \ln r)(c_0^- + d_0^- \theta) + \sum_{n=1}^{\infty} (a_n^- r^n + b_n^- r^{-n}) \\ &\quad \times (c_n^- \cos n\theta + d_n^- \sin n\theta) \end{aligned}$$

Because of the symmetry, φ must be an even function of θ . This suggests that $d_0^\pm = d_n^\pm = 0$, thus

$$\begin{aligned} \varphi^+(r, \theta) &= (a_0^+ + b_0^+ \ln r) + \sum_{n=1}^{\infty} (a_n^+ r^n + b_n^+ r^{-n}) \cos n\theta \\ \varphi^-(r, \theta) &= (a_0^- + b_0^- \ln r) + \sum_{n=1}^{\infty} (a_n^- r^n + b_n^- r^{-n}) \cos n\theta \end{aligned}$$

By using the same reference potential point as used in the rigid case, we have $b_0^+ = b_n^+ = a_0^- = 0$. In the plate, when $r \rightarrow \infty$ the perturbed field vanishes; this requires that $a_0^- = b_0^- = a_n^- = 0$, so

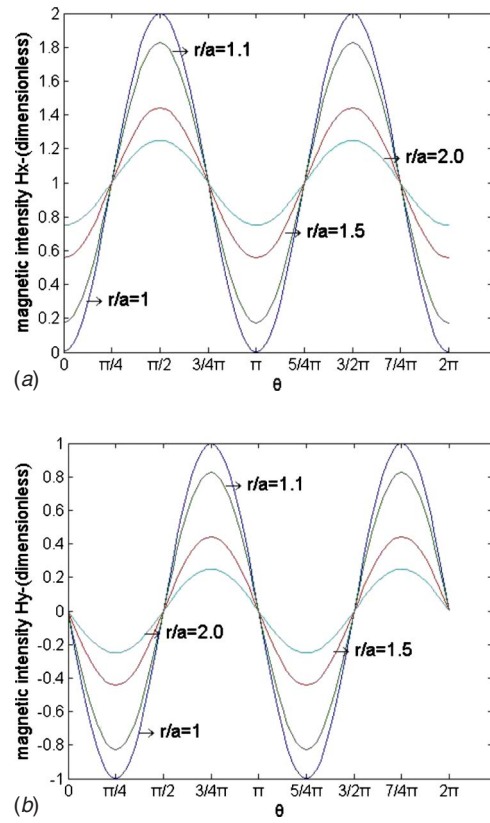


Fig. 6 The rigid intensities for various r/a ratios: (a) H_x and (b) H_y

$$\varphi^+(r, \theta) = \sum_{n=1}^{\infty} a_n^+ r^n \cos n\theta \quad (25)$$

$$\varphi^-(r, \theta) = \sum_{n=1}^{\infty} b_n^- r^{-n} \cos n\theta \quad (26)$$

Substituting Eqs. (25) and (26) into Eq. (24), finally the potentials are

$$\varphi^+(r, \theta) = 0,$$

$$\varphi^-(r, \theta) = -\frac{1}{E}(7+3\nu)p \cdot \frac{B_0}{\mu_0 \mu_r} \cdot \frac{\mu_r-1}{\mu_r+1} \cdot \frac{a^4}{6} \cdot \frac{1}{r^3} \cos 3\theta$$

It indicates that a perturbed magnetic field is generated in the plate by the stress but not in the hole. In Descartes coordinate system, the potential in the plate has form of

$$\begin{aligned} \varphi^-(x, y) &= -\frac{1}{E}(7+3\nu)p \cdot \frac{B_0}{\mu_0 \mu_r} \cdot \frac{\mu_r-1}{\mu_r+1} \\ &\quad \cdot \frac{a^4}{6} \left[4 \frac{x^3}{(x^2+y^2)^3} - 3 \frac{x}{(x^2+y^2)^2} \right] \end{aligned}$$

By $h_x^- = -(\partial \varphi^- / \partial x)$ and $h_y^- = -(\partial \varphi^- / \partial y)$, the intensities of the perturbed magnetic field are

$$h_x^- = -\frac{1}{E}(7+3\nu)p \cdot \frac{B_0}{\mu_0 \mu_r} \cdot \frac{\mu_r-1}{\mu_r+1} \cdot \frac{a^4}{6} \cdot \left(\frac{3}{r^4} - 24 \frac{x^2}{r^6} + 24 \frac{x^4}{r^8} \right) \quad (27a)$$

$$h_y^- = \frac{1}{E}(7+3\nu)p \cdot \frac{B_0}{\mu_0 \mu_r} \cdot \frac{\mu_r-1}{\mu_r+1} \cdot \frac{a^4}{6} \left(12 \frac{xy}{r^6} - 24 \frac{x^3 y}{r^8} \right) \quad (27b)$$

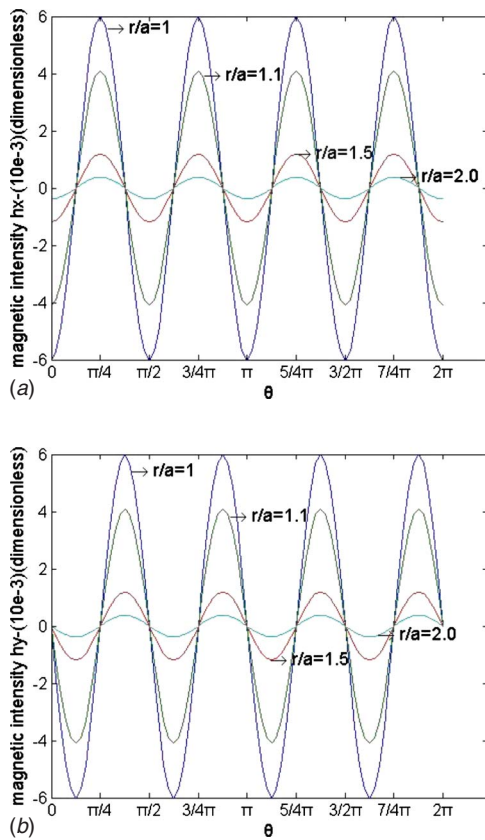


Fig. 7 The perturbed intensities for various r/a ratios: (a) h_x^- and (b) h_y^-

As shown by Eqs. (27a) and (27b) the intensities of the perturbed magnetic field depend on Young's modulus E and Poisson's ratio ν of the plate, and is proportional to the applied stress p . Obviously if there is no hole ($a=0$) or no stress ($p=0$), the perturbed magnetic field will not arise. This is consistent with that discussed in Sec. 3.

5 Results and Discussion

5.1 Comparison Between the Rigid and the Perturbed Fields. The following parameters were used to calculate the field intensities: the radius $a=1$ m, the stress $p=300$ MPa, Poisson's ratio $\nu=0.3$, and Young's modulus $E=200$ GPa. For convenience, the same term $(B_0/\mu_0\mu_r) \cdot [(\mu_r-1)/(\mu_r+1)]$ in Eqs. (22), (27a), and (27b) were not taken into the calculation.

Figures 3(a) and 3(b) show the rigid magnetic field intensity components H_x^- and H_y^- , respectively. There are two maximum values at $\theta=90$ deg and 270 deg, respectively, while two minimum at $\theta=0$ deg and 180 deg along the hole boundary for H_x^- . The y component almost has the same shape as the x component but it rotates 45 deg anticlockwise.

The perturbed magnetic intensity components h_x^- and h_y^- were computed by Eqs. (27a) and (27b) and plotted in Figures 4(a) and 4(b), respectively. It shows that the intensities of the perturbed magnetic field are three orders less than the rigid. This is because the term $(1/E)(7+3\nu)p$ in Eqs. (27a) and (27b) is in the order of 10^{-3} usually. Furthermore, h_x^- has four peaks (at $\theta=45$ deg,

135 deg, 225 deg, and 315 deg) and four valleys (at $\theta=90$ deg, 180 deg, and 270 deg). h_y^- is similar to h_x^- but it rotates 22.5 deg anticlockwise. Those features can be more clearly observed in Figs. 5(a) and 5(b), in which the values of h_x^- and h_y^- were multiplied by 10^3 in order to make them being comparable.

5.2 Distribution of the Magnetic Field Intensities. Figures 6 and 7 show the change of the magnetic intensity along the concentric circles of the hole with various radii before and after deformation, respectively. It found that with ratio r/a increasing the, the perturbed magnetic field intensities decay more rapidly than the rigid state. This suggests that it is easier to observe the perturbed magnetic field at the location close to the hole.

6 Conclusions

In this paper, a set of equations to solve the perturbed magnetic fields induced by mechanical stress in a magnetoelastic solid was presented for the case of weak external magnetic field. The perturbed magnetic fields of infinite plates subjected to tensile stress were derived analytically. Some conclusions were drawn as follows.

- (1) Within the presented approach, the perturbed magnetic field induced by mechanical stress is dominated by the displacement gradient on the boundary of the magnetoelastic solid.
- (2) The perturbed magnetic field intensity is proportional to the applied stress, and is three orders less than the rigid field. The perturbed field has significant different distribution pattern compared with the rigid.
- (3) The perturbed magnetic field reaches its maximum intensity in the boundary region of the hole and its intensity decays faster than the rigid in the radial direction.

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Dynamics of General Constrained Robots Derived from Rigid Bodies

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A systematic approach for deriving the dynamical expression of general constrained robots is developed in this paper. This approach uses rigid-body dynamics and two kinematics-based mapping matrices to form the dynamics of complex robots in closed form. This feature enables the developed modeling approach to be rigorous in nature, since every actuator and gear-head can be separated into rigid bodies and no assumption about approximation beyond rigid-body dynamics is made. The two kinematics-based mapping matrices are used to govern the velocity and force transformations among three configuration spaces, namely, general joint space, general task space, and extended subsystems space. Consequently, the derived dynamics of general constrained robots maintain the same form and main properties as the conventional single-arm constrained robots. This approach is particularly useful for robots with hyper degrees of freedom. Five examples are given. [DOI: 10.1115/1.2839633]

Keywords: robot dynamics, robot simulation and control, rigid body dynamics, robots with closed chains, general constrained robots

1 Introduction

The conventional Lagrangian dynamic model of a constrained robot can be written as

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \boldsymbol{\tau} - \mathbf{J}^T \mathbf{F} \quad (1)$$

when expressed in the joint space or

$$\mathbf{M}_c \dot{\mathbf{v}} + \mathbf{C}_c \mathbf{v} + \mathbf{G}_c = \mathbf{u} - \mathbf{F} \quad (2)$$

when expressed in the Cartesian space [1], where

$$\mathbf{M}_c = \mathbf{J}^{-T} \mathbf{M}(\mathbf{q}) \mathbf{J}^{-1}$$

$$\mathbf{C}_c = \mathbf{J}^{-T} \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \mathbf{J}^{-1} + \mathbf{J}^{-T} \mathbf{M}(\mathbf{q}) \frac{d}{dt} (\mathbf{J}^{-1})$$

$$\mathbf{G}_c = \mathbf{J}^{-T} \mathbf{G}(\mathbf{q})$$

$$\mathbf{u} = \mathbf{J}^{-T} \boldsymbol{\tau}$$

with the Jacobian matrix $\mathbf{J}$ that maps from $\dot{\mathbf{q}}$ to $\mathbf{v}$ being invertible.

It is well known that the total numbers of addition and multiplication of a standard compact-in-form Lagrangian dynamical expression are proportional to the fourth power of the degrees of freedom of a robot [2,3]. This fact challenges both simulation and control of hyper degree-of-freedom (DOF) robots. With respect to this difficulty, a novel systematic approach for dynamical modeling by using the dynamics of joints and of rigid bodies was proposed in [4]. The joint dynamics, however, are based on the same two assumptions in [5]. Although the second assumption is relatively reasonable, the first assumption states that “the kinetic energy of the (motor) rotor is due mainly to its own rotation. Equivalently, the motion of the rotor is a pure rotation with respect to an inertial frame.” Although this assumption keeps the main dynamic effect of the motor rotor, it ignores all secondary dynamic effects of the motor rotor, such as the gyro effect. Therefore, although being roughly acceptable for industrial applications, this approximation possesses limitations when rigorous modeling is concerned. In this paper, the two assumptions in [5] are removed. Consider the fact that a robotic joint is generally formed by a few

rigid bodies. Therefore, only the dynamics of rigid bodies incorporating two kinematics-based mapping matrices are used to form the dynamic equation of general constrained robots. Thus, the developed modeling approach possesses a rigorous basis without using approximations beyond rigid-body dynamics. As illustrated in Fig. 1, the two mapping matrices transform velocities and forces among three configuration spaces, namely *general joint space* $\mathcal{A}$, *general task space* $\mathcal{O}$, and *extended subsystems space* $\mathcal{S}$, and also impose the original coupling (constraints) on the systems. The derived dynamic equation of general constrained robots possesses a closed form resembling the conventional Lagrangian formulation (2), while preserving main properties. As a special treatment, this modeling technique even allows flexible joints to be included in the modeling framework.

This paper is organized as follows: In Sec. 2, the modeling approach for general constrained robots is presented in which three configuration spaces and two mapping matrices are defined. In Sec. 3, the modeling procedure is summarized, followed by five examples.

2 General Constrained Robots

This section presents the procedure for deriving the dynamic expression of general constrained robots, such as the robot illustrated in Fig. 2.

DEFINITION 1. A joint is one bearing or a pair of bearings that permits certain degree-of-freedom relative motion between two rigid bodies. A joint can be actuated or unactuated. Three elementary types of joints are as follows:

- A prismatic joint permits a one-degree-of-freedom translational relative motion between two rigid bodies.
- A revolute joint permit a one-degree-of-freedom rotational relative motion between two rigid bodies.
- A spherical joint permit a three-degree-of-freedom rotational relative motion between two rigid bodies.

DEFINITION 2. A general constrained robot is a base-fixed robot comprised of rigid bodies connected with joints by Definition 1.

2.1 General Constraints. Without loss of generality, it is assumed that a general constrained robot has n_1 single-DOF prismatic or revolute joints and n_3 three-DOF spherical joints. Among the n_1 single-DOF joints, n_{1a} joints are actuated and n_{1u} joints are

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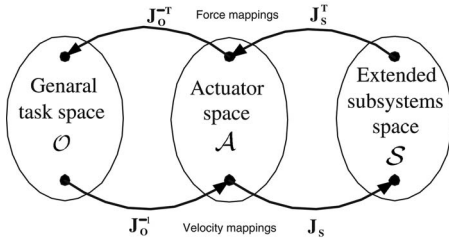


Fig. 1 Velocity and force mappings among three configuration spaces

unactuated. Among the n_3 three-DOF spherical joints, n_{3a} joints are actuated and n_{3u} joints are unactuated. It makes

$$n_1 = n_{1a} + n_{1u} \quad (3)$$

$$n_3 = n_{3a} + n_{3u} \quad (4)$$

Furthermore, it is assumed that the entire robotic system has m motion degrees of freedom. This implies that there exist

$$n_c = (n_1 + 3n_3) - m = [n_{1a} + n_{1u} + 3(n_{3a} + n_{3u})] - m > 0 \quad (5)$$

overall constraints inside the system including the operational constraints and the inherent mechanical constraints.

Two types of coordinate systems are used throughout this paper. The first type is a coordinate system consisting of three mutually orthogonal unit axes as the basis, and the second type is a single-axis coordinate system. The three-unit-axis orthogonal coordinate system is used to measure the linear/angular velocities¹ and the forces/moments of rigid links, as well as the angular velocities and the moments of the three-DOF spherical joints. The single-axis coordinate system is used to measure the linear/rotational velocities and the forces/torques of the single-DOF joints.

Let $[J_a, J_u]$, with $J_a \in \mathbb{R}^{n_c \times (n_{1a} + 3n_{3a})}$ and $J_u \in \mathbb{R}^{n_c \times (n_{1u} + 3n_{3u})}$, be a full row-rank matrix characterizing the n_c constraints as

$$J_a \dot{q}_a + J_u \dot{q}_u = 0 \quad (6)$$

where $\dot{q}_a \in \mathbb{R}^{n_{1a} + 3n_{3a}}$ denotes the velocity coordinates of all actuated joints and $\dot{q}_u \in \mathbb{R}^{n_{1u} + 3n_{3u}}$ denotes the velocity coordinates of all unactuated joints. Let n_p denote the rank of matrix J_u with

$$n_p \leq \min\{n_c, n_{1u} + 3n_{3u}\} \quad (7)$$

Premultiplying $\dot{q}_a$ by a reordering matrix $R_a \in \mathbb{R}^{(n_{1a} + 3n_{3a}) \times (n_{1a} + 3n_{3a})}$ forms $R_a \dot{q}_a = [\dot{q}_{a1}^T, \dot{q}_{a2}^T]^T$ and premultiplying $\dot{q}_u$ by a reordering matrix $R_u \in \mathbb{R}^{(n_{1u} + 3n_{3u}) \times (n_{1u} + 3n_{3u})}$ forms $R_u \dot{q}_u = [\dot{q}_{u1}^T, \dot{q}_{u2}^T]^T$. Equation (6) can be rewritten as

$$J_{u1} \dot{q}_{u1} + J_{a1} \dot{q}_{a1} + J_{a2} \dot{q}_{a2} + J_{u2} \dot{q}_{u2} = 0 \quad (8)$$

subject to

1. $J_{u1} \in \mathbb{R}^{n_c \times n_p}$ and $J_{a1} \in \mathbb{R}^{n_c \times (n_c - n_p)}$ are of full column rank.
2. $[J_{u1}, J_{a1}] \in \mathbb{R}^{n_c \times n_c}$ is invertible.

The reordering process intends to find the dependent velocity coordinates in the joints imposed by the n_c overall constraints. First, it finds n_p independent columns in J_u to form a new matrix J_{u1} . Then, it finds $n_c - n_p$ complementary independent columns in J_a to form a new matrix J_{a1} . Subject to n_c overall constraints, the matrix J_{u1} defines the n_p dependent velocity coordinates in the unactuated joints. The remaining $n_c - n_p$ dependent velocity coordinates are defined by J_{a1} for the actuated joints. Note that the $n_c - n_p$ constraints imposed on the actuated joints result in the same

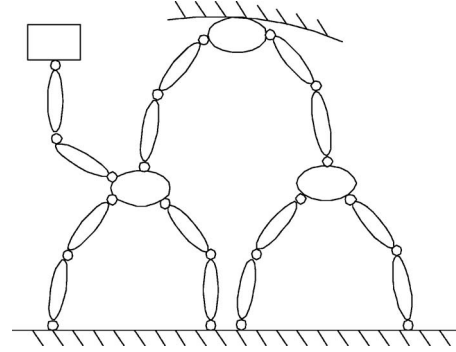


Fig. 2 A general constrained robotic system

number of dimensions for the general constraint forces.²

It follows from (8) that

$$\begin{bmatrix} \dot{q}_{a1} \\ \dot{q}_{a2} \end{bmatrix} = -[J_{u1} \quad J_{a1}]^{-1} [J_{a2} \quad J_{u2}] \begin{bmatrix} \dot{q}_{a2} \\ \dot{q}_{u2} \end{bmatrix} = - \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_{a2} \\ \dot{q}_{u2} \end{bmatrix} \quad (9)$$

Equation (9) indicates that the joint velocity coordinates $\dot{q}_{a2} \in \mathbb{R}^{[(n_{1a} + 3n_{3a}) - (n_c - n_p)]}$ of the actuated joints and $\dot{q}_{u2} \in \mathbb{R}^{[(n_{1u} + 3n_{3u}) - n_p]}$ of the unactuated joints form the independent joint velocity coordinates of the system. As mentioned above, n_p -dimensional constraints are imposed on the n_p unactuated (passive) joints as

$$\dot{q}_{u1} + J_{11} \dot{q}_{a2} + J_{12} \dot{q}_{u2} = 0 \quad (10)$$

and the remaining $(n_c - n_p)$ -dimensional constraints are imposed on the $n_c - n_p$ actuated joints as

$$\dot{q}_{a1} + J_{21} \dot{q}_{a2} + J_{22} \dot{q}_{u2} = 0 \quad (11)$$

Premultiplying (11) by a $(n_c - n_p) \times (n_c - n_p)$ full-rank matrix, denoted as T_c , yields

$$T_c [I_{(n_c - n_p)} \quad J_{21} \quad J_{22}] \begin{bmatrix} \dot{q}_{a1} \\ \dot{q}_{a2} \\ \dot{q}_{u2} \end{bmatrix} \stackrel{\text{def}}{=} J_f \begin{bmatrix} \dot{q}_{a1} \\ \dot{q}_{a2} \\ \dot{q}_{u2} \end{bmatrix} = 0 \quad (12)$$

with

$$J_f = T_c [[I_{(n_c - n_p)} \quad J_{21}]] R_a \quad J_{22} \quad (13)$$

where matrix T_c maps the $(n_c - n_p)$ -dimensional constraints defined by (11) to where the $(n_c - n_p)$ -dimensional general constraint force is defined.³ Note that matrix $J_f \in \mathbb{R}^{(n_c - n_p) \times (n_{1a} + 3n_{3a} + n_{1u} + 3n_{3u} - n_p)}$ is of full row rank.

Define the system dimensions as

$$n \stackrel{\text{def}}{=} n_1 + 3n_3 - n_p = (n_{1a} + 3n_{3a} + n_{1u} + 3n_{3u}) - n_p = m + (n_c - n_p) \quad (14)$$

The system dimensions include m motion degrees of freedom and $n_c - n_p$ dimensions for the general constraint forces.

Remark 2.1. The n_c constraints of a robotic system represent the pure mechanical constraints inside the robotic system (such as the constraints among the linkages) and the constraints between the robot and the environment. Among the n_c overall constraints, n_p constraints are satisfied by releasing the corresponding motion

²The general constraint forces in this paper refer to all the constraint forces that can be directly regulated by actuators without affecting the motion of the systems. The general constraint forces include the conventional constraint forces for single-arm robots and the internal forces for coordinated multiple-arm robots.

³Matrix T_c can be identity if the constraint equation (11) is directly defined on the operations.

¹Linear velocity refers to point velocity of the origin of a corresponding frame.

coordinates associated with the unactuated joints (as performed by (10)). The remaining $n_c - n_p$ constraints appear in the ultimate dynamic equation.

2.2 General Joint Space $\mathcal{A}$. The general joint space $\mathcal{A}$ comprises $n_{1a} + 3n_{3a}$ actuated dimensions and $n_{1u} + 3n_{3u} - n_p$ unactuated dimensions and, therefore, possesses n dimensions.

Let

$$\dot{\mathbf{q}}_A = \begin{bmatrix} \dot{\mathbf{q}}_a \\ \dot{\mathbf{q}}_{u2} \end{bmatrix} \in \mathbb{R}^n \quad (15)$$

be the velocity coordinates and let

$$\boldsymbol{\tau}_A = \begin{bmatrix} \boldsymbol{\tau}_a \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^n \quad (16)$$

be the corresponding torque coordinates, in the *general joint space* $\mathcal{A}$, where $\boldsymbol{\tau}_a$ in $\mathbb{R}^{n_{1a} + 3n_{3a}}$ denotes the joint control forces/torques.

2.3 General Task Space $\mathcal{O}$. The general task space $\mathcal{O}$ comprises an m -dimensional motion space and an $(n_c - n_p)$ -dimensional constraint force space. Thus, it possesses n dimensions, and is analogous to the Cartesian space. Configuration space $\mathcal{O}$ is application oriented.

In view of (12) and (15), the constraint force is characterized by $\mathbf{J}_f \dot{\mathbf{q}}_A = \mathbf{0}$. Accordingly, the constraint force transferred to the general joint space $\mathcal{A}$ can be expressed as $\mathbf{J}_f^T \boldsymbol{\eta}_f$, where $\boldsymbol{\eta}_f \in \mathbb{R}^{n_c - n_p}$ denotes the general constraint force coordinates that do not transfer power. Besides the general constraint force, the dynamic contact force coordinates in the motion space [6] can be denoted as $\boldsymbol{\eta}_m \in \mathbb{R}^m$, which represents the forces that are state dependent. Examples include frictional forces and contact forces with compliant environments. As a result, the overall general constraint force coordinates and the dynamic contact force coordinates converted to the general joint space $\mathcal{A}$ can be written as

$$\boldsymbol{\tau}^* = [\mathbf{J}_f^T \quad \mathbf{D}_m^T] \begin{bmatrix} \boldsymbol{\eta}_f \\ \boldsymbol{\eta}_m \end{bmatrix} \quad (17)$$

where $\mathbf{D}_m \in \mathbb{R}^{m \times n}$ is a matrix.

In space $\mathcal{O}$, let $\mathbf{v}_m \in \mathbb{R}^m$ be the independent velocity coordinates subject to

$$\mathbf{v}_m = \mathbf{J}_m \dot{\mathbf{q}}_A \quad (18)$$

where $\mathbf{J}_m \in \mathbb{R}^{m \times n}$ is of full row rank.

Because the rows of $\mathbf{J}_f$ span the space for the general constraint force and the rows of $\mathbf{J}_m$ span the space for the free motion, the orthogonality of $\mathbf{J}_f$ and $\mathbf{J}_m$ (i.e., $\mathbf{J}_f^T \mathbf{J}_m = \mathbf{0}$) can be ensured [6]. Furthermore, consider the fact that both $\mathbf{J}_m$ and $\mathbf{J}_f$ are of full row rank, the composed matrix

$$\mathbf{J}_O = \begin{bmatrix} \mathbf{J}_m \\ \mathbf{J}_f \end{bmatrix} \in \mathbb{R}^{n \times n}$$

is of full rank and, therefore, is invertible such that

$$\mathbf{J}_O^{-1} = \begin{bmatrix} \mathbf{J}_m \\ \mathbf{J}_f \end{bmatrix}^{-1} = [\mathbf{J}_\Omega \quad \mathbf{J}_\Psi] \quad (19)$$

where $\mathbf{J}_\Omega \in \mathbb{R}^{n \times m}$ and $\mathbf{J}_\Psi \in \mathbb{R}^{n \times (n-m)}$ are two matrices with full column rank.

In view of (12), (15), and (18), it follows that

$$\mathbf{v}_O \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{v}_m \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_m \\ \mathbf{J}_f \end{bmatrix} \dot{\mathbf{q}}_A \quad (20)$$

Consequently, it yields

$$\dot{\mathbf{q}}_A = \mathbf{J}_\Omega \mathbf{v}_m \quad (21)$$

2.4 Extended Subsystems Space $\mathcal{S}$. The extended subsystems space $\mathcal{S}$ in which the dynamics of rigid bodies are ex-

pressed casts the kernel of the proposed approach for dynamic modeling. This is due to the fact that the separated dynamics of rigid bodies that comprise a robotic system are much simpler than the standard Lagrangian dynamical expression of the aggregated (coupled) robotic system, particularly for robots with high degrees of freedom [7].

Three types of subsystems, namely, the rigid links (rigid bodies), the single-DOF prismatic or revolute joints, and the three-DOF spherical joints, are studied. Let Φ be a set containing all frames each is fixed to a corresponding rigid body, Θ_1 be a set containing the sequential numbers of all single-DOF joints, and Θ_3 be a set containing the sequential numbers of all three-DOF spherical joints.

Define a body-frame $\{\mathbf{B}\} \in \Phi$ referenced linear/angular velocity vector as

$${}^B \mathbf{V} = \begin{bmatrix} {}^B \mathbf{R}_I \mathbf{v}_B \\ {}^B \mathbf{R}_I \boldsymbol{\omega}_B \end{bmatrix} \in \mathbb{R}^6 \quad (22)$$

where ${}^B \mathbf{R}_I \in \mathbb{R}^{3 \times 3}$ is a rotation matrix that transforms a three-dimensional vector expressed in inertial frame $\{\mathbf{I}\}$ to the same vector expressed in frame $\{\mathbf{B}\}$, and $\mathbf{v}_B \in \mathbb{R}^3$ and $\boldsymbol{\omega}_B \in \mathbb{R}^3$ denote the linear and angular velocities of frame $\{\mathbf{B}\}$ and expressed in inertial frame $\{\mathbf{I}\}$.

Accordingly, define a body-frame $\{\mathbf{B}\} \in \Phi$ referenced force/moment vector as

$${}^B \mathbf{F} = \begin{bmatrix} {}^B \mathbf{R}_I \mathbf{f}_B \\ {}^B \mathbf{R}_I \mathbf{m}_B \end{bmatrix} \in \mathbb{R}^6 \quad (23)$$

where $\mathbf{f}_B \in \mathbb{R}^3$ and $\mathbf{m}_B \in \mathbb{R}^3$ denote a force and a moment referenced by frame $\{\mathbf{B}\}$ and expressed in inertial frame $\{\mathbf{I}\}$.

Let ${}^B \mathbf{F}^* \in \mathbb{R}^6$ be the net force/moment of a rigid body, referenced to the body frame $\{\mathbf{B}\}$. The dynamics of this rigid body expressed in the body frame $\{\mathbf{B}\} \in \Phi$ can be written as [7]

$$\mathbf{M}_B \frac{d}{dt} ({}^B \mathbf{V}) + \mathbf{C}_B {}^B \mathbf{V} + \mathbf{G}_B = {}^B \mathbf{F}^* \quad (24)$$

where the explicit expressions of $\mathbf{M}_B \in \mathbb{R}^{6 \times 6}$, $\mathbf{C}_B \in \mathbb{R}^{6 \times 6}$, and $\mathbf{G}_B \in \mathbb{R}^6$ are given by [7] (p. 418) with $\mathbf{M}_B$ being constant and symmetric and $\mathbf{C}_B$ being skew symmetric.

The force/torque relation of the j th single-DOF joint is

$$\xi_{1j}(t) = \tau_{1j} - f_{p1j}, \quad j \in \Theta_1 \quad (25)$$

where $\xi_{1j}(t) \in \mathbb{R}$ is the frictional force/torque, $\tau_{1j} \in \mathbb{R}$ is the joint control force/torque,⁴ and $-f_{p1j} \in \mathbb{R}$ is the projected force/moment from the links along the joint axis.

The moment relation of the i th three-DOF spherical joint is

$$\xi_{3i}(t) = \tau_{3i} - \mathbf{m}_{p3i}, \quad i \in \Theta_3 \quad (26)$$

where $\xi_{3i}(t) \in \mathbb{R}^3$ denotes the frictional moment, $\tau_{3i} \in \mathbb{R}^3$ is the joint control torques,⁵ and $-\mathbf{m}_{p3i} \in \mathbb{R}^3$ is the projected moment from the links along the three joint axes.

The extended velocity and net force/moment coordinates in space $\mathcal{S}$ are expressed as

$$\mathcal{V} \stackrel{\text{def}}{=} [\cdots, {}^B \mathbf{V}^T, \cdots]^T, \{\mathbf{B}\} \in \Phi \quad (27)$$

$$\mathcal{F}^* \stackrel{\text{def}}{=} [\cdots, {}^B \mathbf{F}^{*T}, \cdots]^T, \{\mathbf{B}\} \in \Phi \quad (28)$$

In view of (24), it follows from the definitions of $\mathcal{V}$ and $\mathcal{F}^*$ that

$$\mathcal{F}^* = \mathcal{M} \dot{\mathcal{V}} + \mathcal{C} \mathcal{V} + \mathcal{G} \quad (29)$$

where

⁴ $\tau_{1j} = 0$ holds for a unactuated joint.

⁵ $\tau_{3i} = 0$ holds for a unactuated spherical joint.

$$\mathcal{M} = \text{diag}(\cdots, \mathbf{M}_B, \cdots), \{\mathbf{B}\} \in \Phi$$

$$\mathcal{C} = \text{diag}(\cdots, \mathbf{C}_B, \cdots), \{\mathbf{B}\} \in \Phi$$

$$\mathcal{G} = [\cdots, \mathbf{G}_B^T, \cdots]^T, \{\mathbf{B}\} \in \Phi$$

Based on the definition of $\dot{\mathbf{q}}_A \in \mathbb{R}^n$ in space $\mathcal{A}$, the extended velocity $\mathcal{V}$ in space $\mathcal{S}$ can be expressed as

$$\mathcal{V} = \mathbf{J}_S \dot{\mathbf{q}}_A \quad (30)$$

The physical meaning of (30) is that the velocities of all rigid bodies are completely dependent on the velocities in space $\mathcal{A}$. Thus, Eq. (30) imposes kinematic constraints on (24).

With $\boldsymbol{\eta}_f = 0$ and $\boldsymbol{\eta}_m = 0$, the power received by all rigid links equals the power generated by all joints, i.e.,

$$\sum_{\{\mathbf{B}\} \in \Phi} \mathbf{B}^T \mathbf{V}^T \mathbf{F}^* = \sum_{j \in \Theta_1} \{\dot{q}_{1j} f_{p1j}\} + \sum_{i \in \Theta_3} \{\dot{q}_{3i} \mathbf{m}_{p3i}\} \quad (31)$$

Substituting f_{p1j} from (25) and $\mathbf{m}_{p3i}$ from (26) into (31) yields

$$\begin{aligned} \sum_{\{\mathbf{B}\} \in \Phi} \mathbf{B}^T \mathbf{V}^T \mathbf{F}^* &= \sum_{j \in \Theta_1} \{\dot{q}_{1j} \tau_{1j}\} + \sum_{i \in \Theta_3} \{\dot{q}_{3i}^T \tau_{3i}\} - \sum_{j \in \Theta_1} \{\dot{q}_{1j} \xi_{1j}\} \\ &\quad - \sum_{i \in \Theta_3} \{\dot{q}_{3i}^T \xi_{3i}\} \end{aligned} \quad (32)$$

Note that the definitions of $\mathcal{V}$ and $\mathcal{F}^*$ ensures

$$\sum_{\{\mathbf{B}\} \in \Phi} \mathbf{B}^T \mathbf{V}^T \mathbf{F}^* = \mathcal{V}^T \mathcal{F}^* \quad (33)$$

Meanwhile,

$$\sum_{j \in \Theta_1} \{\dot{q}_{1j} \tau_{1j}\} + \sum_{i \in \Theta_3} \{\dot{q}_{3i}^T \tau_{3i}\} = \dot{\mathbf{q}}_A^T \boldsymbol{\tau}_A \quad (34)$$

is valid because $\tau_{1j} = 0$ or $\tau_{3i} = 0$ for a unactuated joint. Moreover, rewrite $\sum_{j \in \Theta_1} \{\dot{q}_{1j} \xi_{1j}\} + \sum_{i \in \Theta_3} \{\dot{q}_{3i}^T \xi_{3i}\}$ as

$$\begin{aligned} \sum_{j \in \Theta_1} \{\dot{q}_{1j} \xi_{1j}\} + \sum_{i \in \Theta_3} \{\dot{q}_{3i}^T \xi_{3i}\} &= \dot{\mathbf{q}}_{a1}^T \xi_{a1} + \dot{\mathbf{q}}_{a2}^T \xi_{a2} + \dot{\mathbf{q}}_{u1}^T \xi_{u1} + \dot{\mathbf{q}}_{u2}^T \xi_{u2} \\ &= \dot{\mathbf{q}}_A^T \xi_q \end{aligned} \quad (35)$$

in view of (10), where vectors ξ_{a1} , ξ_{a2} , ξ_{u1} , and ξ_{u2} contain frictional forces/torques/moments corresponding to $\dot{\mathbf{q}}_{a1}$, $\dot{\mathbf{q}}_{a2}$, $\dot{\mathbf{q}}_{u1}$, and $\dot{\mathbf{q}}_{u2}$, respectively, and

$$\xi_q = \begin{bmatrix} \mathbf{R}_a^T \\ \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & -\mathbf{J}_{11} & -\mathbf{J}_{12} \\ 0 & 0 & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \xi_{a1} \\ \xi_{a2} \\ \xi_{u1} \\ \xi_{u2} \end{bmatrix} \quad (36)$$

Thus, substituting (33)–(35) into (32) yields

$$\mathcal{V}^T \mathcal{F}^* + \dot{\mathbf{q}}_A^T \xi_q = \dot{\mathbf{q}}_A^T \boldsymbol{\tau}_A \quad (37)$$

In view of (30) and (37), it follows that

$$\dot{\mathbf{q}}_A^T \mathbf{J}_S^T \mathcal{F}^* + \dot{\mathbf{q}}_A^T \xi_q = \dot{\mathbf{q}}_A^T \boldsymbol{\tau}_A \quad (38)$$

In a particular case that only the k th element of $\dot{\mathbf{q}}_A$ is nonzero (i.e. mathematically, $\dot{q}_k \neq 0$ and $\dot{q}_j = 0$ for $j = 1, 2, \dots, k-1, k+1, \dots, n$), it follows from (38) that

$$(\mathbf{J}_S^T \mathcal{F}^*)_k + (\xi_q)_k = \tau_k \quad (39)$$

where $(\mathbf{J}_S^T \mathcal{F}^*)_k$, $(\xi_q)_k$, and τ_k denote the k th elements of $\mathbf{J}_S^T \mathcal{F}^*$, ξ_q , and $\boldsymbol{\tau}_A$, respectively. Equation (39) gives a transformation about forces from space $\mathcal{S}$ to space $\mathcal{A}$. Because the transformation depends on the configuration only, (39) will be valid for all $k \in \{1, n\}$. Thus, it yields

$$\mathbf{J}_S^T \mathcal{F}^* + \xi_q = \boldsymbol{\tau}_A \quad (40)$$

Equation (40) demonstrates how the net forces/moments of a ro-

Table 1 Comparison between (2) and (42)

Eq. (2)	Eq. (42)
$\mathbf{M}(\mathbf{q})$	$\mathbf{J}_S^T \mathcal{M} \mathbf{J}_S$
$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$	$\mathbf{J}_S^T \mathcal{C} \mathbf{J}_S$
$\mathbf{J}^{-T} \mathbf{M}(\mathbf{q}) \mathbf{J}^{-1}$	$\mathbf{J}_O^T \mathbf{J}_S^T \mathcal{M} \mathbf{J}_S \mathbf{J}_O^{-1}$
$\mathbf{J}^{-T} \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \mathbf{J}^{-1} + \mathbf{J}^{-T} \mathbf{M}(\mathbf{q}) (d/dt) (\mathbf{J}^{-1})$	$\mathbf{J}_O^T \mathbf{J}_S^T \mathcal{C} \mathbf{J}_S \mathbf{J}_O^{-1} + \mathbf{J}_O^T \mathbf{J}_S^T \mathcal{M} (d/dt) (\mathbf{J}_S \mathbf{J}_O^{-1})$
$\mathbf{J}^{-T} \mathbf{G}(\mathbf{q})$	$\mathbf{J}_O^T [\mathbf{J}_S^T \mathcal{G} + \xi_q + \mathbf{D}_m^T \boldsymbol{\eta}_m]$
$\mathbf{J}^{-T} \boldsymbol{\tau}$	$\mathbf{J}_O^T \boldsymbol{\tau}_A$
$\mathbf{F}$	$\begin{bmatrix} 0 \\ \boldsymbol{\eta}_f \end{bmatrix}$

botic system are mapped into the *general joint space* $\mathcal{A}$. The actual joint torques, however, will include the torques that are transferred from the general constraint force coordinates $\boldsymbol{\eta}_f$ and from the dynamic contact force coordinates $\boldsymbol{\eta}_m$ as formulated by (17). Therefore, the force transformation equation should be rewritten as

$$\mathbf{J}_S^T \mathcal{F}^* + \xi_q = \boldsymbol{\tau}_A - \boldsymbol{\tau}^* \quad (41)$$

2.5 Dynamic Model. Based on (17), (20), (29), (30), and (41), the dynamic model of a general constrained robot expressed in space $\mathcal{O}$ is written as

$$\mathcal{M}_G \begin{bmatrix} \dot{\mathbf{v}}_m \\ 0 \end{bmatrix} + \mathcal{C}_G \begin{bmatrix} \mathbf{v}_m \\ 0 \end{bmatrix} + \mathcal{G}_G = \mathbf{J}_O^T \boldsymbol{\tau}_A - \begin{bmatrix} 0 \\ \boldsymbol{\eta}_f \end{bmatrix} \quad (42)$$

where

$$\mathcal{M}_G = \mathbf{J}_O^T \mathbf{J}_S^T \mathcal{M} \mathbf{J}_S \mathbf{J}_O^{-1}$$

$$\mathcal{C}_G = \left[\mathbf{J}_O^T \mathbf{J}_S^T \mathcal{C} \mathbf{J}_S \mathbf{J}_O^{-1} + \mathbf{J}_O^T \mathbf{J}_S^T \mathcal{M} \frac{d}{dt} (\mathbf{J}_S \mathbf{J}_O^{-1}) \right]$$

$$\mathcal{G}_G = \mathbf{J}_O^T [\mathbf{J}_S^T \mathcal{G} + \xi_q + \mathbf{D}_m^T \boldsymbol{\eta}_m]$$

A comparison between (2) and (42) is given in Table 1.

Remark 2.2. In Eq. (42), $\mathbf{v}_m \in \mathbb{R}^m$ denotes the independent velocity coordinates in the general task space $\mathcal{O}$. Among the m dimensions of $\mathbf{v}_m \in \mathbb{R}^m$, there exist $(n_{1u} + 3n_{3u} - n_p)$ unactuated dimensions. Therefore,

$$m \geq n_{1u} + 3n_{3u} - n_p \quad (43)$$

holds. Meanwhile, the dimensions for the general constraint force are $n_c - n_p$.

Remark 2.3. In simulations, the independent variables of integration are $\dot{\mathbf{v}}_m \in \mathbb{R}^m$. The effective velocity coordinates of the entire system are denoted as $[\mathbf{v}_m^T, \mathbf{0}^T]^T \in \mathbb{R}^n$, where $\mathbf{v}_m \in \mathbb{R}^m$ is the integral of $\dot{\mathbf{v}}_m \in \mathbb{R}^m$.

Remark 2.4. In view of the dynamic equation of the general constraint robots represented by (42), two groups of components are needed. The first group consists of two kinematics mapping matrices $\mathbf{J}_O$ and $\mathbf{J}_S$, and the second group consists of $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{G}$, the matrices and vector formed from the dynamics of all rigid bodies [7]. Because the dynamic formulation in space $\mathcal{S}$ is standard, the task needs to be done for a particular application is to find the two kinematics-based mapping matrices $\mathbf{J}_O$ and $\mathbf{J}_S$.

Remark 2.5. Note that $\mathcal{M}$ is time invariant and symmetric, and $\mathcal{C}$ is skew symmetric. Therefore, it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\mathbf{J}_O^T \mathbf{J}_S^T \mathcal{M} \mathbf{J}_S \mathbf{J}_O^{-1}) - \left[\mathbf{J}_O^T \mathbf{J}_S^T \mathcal{C} \mathbf{J}_S \mathbf{J}_O^{-1} + \mathbf{J}_O^T \mathbf{J}_S^T \mathcal{M} \frac{d}{dt} (\mathbf{J}_S \mathbf{J}_O^{-1}) \right] \\ &= \frac{1}{2} \frac{d}{dt} (\mathbf{J}_O^T \mathbf{J}_S^T) \mathcal{M} \mathbf{J}_S \mathbf{J}_O^{-1} - \frac{1}{2} \mathbf{J}_O^T \mathbf{J}_S^T \mathcal{M} \frac{d}{dt} (\mathbf{J}_S \mathbf{J}_O^{-1}) - \mathbf{J}_O^T \mathbf{J}_S^T \mathcal{C} \mathbf{J}_S \mathbf{J}_O^{-1} \end{aligned}$$

is skew symmetric.

Remark 2.6. Newton–Euler formulation has been well known for its recursive forms in either forward dynamics [8,9] or inverse dynamics [7,10]. The dynamic formulation presented in this paper can be considered as a closed form of the Newton–Euler formulation that may have exactly the same application forum as the Lagrangian formulation. In other words, the closed form of the Newton–Euler formulation is applicable to both forward-dynamics-based simulations and inverse-dynamics-based controls [11,12].

3 Modeling Procedure and Examples

3.1 Modeling Procedure. In this section, the modeling approach presented in the last section is summarized into the following steps.

Step 1. For a given base-fixed (or equivalent) robotic system, count the number of single-DOF prismatic or revolute joints and the number of three-DOF spherical joints. It yields n_1 and n_3 . Among the n_1 single-DOF prismatic or revolute joints, count the number of actuated joints and the number of unactuated joints. It yields n_{1a} and n_{1u} . Among the n_3 three-DOF spherical joints, count the number of actuated joints and the number of unactuated joints. It yields n_{3a} and n_{3u} .

Step 2. Determine the number of motion degrees of freedom of the system and obtain m . Calculate n_c (the dimensions of the constraints) in terms of (5).

Step 3. Specify the n_c overall constraints. Assign appropriate coordinate frames in space $\mathcal{O}$ for motion/force descriptions. Furthermore, assign at least one coordinate frame to each rigid body.

Step 4. Specify $\dot{\mathbf{q}}_a \in \mathbb{R}^{n_{1a}+3n_{3a}}$ and $\dot{\mathbf{q}}_u \in \mathbb{R}^{n_{1u}+3n_{3u}}$, and form $\mathbf{J}_a$ and $\mathbf{J}_u$ subject to (6) accordingly. It yields n_p , the rank of $\mathbf{J}_u$.

Step 5. Find two reordering matrices $\mathbf{R}_a$ and $\mathbf{R}_u$ for $\mathbf{J}_a$ and $\mathbf{J}_u$, respectively, to form $\mathbf{J}_{a1}$, $\mathbf{J}_{a2}$, $\mathbf{J}_{u1}$, and $\mathbf{J}_{u2}$ such that $[\mathbf{J}_{u1}, \mathbf{J}_{a1}] \in \mathbb{R}^{n_c \times n_c}$ is invertible.

Step 6. Calculate $\mathbf{J}_{11}$, $\mathbf{J}_{12}$, $\mathbf{J}_{21}$, and $\mathbf{J}_{22}$ from (9).

Step 7. Specify the constraint force coordinates $\boldsymbol{\eta}_f \in \mathbb{R}^{n_c - n_p}$, determine the matrix $\mathbf{T}_c$, and then form $\mathbf{J}_f$ in terms of (13).

Step 8. Determine n in terms of (14) and form $\dot{\mathbf{q}}_A$ and $\boldsymbol{\tau}_A$ in terms of (15) and (16) in space $\mathcal{A}$.

Step 9. Specify the dynamic contact force coordinates $\boldsymbol{\eta}_m$, and form the transformation matrix $\mathbf{D}_m$, accordingly.

Step 10. Specify the m -dimensional independent velocity coordinates $\mathbf{v}_m$ in space $\mathcal{O}$ and form the full row-rank mapping matrix $\mathbf{J}_m$. Construct

$$\mathbf{J}_O = \begin{bmatrix} \mathbf{J}_m \\ \mathbf{J}_f \end{bmatrix}.$$

Step 11. Construct the velocity mapping matrix $\mathbf{J}_S$ in terms of (30).

Step 12. Write the block diagonal matrices and vector $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{G}$ in space $\mathcal{S}$. Finally, form the dynamic model (42).

Five examples are presented below to demonstrate each step in detail.

3.2 Single-Arm Constrained Direct-Drive Manipulators.

The first example is a six-single-DOF-joint direct-drive robot manipulator grasping an object in contact with a plane, as illustrated in Fig. 3. Being a direct-drive robot, the system has $n_1 = n_{1a} = 6$ actuated single-DOF joints, $n_{1u} = 0$ unactuated single-DOF joint, and $n_3 = 0$ three-DOF spherical joint in step 1.

Because the robot end effector is in contact with a plane, it

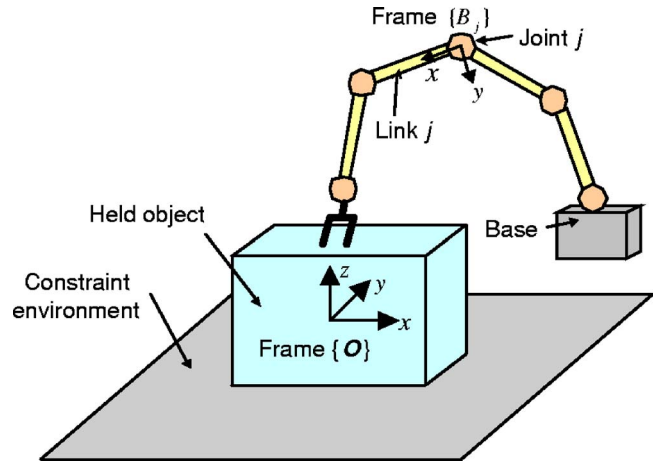


Fig. 3 Single-arm constrained robot

yields $m=3$ (the linear motion along the two tangential directions of the plane and the rotational motion along the normal vector of the plane). Thus, it follows that $n_c = 6 + 0 - 3 = 3$ from (5) in step 2.

In step 3, the linear motion along the two tangential directions of the contact plane and the rotational motion along the normal vector of the contact plane specify the motion space. The linear motion along the normal vector of the contact plane and the rotational motion along the two tangential directions of the contact plane specify the constraints. A frame $\{O\}$ is fixed to the robot end effector in a way that its x - and y -axes lie on the contact plane and its z -axis coincides with the normal vector of the contact plane. Meanwhile, the six joints are numbered sequentially from the base toward the end effector with the j th joint connecting the j th link with the $j-1$ th link, $j=1, \dots, 6$. There are six auxiliary frames $\{B_j\}$, $j=1, 2, \dots, 6$, each is fixed to link j with its z -axis, coinciding with the j th joint.

In step 4, specify $\dot{\mathbf{q}}_a = [\dot{q}_1, \dots, \dot{q}_6]^T \in \mathbb{R}^6$ and $\dot{\mathbf{q}}_u = 0$, and form $\mathbf{J}_a$ as

$$\mathbf{J}_a = \mathbf{T}_j \mathbf{J}_q \quad (44)$$

with

$$\mathbf{T}_f = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 6}$$

$$\mathbf{J}_q = [\mathbf{B}_1 \mathbf{U}_O^T \mathbf{z}_1, \mathbf{B}_2 \mathbf{U}_O^T \mathbf{z}_2, \dots, \mathbf{B}_6 \mathbf{U}_O^T \mathbf{z}_6] \in \mathbb{R}^{6 \times 6}$$

where

$$\mathbf{A} \mathbf{U}_B = \begin{bmatrix} \mathbf{A} \mathbf{R}_B & \mathbf{0}_{3 \times 3} \\ (\mathbf{A} \mathbf{r}_B \times) \mathbf{A} \mathbf{R}_B & \mathbf{A} \mathbf{R}_B \end{bmatrix} \in \mathbb{R}^{6 \times 6}$$

denotes a force/moment transformation matrix that transforms a force/moment measured and expressed in frame $\{B\}$ to the force/moment measured and expressed in frame $\{A\}$, with $\mathbf{A} \mathbf{R}_B \in \mathbb{R}^{3 \times 3}$ being a rotation matrix that transforms a 3×1 vector expressed in frame $\{B\}$ to the same vector expressed in frame $\{A\}$ and $\mathbf{A} \mathbf{r}_B \in \mathbb{R}^3$ being a vector pointing from the origin of frame $\{A\}$ to the origin of frame $\{B\}$ and expressed in frame $\{A\}$, and $\mathbf{z}_j = [0 \ 0 \ 1 \ 0 \ 0 \ 0]^T$ for a prismatic joint and $\mathbf{z}_j = [0 \ 0 \ 0 \ 0 \ 0 \ 1]^T$ for a revolute joint, $j=1, \dots, 6$. Because there is no unactuated joint in the system, it yields $\mathbf{J}_u = 0$ and, therefore, $n_p = 0$.

In step 5, find $\mathbf{R}_a \in \mathbb{R}^{6 \times 6}$ to form $\mathbf{J}_a \mathbf{R}_a^T = [\mathbf{J}_{a1} \in \mathbb{R}^{3 \times 3}, \mathbf{J}_{a2} \in \mathbb{R}^{3 \times 3}]$ such that $\mathbf{J}_{a1}$ is invertible.

In step 6, $\dot{\mathbf{q}}_{a2} \in \mathbb{R}^3$ being the independent velocity coordinates leads to

$$\mathbf{J}_{21} = \mathbf{J}_{a1}^{-1} \mathbf{J}_{a2} \in \mathbb{R}^{3 \times 3}$$

with $\mathbf{J}_{11}=0$, $\mathbf{J}_{12}=0$, and $\mathbf{J}_{22}=0$ in (9).

In step 7, the three-dimensional constraint forces are specified as the force along the z -axis and the two moments along the x - and y -axes of frame $\{\mathbf{O}\}$. It follows that

$$\boldsymbol{\eta}_f = \begin{bmatrix} f_z \\ m_x \\ m_y \end{bmatrix} \in \mathbb{R}^3$$

where $f_z \in \mathbb{R}$ denotes the force along the z -axis of frame $\{\mathbf{O}\}$ and $m_x \in \mathbb{R}$ and $m_y \in \mathbb{R}$ denote the moments along the x - and y -axes of frame $\{\mathbf{O}\}$. Moreover, specify $\mathbf{T}_c = \mathbf{J}_{a1}$ leading to

$$\mathbf{J}_f = \mathbf{J}_a \in \mathbb{R}^{3 \times 6}$$

from (13).

In step 8, $n=6$ is obtained from (14). $\dot{\mathbf{q}}_A = [\dot{q}_1, \dots, \dot{q}_6]^T \in \mathbb{R}^6$ and $\boldsymbol{\tau}_A = [\tau_1, \dots, \tau_6]^T \in \mathbb{R}^6$ represent the velocities and control torques in space $\mathcal{A}$. In step 9, the dynamic contact forces are the frictional forces and torque in the motion space. It yields

$$\mathbf{D}_m = \mathbf{T}_m \mathbf{J}_q \in \mathbb{R}^{3 \times 6} \quad (45)$$

with

$$\mathbf{T}_m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 6}$$

In step 10, it follows that

$$\mathbf{v}_m = \begin{bmatrix} v_x \\ v_y \\ \omega_z \end{bmatrix} \in \mathbb{R}^3$$

where $v_x \in \mathbb{R}$ and $v_y \in \mathbb{R}$ denote the linear velocities of the end effector along the x - and y -axes of frame $\{\mathbf{O}\}$ and $\omega_z \in \mathbb{R}$ denotes the angular velocity of the end effector along the z -axis of frame $\{\mathbf{O}\}$. Accordingly, it yields

$$\mathbf{J}_m = \mathbf{T}_m \mathbf{J}_q \in \mathbb{R}^{3 \times 6}$$

to form

$$\mathbf{J}_O = \begin{bmatrix} \mathbf{J}_m \\ \mathbf{J}_f \end{bmatrix} \in \mathbb{R}^{6 \times 6}.$$

In step 11, the velocity mapping matrix $\mathbf{J}_S$ is designed as

$$\mathbf{J}_S = \mathbf{J}_L \in \mathbb{R}^{42 \times 6}$$

with

$$\mathbf{J}_L = \begin{bmatrix} \mathbf{z}_1 & & & & & \\ \mathbf{B}_1 \mathbf{U}_{\mathbf{B}_2}^T \mathbf{z}_1 & \mathbf{z}_2 & & & & \\ \vdots & \ddots & \ddots & & & \\ \mathbf{B}_1 \mathbf{U}_{\mathbf{B}_6}^T \mathbf{z}_1 & \mathbf{B}_2 \mathbf{U}_{\mathbf{B}_6}^T \mathbf{z}_2 & \cdots & \mathbf{z}_6 & & \\ \mathbf{B}_1 \mathbf{U}_{\mathbf{O}}^T \mathbf{z}_1 & \mathbf{B}_2 \mathbf{U}_{\mathbf{O}}^T \mathbf{z}_2 & \cdots & \mathbf{B}_6 \mathbf{U}_{\mathbf{O}}^T \mathbf{z}_6 & & \end{bmatrix} \in \mathbb{R}^{42 \times 6} \quad (46)$$

In step 12, $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{G}$ in space $\mathcal{S}$ are obtained as

$$\mathcal{M} = \text{diag}(\mathbf{M}_{\mathbf{B}_1}, \dots, \mathbf{M}_{\mathbf{B}_6}, \mathbf{M}_{\mathbf{O}})$$

$$\mathcal{C} = \text{diag}(\mathbf{C}_{\mathbf{B}_1}, \dots, \mathbf{C}_{\mathbf{B}_6}, \mathbf{C}_{\mathbf{O}})$$

$$\mathcal{G} = [\mathbf{G}_{\mathbf{B}_1}^T, \dots, \mathbf{G}_{\mathbf{B}_6}^T, \mathbf{G}_{\mathbf{O}}^T]^T$$

3.3 Single-Arm Constrained Manipulators With Gears.

When a gear box is placed between each motor and its output shaft, as illustrated in Fig. 4, each actuated joint is split into two joints by Definition 1. Assume that the gear box has only two

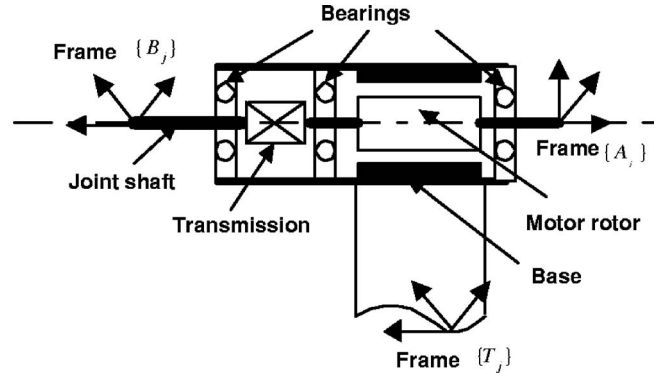


Fig. 4 A joint assembly with motor and transmission

gears: one for input and another for output. As far as multiple gearing stages are concerned, an actuated joint can be split into multiple joints.

It follows that $n_{1a}=6$, $n_{1u}=6$, and $n_3=0$ in step 1. In step 2, it yields $m=3$ (the linear motion along the two tangential directions of the plane and the rotational motion along the normal vector of the plane). Thus, it follows $n_c=(n_1+3n_3)-m=n_{1a}+n_{1u}-m=6+6-3=9$ from (5).

In step 3, the nine overall constraints comprise three constraints at the end effector and six constraints at the gear boxes. The three constraints at the end effector constrain the linear motion along the normal vector of the contact plane and the rotational motion along the two tangential directions of the contact plane. The six constraints at the gear boxes constrain the motion between the input gears and the output gears. Moreover, a frame $\{\mathbf{O}\}$ is fixed to the robot end effector in a way that its x - and y -axes lie on the contact plane and its z -axis coincides with the normal vector of the contact plane. At each actuator, three frames, denoted as $\{\mathbf{T}_j\}$, $\{\mathbf{A}_j\}$, and $\{\mathbf{B}_j\}$, $j \in \{1, 6\}$, are fixed to the motor base, to the motor rotor, and to the output shaft, respectively, with their z -axes coinciding with the corresponding joint axes, as illustrated in Fig. 4.

In step 4, let $\dot{q}_{aj} \in \mathbb{R}$ be the motor velocity measured by frame $\{\mathbf{A}_j\}$ with respect to frame $\{\mathbf{T}_j\}$. Let $\dot{q}_{uj} \in \mathbb{R}$ be the output shaft velocity measured by frame $\{\mathbf{B}_j\}$ with respect to frame $\{\mathbf{T}_j\}$. Specify $\dot{\mathbf{q}}_a = [\dot{q}_{a1}, \dots, \dot{q}_{a6}]^T \in \mathbb{R}^6$ and $\dot{\mathbf{q}}_u = [\dot{q}_{u1}, \dots, \dot{q}_{u6}]^T \in \mathbb{R}^6$. Then, it follows from (6) that

$$\mathbf{J}_a = \begin{bmatrix} \mathbf{0}_{3 \times 6} \\ -\mathbf{I}_6 \end{bmatrix} \in \mathbb{R}^{9 \times 6} \quad (47)$$

$$\mathbf{J}_u = \begin{bmatrix} \mathbf{T}_f \mathbf{J}_q \\ \text{diag}(k_{g1}, \dots, k_{g6}) \end{bmatrix} \in \mathbb{R}^{9 \times 6} \quad (48)$$

where

$$\mathbf{T}_f = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 6}$$

$$\mathbf{J}_q = [\mathbf{B}_1 \mathbf{U}_{\mathbf{O}}^T \mathbf{z}_1, \mathbf{B}_2 \mathbf{U}_{\mathbf{O}}^T \mathbf{z}_2, \dots, \mathbf{B}_6 \mathbf{U}_{\mathbf{O}}^T \mathbf{z}_6] \in \mathbb{R}^{6 \times 6}$$

In (48), $k_{gj} > 0$ denotes the gear ratio of the j th gear box, $j \in \{1, 6\}$. Moreover, it yields $n_p=6$, the rank of $\mathbf{J}_u$. Furthermore, the constraints at the six actuators can be expressed by

$$\dot{q}_{aj} = k_{gj} \dot{q}_{uj}, \quad j \in \{1, 6\} \quad (49)$$

in view of (6), (47), and (48).

In step 5, find $\mathbf{R}_a \in \mathbb{R}^{6 \times 6}$ to form $\mathbf{J}_a \mathbf{R}_a^T = [\mathbf{J}_{a1} \in \mathbb{R}^{9 \times 3}, \mathbf{J}_{a2} \in \mathbb{R}^{9 \times 3}]$ with

$$\mathbf{J}_{a1} = \mathbf{J}_a \mathbf{R}_a^T \begin{bmatrix} \mathbf{I}_3 \\ \mathbf{0}_{3 \times 3} \end{bmatrix}$$

$$\mathbf{J}_{a2} = \mathbf{J}_a \mathbf{R}_a^T \begin{bmatrix} \mathbf{0}_{3 \times 3} \\ \mathbf{I}_3 \end{bmatrix}$$

such that $[\mathbf{J}_u, \mathbf{J}_{a1}] \in \mathbb{R}^{9 \times 9}$ is invertible.

In step 6, $\dot{\mathbf{q}}_{a2} \in \mathbb{R}^3$ being the independent velocity coordinates leads to

$$\begin{bmatrix} \mathbf{J}_{11} \\ \mathbf{J}_{21} \end{bmatrix} = [\mathbf{J}_u \quad \mathbf{J}_{a1}]^{-1} \mathbf{J}_{a2} \in \mathbb{R}^{9 \times 3}$$

with $\mathbf{J}_{11} \in \mathbb{R}^{6 \times 3}$ and

$$\mathbf{J}_{21} = \left(\mathbf{T}_f \mathbf{J}_q \text{diag}(k_{g1}, \dots, k_{g6})^{-1} \mathbf{R}_a^T \begin{bmatrix} \mathbf{I}_3 \\ \mathbf{0}_{3 \times 3} \end{bmatrix} \right)^{-1} \\ \times \mathbf{T}_f \mathbf{J}_q \text{diag}(k_{g1}, \dots, k_{g6})^{-1} \mathbf{R}_a^T \begin{bmatrix} \mathbf{0}_{3 \times 3} \\ \mathbf{I}_3 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

Meanwhile, the fact that $\dot{\mathbf{q}}_{u2}=0$ leads to $\mathbf{J}_{12}=0$ and $\mathbf{J}_{22}=0$ in (9).

In step 7, the three-dimensional constraint force coordinates are specified as

$$\boldsymbol{\eta}_f = \begin{bmatrix} f_z \\ m_x \\ m_y \end{bmatrix} \in \mathbb{R}^3$$

where $f_z \in \mathbb{R}$ denotes the force along the z -axis of frame $\{\mathbf{O}\}$ and $m_x \in \mathbb{R}$ and $m_y \in \mathbb{R}$ denote the moments along the x - and y -axes of frame $\{\mathbf{O}\}$. Moreover, specify

$$\mathbf{T}_c = \mathbf{T}_f \mathbf{J}_q \text{diag}(k_{g1}, \dots, k_{g6})^{-1} \mathbf{R}_a^T \begin{bmatrix} \mathbf{I}_3 \\ \mathbf{0}_{3 \times 3} \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

leading to

$$\mathbf{J}_f = \mathbf{T}_f \mathbf{J}_q \text{diag}(k_{g1}, \dots, k_{g6})^{-1} \in \mathbb{R}^{3 \times 6}$$

from (13).

In step 8, $n=6$ is obtained from (14). Meanwhile, $\dot{\mathbf{q}}_A = \dot{\mathbf{q}}_a \in \mathbb{R}^6$ and $\boldsymbol{\tau}_A = [\tau_1, \dots, \tau_6]^T \in \mathbb{R}^6$ represent the velocities and control torques in space $\mathcal{A}$.

In step 9, the dynamic contact forces are the frictional forces and torque in the motion space. It yields

$$\mathbf{D}_m = \mathbf{T}_m \mathbf{J}_q \text{diag}(k_{g1}, \dots, k_{g6})^{-1} \in \mathbb{R}^{3 \times 6}$$

where $\mathbf{T}_m \in \mathbb{R}^{3 \times 6}$ is defined in (45) and $\mathbf{J}_q \in \mathbb{R}^{6 \times 6}$ is defined in (44).

In step 10, it follows that

$$\mathbf{v}_m = \begin{bmatrix} v_x \\ v_y \\ \omega_z \end{bmatrix} \in \mathbb{R}^3$$

where $v_x \in \mathbb{R}$ and $v_y \in \mathbb{R}$ denote the linear velocities of the end effector along the x - and y -axes of frame $\{\mathbf{O}\}$ and $\omega_z \in \mathbb{R}$ denotes the angular velocity of the end effector along the z -axis of frame $\{\mathbf{O}\}$. Accordingly, it yields

$$\mathbf{J}_m = \mathbf{T}_m \mathbf{J}_q \text{diag}(k_{g1}, \dots, k_{g6})^{-1} \in \mathbb{R}^{3 \times 6}$$

to form

$$\mathbf{J}_O = \begin{bmatrix} \mathbf{J}_m \\ \mathbf{J}_f \end{bmatrix} \in \mathbb{R}^{6 \times 6}$$

In step 11, the velocity mapping matrix $\mathbf{J}_S$ is designed as

$$\mathbf{J}_S = \begin{bmatrix} \mathbf{J}_L \text{diag}(k_{g1}, \dots, k_{g6})^{-1} \\ \mathbf{J}_A \end{bmatrix} \in \mathbb{R}^{78 \times 6}$$

with $\mathbf{J}_L$ being defined in (46) and

$$\mathbf{J}_A = \begin{bmatrix} \mathbf{z}_1 & & & & & \\ \mathbf{A}_1 \mathbf{U}_{A_2}^T \mathbf{z}_1 & \mathbf{z}_2 & & & & \\ \vdots & \ddots & \ddots & & & \\ \mathbf{A}_1 \mathbf{U}_{A_6}^T \mathbf{z}_1 & \mathbf{A}_2 \mathbf{U}_{A_6}^T \mathbf{z}_2 & \dots & \mathbf{z}_6 & & \end{bmatrix} \in \mathbb{R}^{36 \times 6} \quad (50)$$

In step 12, $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{G}$ in space $\mathcal{S}$ are obtained as

$$\mathcal{M} = \text{diag}(\mathbf{M}_{B_1}, \dots, \mathbf{M}_{B_6}, \mathbf{M}_O, \mathbf{M}_{A_1}, \dots, \mathbf{M}_{A_6}) \quad (51)$$

$$\mathcal{C} = \text{diag}(\mathbf{C}_{B_1}, \dots, \mathbf{C}_{B_6}, \mathbf{C}_O, \mathbf{C}_{A_1}, \dots, \mathbf{C}_{A_6}) \quad (52)$$

$$\mathcal{G} = [\mathbf{G}_{B_1}^T, \dots, \mathbf{G}_{B_6}^T, \mathbf{G}_O^T, \mathbf{G}_{A_1}^T, \dots, \mathbf{G}_{A_6}^T]^T \quad (53)$$

3.4 Single-Arm Constrained Manipulators With Flexible Joints. When the gear box associated with each actuator is replaced by a torsional spring, the constrained robot in Fig. 3 becomes a constrained flexible-joint robot [5]. It follows that $n_{1a}=6$, $n_{1u}=6$, and $n_3=0$ in step 1.

In step 2, because the relative motion between each motor rotor and its output shaft is permitted, it follows that $m=9$ (the linear motion along the two tangential directions of the contact plane, the rotational motion along the normal vector of the plane, and the six positions of the motor rotors). Thus, it follows $n_c=(n_1+3n_3)-m=6+6-9=3$ from (5).

In step 3, the three constraints at the end effector constrain the linear motion along the normal vector of the contact plane and the rotational motion along the two tangential directions of the contact plane. A frame $\{\mathbf{O}\}$ is fixed to the robot end effector in a way that its x - and y -axes lie on the contact plane and its z -axis coincides with the normal vector of the contact plane. At each actuator, three frames, denoted as $\{\mathbf{T}_j\}$, $\{\mathbf{A}_j\}$, and $\{\mathbf{B}_j\}$, $j \in \{1, 6\}$, are fixed to the motor base, to the motor rotor, and to the output shaft, respectively, with their z -axes coinciding with the corresponding joint axes, as illustrated in Fig. 4.

In step 4, let $\dot{q}_{aj} \in \mathbb{R}$ be the motor velocity measured by frame $\{\mathbf{A}_j\}$ with respect to frame $\{\mathbf{T}_j\}$. Let $\dot{q}_{ujj} \in \mathbb{R}$ be the output shaft velocity measured by frame $\{\mathbf{B}_j\}$ with respect to frame $\{\mathbf{T}_j\}$. Specify $\dot{\mathbf{q}}_a = [\dot{q}_{a1}, \dots, \dot{q}_{a6}]^T \in \mathbb{R}^6$ and $\dot{\mathbf{q}}_u = [\dot{q}_{u1}, \dots, \dot{q}_{u6}]^T \in \mathbb{R}^6$. Then, it follows that $\mathbf{J}_a=0$ and

$$\mathbf{J}_u = \mathbf{T}_f \mathbf{J}_q \in \mathbb{R}^{3 \times 6}$$

where $\mathbf{T}_f \in \mathbb{R}^{3 \times 6}$ and $\mathbf{J}_q \in \mathbb{R}^{6 \times 6}$ are defined in (44). Thus, it yields $n_p=3$, which is the rank of $\mathbf{J}_u$.

In step 5, Let $\mathbf{R}_u \in \mathbb{R}^{6 \times 6}$ be a reordering matrix such that

$$\begin{bmatrix} \dot{\mathbf{q}}_{u1} \in \mathbb{R}^3 \\ \dot{\mathbf{q}}_{u2} \in \mathbb{R}^3 \end{bmatrix} = \mathbf{R}_u \dot{\mathbf{q}}_u$$

$$[\mathbf{J}_{u1} \in \mathbb{R}^{3 \times 3}, \mathbf{J}_{u2} \in \mathbb{R}^{3 \times 3}] = \mathbf{J}_u \mathbf{R}_u^T$$

with $\mathbf{J}_{u1}$ being invertible.

In step 6, calculate $\dot{\mathbf{q}}_{u1} = -\mathbf{J}_{u1}^{-1} \mathbf{J}_{u2} \dot{\mathbf{q}}_{u2} = -\mathbf{J}_{12} \dot{\mathbf{q}}_{u2}$ with $\mathbf{J}_{11}=0$, $\mathbf{J}_{21}=0$, and $\mathbf{J}_{22}=0$ in terms of (9). In step 7, because $n_c-n_p=0$, the constraint force/moments at the end effector are not reflected in the dynamic equation with $\boldsymbol{\eta}_f=0$.

In step 8, $n=9$ is obtained from (14). Meanwhile, it follows that

$$\dot{\mathbf{q}}_A = \begin{bmatrix} \dot{\mathbf{q}}_a \\ \dot{\mathbf{q}}_{u2} \end{bmatrix} \in \mathbb{R}^9$$

$$\boldsymbol{\tau}_A = \begin{bmatrix} \boldsymbol{\tau}_a \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^9$$

In step 9, the dynamic contact forces include not only the frictional forces and torque at the contact plane, but also the six torsional torques at the six actuators. It follows that

$$\boldsymbol{\eta}_m = \begin{bmatrix} f_{fx} \\ f_{fy} \\ m_{fz} \\ k_{s1}(q_{a1} - q_{u1}) \\ \vdots \\ k_{s6}(q_{a6} - q_{u6}) \end{bmatrix} \in \mathbb{R}^9$$

and

$$\mathbf{D}_m = \begin{bmatrix} \mathbf{0}_{3 \times 6} & \mathbf{T}_m \mathbf{J}_q \mathbf{R}_u^T \begin{bmatrix} -\mathbf{J}_{12} \\ \mathbf{I}_3 \end{bmatrix} \\ \mathbf{I}_6 & -\mathbf{R}_u^T \begin{bmatrix} -\mathbf{J}_{12} \\ \mathbf{I}_3 \end{bmatrix} \end{bmatrix}$$

where f_{fx} , f_{fy} , and m_{fz} denote the frictional forces along the x - and y -axes and the frictional torque along the z -axis of the contact plane; $\mathbf{T}_m \in \mathbb{R}^{3 \times 6}$ is defined in (45) and $\mathbf{J}_q \in \mathbb{R}^{6 \times 6}$ is defined in (44).

In step 10, it follows that

$$\mathbf{v}_m = \begin{bmatrix} v_x \\ v_y \\ \omega_z \\ \dot{\mathbf{q}}_a \end{bmatrix} \in \mathbb{R}^9$$

where $v_x \in \mathbb{R}$ and $v_y \in \mathbb{R}$ denote the linear velocities of the end effector along the x - and y -axes of frame $\{\mathbf{O}\}$ and $\omega_z \in \mathbb{R}$ denotes the angular velocity of the end effector along the z axis of frame $\{\mathbf{O}\}$. Accordingly, it yields

$$\mathbf{J}_m = \begin{bmatrix} \mathbf{0}_{3 \times 6} & \mathbf{T}_m \mathbf{J}_q \mathbf{R}_u^T \begin{bmatrix} -\mathbf{J}_{12} \\ \mathbf{I}_3 \end{bmatrix} \\ \mathbf{I}_6 & \mathbf{0}_{6 \times 3} \end{bmatrix}$$

and furthermore,

$$\mathbf{J}_O = \mathbf{J}_m \in \mathbb{R}^{9 \times 9}$$

In step 11, the velocity mapping matrix $\mathbf{J}_S$ is designed as

$$\mathbf{J}_S = \begin{bmatrix} \mathbf{0}_{42 \times 6} & \mathbf{J}_L \mathbf{R}_u^T \begin{bmatrix} -\mathbf{J}_{12} \\ \mathbf{I}_3 \end{bmatrix} \\ \mathbf{J}_A & \mathbf{0}_{36 \times 3} \end{bmatrix} \in \mathbb{R}^{78 \times 9}$$

with $\mathbf{J}_L \in \mathbb{R}^{42 \times 6}$ and $\mathbf{J}_A \in \mathbb{R}^{36 \times 6}$ being defined in (46) and (50), respectively. In step 12, $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{G}$ in space $\mathcal{S}$ are defined in Eqs. (51)–(53).

3.5 Coordinated Multiple Manipulators. Consider a system comprised of h direct-drive manipulators grasping a rigid object moving in free space without kinematic singularity. Each manipulator has six actuated single-DOF joints. It follows that $n_1 = n_{1a} = 6h$, $n_{1u} = 0$, and $n_3 = 0$ in step 1. Meanwhile, it yields $m = 6$ and $n_c = 6(h-1)$ in step 2.

In step 3, the six-dimensional motion belongs to the held object and the $6(h-1)$ -dimensional constraints are imposed on the end effectors of the h manipulators. Frame $\{\mathbf{O}\}$ is fixed to the held object. Frame $\{\mathbf{B}_{ij}\}$ is assigned to the j th link of the i th manipulator, $i = 1, 2, \dots, h$, $j = 1, 2, \dots, 6$.

In step 4, $\dot{\mathbf{q}}_a = [\dot{\mathbf{q}}_1^T, \dot{\mathbf{q}}_2^T, \dots, \dot{\mathbf{q}}_h^T]^T \in \mathbb{R}^{6h}$ with $\dot{\mathbf{q}}_i = [\dot{q}_{i1}, \dot{q}_{i2}, \dots, \dot{q}_{i6}]^T \in \mathbb{R}^6$ and $\dot{\mathbf{q}}_u = 0$ are specified. Furthermore, $\mathbf{J}_a$ is formed as

$$\mathbf{J}_a = \mathbf{T}_{hf} \mathbf{J}_{hq} \in \mathbb{R}^{6(h-1) \times 6h}$$

with

$$\mathbf{T}_{hf} = \begin{bmatrix} \mathbf{I}_6 & -\mathbf{I}_6 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_6 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\mathbf{I}_6 & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I}_6 & -\mathbf{I}_6 \end{bmatrix} \in \mathbb{R}^{6(h-1) \times 6h}$$

$$\mathbf{J}_{hq} = \text{diag}(\mathbf{J}_1, \mathbf{J}_2, \dots, \mathbf{J}_h) \in \mathbb{R}^{6h \times 6h}$$

where $\mathbf{J}_i$, $i = 1, 2, \dots, h$, is exactly the same as $\mathbf{J}_q$ defined in Secs. 3.1–3.4. Meanwhile, it follows that $\mathbf{J}_u = 0$ and $n_p = 0$.

In step 5, design

$$\mathbf{R}_a = \begin{bmatrix} \mathbf{0}_{6(h-1) \times 6} & \mathbf{I}_{6(h-1)} \\ \mathbf{I}_6 & \mathbf{0}_{6 \times 6(h-1)} \end{bmatrix} \in \mathbb{R}^{6h \times 6h}$$

to form

$$\mathbf{J}_a \mathbf{R}_a^T = [\mathbf{J}_{a1} \in \mathbb{R}^{6(h-1) \times 6(h-1)}, \mathbf{J}_{a2} \in \mathbb{R}^{6(h-1) \times 6}] \in \mathbb{R}^{6(h-1) \times 6h}$$

such that $\mathbf{J}_{a1}$ is of full rank.

In step 6, compute

$$\mathbf{J}_{21} = \mathbf{J}_{a1}^{-1} \mathbf{J}_{a2} \in \mathbb{R}^{6(h-1) \times 6}$$

with $\mathbf{J}_{11} = 0$, $\mathbf{J}_{12} = 0$, and $\mathbf{J}_{22} = 0$ in (9).

In step 7, the constraint forces are located among the end effectors of the h manipulators. Therefore, it follows that

$$\boldsymbol{\eta}_f = \begin{bmatrix} \boldsymbol{\eta}_{12} \\ \boldsymbol{\eta}_{23} \\ \vdots \\ \boldsymbol{\eta}_{(h-1)h} \end{bmatrix} \in \mathbb{R}^{6(h-1)}$$

where $\boldsymbol{\eta}_{(i-1)i} \in \mathbb{R}^6$, $i = 2, \dots, h$, denotes the internal force between the $i-1$ manipulator and the i th manipulator, expressed in frame $\{\mathbf{O}\}$. Consequently, specify $\mathbf{T}_c = \mathbf{J}_{a1} \in \mathbb{R}^{6(h-1) \times 6(h-1)}$ leading to

$$\mathbf{J}_f = \mathbf{J}_a \in \mathbb{R}^{6(h-1) \times 6h}$$

from (13).

In step 8, $n = 6h$ is obtained from (14). Furthermore, it yields

$$\dot{\mathbf{q}}_A = \dot{\mathbf{q}}_a \in \mathbb{R}^{6h}$$

and

$$\boldsymbol{\tau}_A = [\boldsymbol{\tau}_1^T, \boldsymbol{\tau}_2^T, \dots, \boldsymbol{\tau}_h^T]^T \in \mathbb{R}^{6h}$$

with $\boldsymbol{\tau}_i = [\tau_{i1}, \tau_{i2}, \dots, \tau_{i6}] \in \mathbb{R}^6$.

Because all the end effectors are rigidly holding a rigid object, there is no dynamic contact force. It gives $\boldsymbol{\eta}_m = 0$ in step 9.

In step 10, the independent velocity coordinates $\mathbf{v}_m$ in space $\mathcal{O}$ are specified as $\mathbf{v}_m = \mathbf{0}_V \in \mathbb{R}^6$, where $\mathbf{0}_V$ denotes the linear/angular velocities of and expressed in frame $\{\mathbf{O}\}$ [7]. Accordingly, $\mathbf{J}_m$ is obtained as

$$\mathbf{J}_m = \mathcal{H} \mathbf{J}_{hq} \in \mathbb{R}^{6 \times 6h}$$

where

$$\mathcal{H} = [\gamma_1 \mathbf{I}_6, \gamma_2 \mathbf{I}_6, \dots, \gamma_h \mathbf{I}_6] \in \mathbb{R}^{6 \times 6h}$$

with $\gamma_i \geq 0$ subject to $\sum_{i=1}^h \gamma_i = 1$. Finally, it forms

$$\mathbf{J}_O = \begin{bmatrix} \mathbf{J}_m \\ \mathbf{J}_f \end{bmatrix} \in \mathbb{R}^{6h \times 6h}$$

In step 11, the velocity mapping matrix $\mathbf{J}_S$ is obtained as

$$\mathbf{J}_S = \begin{bmatrix} \text{diag}(\mathbf{J}_{S1}, \mathbf{J}_{S2}, \dots, \mathbf{J}_{Sh}) \\ \mathcal{H} \mathbf{J}_{hq} \end{bmatrix} \in \mathbb{R}^{(36h+6) \times 6h}$$

with

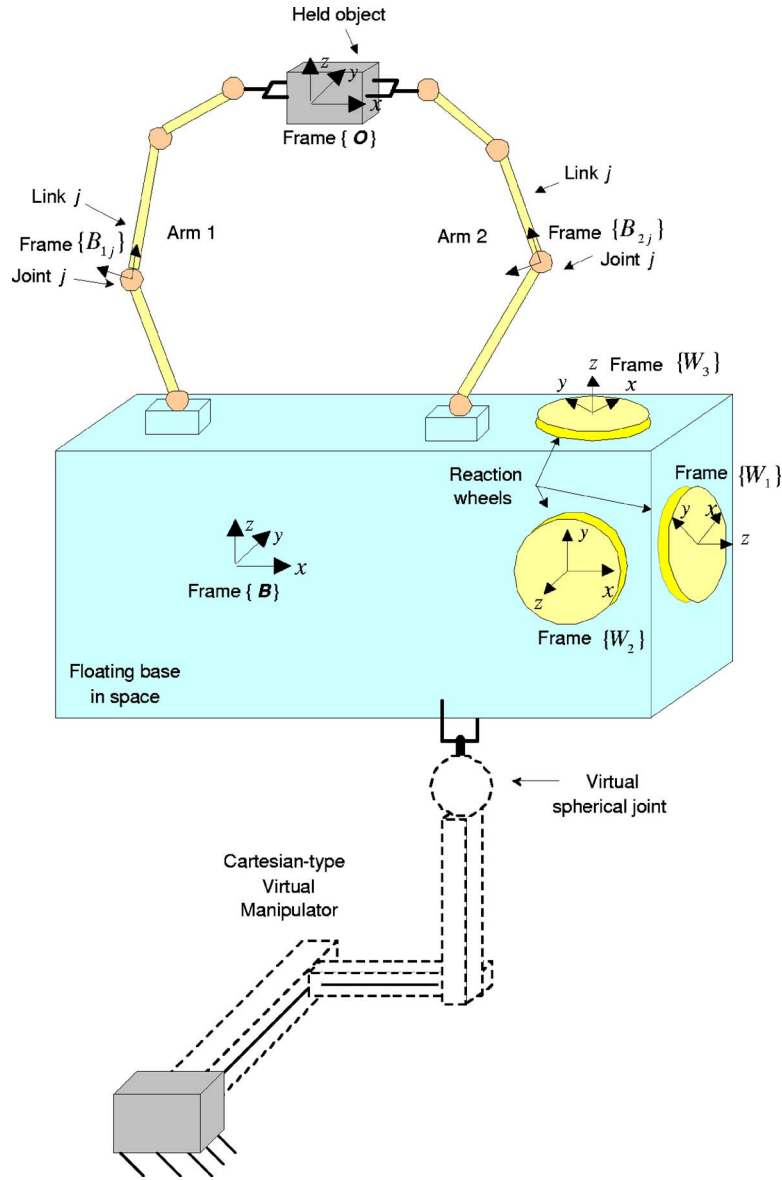


Fig. 5 A space robot with two arms holding a rigid object

$$\mathbf{J}_{Si} = \begin{bmatrix} \mathbf{z}_{i1} & & & & & \\ \mathbf{B}_{i1}\mathbf{U}_{\mathbf{B}_{i2}}^T\mathbf{z}_{i1} & \mathbf{z}_{i2} & & & & \\ \vdots & \ddots & \ddots & & & \\ \mathbf{B}_{i1}\mathbf{U}_{\mathbf{B}_{i6}}^T\mathbf{z}_{i1} & \mathbf{B}_{i2}\mathbf{U}_{\mathbf{B}_{i6}}^T\mathbf{z}_{i2} & \cdots & \mathbf{z}_{i6} & & \end{bmatrix} \in \mathbb{R}^{36 \times 6} \quad (54)$$

for $i=1,2,\dots,h$, where $\mathbf{z}_{ij}=[0\ 0\ 1\ 0\ 0\ 0]^T \in \mathbb{R}^6$ for a prismatic joint and $\mathbf{z}_{ij}=[0\ 0\ 0\ 0\ 1]^T$ for a revolute joint, $j=1,2,\dots,6$.

Finally, in step 12, $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{G}$ in space $\mathcal{S}$ are obtained as

$$\mathcal{M} = \text{diag}$$

$$(\mathbf{M}_{\mathbf{B}_{11}}, \dots, \mathbf{M}_{\mathbf{B}_{16}}, \mathbf{M}_{\mathbf{B}_{21}}, \dots, \mathbf{M}_{\mathbf{B}_{26}}, \dots, \mathbf{M}_{\mathbf{B}_{h1}}, \dots, \mathbf{M}_{\mathbf{B}_{h6}}, \mathbf{M}_{\mathbf{O}})$$

$$\mathcal{C} = \text{diag}(\mathbf{C}_{\mathbf{B}_{11}}, \dots, \mathbf{C}_{\mathbf{B}_{16}}, \mathbf{C}_{\mathbf{B}_{21}}, \dots, \mathbf{C}_{\mathbf{B}_{26}}, \dots, \mathbf{C}_{\mathbf{B}_{h1}}, \dots, \mathbf{C}_{\mathbf{B}_{h6}}, \mathbf{C}_{\mathbf{O}})$$

$$\mathcal{G} = [\mathbf{G}_{\mathbf{B}_{11}}^T, \dots, \mathbf{C}_{\mathbf{B}_{16}}^T, \mathbf{G}_{\mathbf{B}_{21}}^T, \dots, \mathbf{G}_{\mathbf{B}_{26}}^T, \dots, \mathbf{G}_{\mathbf{B}_{h1}}^T, \dots, \mathbf{G}_{\mathbf{B}_{h6}}^T, \mathbf{G}_{\mathbf{O}}^T]^T$$

3.6 Space Robots. A space robot equipped with three reaction wheels and two arms holding a common rigid object, as illustrated in Fig. 5, is taken as an example to detail the design procedure. It

is assumed that each arm has six single-DOF direct-drive joints without subject to kinematic singularity. In order to make the modeling approach applicable, a Cartesian-type virtual manipulator with six degrees of freedom is added between the ground and the space robot base.

It follows that $n_{1a}=2 \times 6 + 3 \times 1 = 15$, $n_{1u}=3 \times 1 = 3$, $n_{3a}=0$, and $n_{3u}=1$, because there are 15 single-DOF actuated joints associated with the two arms and with the three reaction wheels, and there are 3 single-DOF unactuated joints and 1 three-DOF unactuated spherical joint associated with the virtual manipulator. It turns out to be $n_1=n_{1a}+n_{1u}=15+3=18$ and $n_3=n_{3u}=1$ in step 1.

In step 2, it yields $m=15$ and $n_c=n_1+3n_3-m=6$. The 15 motion degrees of freedom include 6 for the base, 6 for the held object, and 3 for the three reaction wheels.

In step 3, the $n_c=6$ overall constraints apply to the two end effectors through the commonly held object. Frame $\{\mathbf{B}\}$ is attached to the base, frame $\{\mathbf{O}\}$ is attached to the held object, frame $\{\mathbf{B}_{ij}\}$, $i \in \{1,2\}$, $j \in \{1,6\}$, is attached to the j th link of the i th arm with its z -axis aligning with the j th joint of the i th arm, and frame $\{\mathbf{W}_k\}$, $k \in \{1,3\}$, is attached to the k th reaction wheel with its

z-axis aligning with the joint axis.

In step 4, specify

$$\dot{\mathbf{q}}_a = [\dot{\mathbf{q}}_{m1}^T, \dot{\mathbf{q}}_{m2}^T, \dot{\mathbf{q}}_w^T]^T \in \mathbb{R}^{15}$$

with $\dot{\mathbf{q}}_{mi} = [\dot{q}_{i1}, \dot{q}_{i2}, \dots, \dot{q}_{i6}]^T \in \mathbb{R}^6, i=1,2$, denoting the joint velocity coordinates of the i th arm and $\dot{\mathbf{q}}_w = [\dot{q}_{w1}, \dot{q}_{w2}, \dot{q}_{w3}]^T \in \mathbb{R}^3$ denoting the joint velocity coordinates of the reaction wheels, and then specify

$$\dot{\mathbf{q}}_u = {}^B\mathbf{V} \in \mathbb{R}^6$$

Because the joint velocities of the virtual manipulator are not measurable, the linear/angular velocities of the base expressed in frame $\{\mathbf{B}\}$ are used instead. The six overall constraints at the held object can be expressed as

$$\mathbf{J}_a = [\mathbf{J}_{m1} \quad -\mathbf{J}_{m2} \quad \mathbf{0}_{6 \times 3}] \in \mathbb{R}^{6 \times 15}$$

$$\mathbf{J}_u = \mathbf{0}_{6 \times 6}$$

in terms of (6), where

$$\mathbf{J}_{mi} = [{}^B\mathbf{U}_{Oz_{i1}}^T, {}^B\mathbf{U}_{Oz_{i2}}^T, \dots, {}^B\mathbf{U}_{Oz_{i6}}^T] \in \mathbb{R}^{6 \times 6}$$

for $i=1,2$. Because $\mathbf{J}_u = \mathbf{0}_{6 \times 6}$, it yields $n_p=0$.

The fact of $n_p=0$ leads to $\mathbf{J}_{a1} = \mathbf{J}_{m1} \in \mathbb{R}^{6 \times 6}$ and $\mathbf{J}_{a2} = [-\mathbf{J}_{m2} \quad \mathbf{0}_{6 \times 3}] \in \mathbb{R}^{6 \times 9}$ in step 5, and

$$\mathbf{J}_{21} = [-\mathbf{J}_{m1}^T \mathbf{J}_{m2} \quad \mathbf{0}_{6 \times 3}] \in \mathbb{R}^{6 \times 9}$$

$$\mathbf{J}_{22} = \mathbf{0}_{6 \times 6}$$

with $\mathbf{J}_{11} = \mathbf{0}$ and $\mathbf{J}_{12} = \mathbf{0}$ in step 6.

In step 7, specify the constraint force coordinates $\boldsymbol{\eta}_f \in \mathbb{R}^6$ as the internal force/moment of the held object. It then determines the matrix

$$\mathbf{T}_c = \mathbf{J}_{m1} \in \mathbb{R}^{6 \times 6}$$

$$\mathbf{J}_f = [\mathbf{J}_a \quad \mathbf{J}_u] \in \mathbb{R}^{6 \times 21}$$

in terms of (13).

In step 8, $n=21$ is obtained from (14). Furthermore, it yields

$$\dot{\mathbf{q}}_A = [\dot{\mathbf{q}}_a^T, \dot{\mathbf{q}}_u^T]^T \in \mathbb{R}^{21}$$

$$\boldsymbol{\tau}_A = [\boldsymbol{\tau}_a^T, \mathbf{0}^T]^T \in \mathbb{R}^{21}$$

where $\dot{\mathbf{q}}_a \in \mathbb{R}^{15}$ and $\dot{\mathbf{q}}_u \in \mathbb{R}^6$ have been defined in step 4, and

$$\boldsymbol{\tau}_a = [\boldsymbol{\tau}_{m1}^T, \boldsymbol{\tau}_{m2}^T, \boldsymbol{\tau}_w^T]^T \in \mathbb{R}^{15}$$

with $\boldsymbol{\tau}_{mi} = [\tau_{i1}, \tau_{i2}, \dots, \tau_{i6}] \in \mathbb{R}^6$ for $i=1,2$ and $\boldsymbol{\tau}_w = [\tau_{w1}, \tau_{w2}, \tau_{w3}]^T \in \mathbb{R}^3$.

Because the two end effectors are rigidly holding an object, there is no dynamic contact force. It gives $\boldsymbol{\eta}_m = \mathbf{0}$ in step 9.

In step 10, the 15 independent velocity coordinates $\mathbf{v}_m$ in space $\mathcal{O}$ are specified as

$$\mathbf{v}_m = [{}^O\mathbf{V}^T, \dot{\mathbf{q}}_w^T, {}^B\mathbf{V}^T]^T \in \mathbb{R}^{15}$$

where ${}^O\mathbf{V} \in \mathbb{R}^6$ denotes the linear/angular velocities of the held object and expressed in frames $\{\mathbf{O}\}$ [7]. Accordingly, $\mathbf{J}_m$ is obtained as

$$\mathbf{J}_m = \begin{bmatrix} \mathbf{J}_{m1} & \mathbf{0}_{6 \times 6} & \mathbf{0}_{6 \times 3} & {}^B\mathbf{U}_O^T \\ \mathbf{0}_{3 \times 6} & \mathbf{0}_{3 \times 6} & \mathbf{I}_3 & \mathbf{0}_{3 \times 6} \\ \mathbf{0}_{6 \times 6} & \mathbf{0}_{6 \times 6} & \mathbf{0}_{6 \times 3} & \mathbf{I}_6 \end{bmatrix} \in \mathbb{R}^{15 \times 21}$$

Finally, it forms

$$\mathbf{J}_O = \begin{bmatrix} \mathbf{J}_m \\ \mathbf{J}_f \end{bmatrix} \in \mathbb{R}^{21 \times 21}$$

In step 11, the velocity mapping matrix $\mathbf{J}_S$ defined by (30) is obtained as

$$\mathbf{J}_S = \begin{bmatrix} \mathbf{J}_{m1} & \mathbf{0}_{6 \times 6} & \mathbf{0}_{6 \times 3} & {}^B\mathbf{U}_O^T \\ \mathbf{J}_{S1} & \mathbf{0}_{36 \times 6} & \mathbf{0}_{36 \times 3} & \mathbf{J}_{B1} \\ \mathbf{0}_{36 \times 6} & \mathbf{J}_{S2} & \mathbf{0}_{36 \times 3} & \mathbf{J}_{B2} \\ \mathbf{0}_{18 \times 6} & \mathbf{0}_{18 \times 6} & \mathbf{J}_W & \mathbf{J}_{BW} \\ \mathbf{0}_{6 \times 6} & \mathbf{0}_{6 \times 6} & \mathbf{0}_{6 \times 3} & \mathbf{I}_6 \end{bmatrix} \in \mathbb{R}^{102 \times 21}$$

with $\mathbf{J}_{Si} \in \mathbb{R}^{36 \times 6}, i=1,2$, being defined in (54) and

$$\mathbf{J}_{Bi} = [{}^B\mathbf{U}_{B_{i1}}, {}^B\mathbf{U}_{B_{i2}}, \dots, {}^B\mathbf{U}_{B_{i6}}]^T \in \mathbb{R}^{36 \times 6}, i=1,2$$

$$\mathbf{J}_W = \text{diag}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) \in \mathbb{R}^{18 \times 3}$$

$$\mathbf{J}_{BW} = [{}^B\mathbf{U}_{W_1}, {}^B\mathbf{U}_{W_2}, {}^B\mathbf{U}_{W_3}]^T \in \mathbb{R}^{18 \times 6}$$

where $\mathbf{z}_j = [0, 0, 0, 0, 0, 1]^T \in \mathbb{R}^6$ for all $j \in \{1, 3\}$.

Finally, in step 12, $\mathcal{M}, \mathcal{C}$, and $\mathcal{G}$ in space $\mathcal{S}$ are obtained as

$$\mathcal{M} = \text{diag}$$

$$(\mathbf{M}_O, \mathbf{M}_{B_{11}}, \dots, \mathbf{M}_{B_{16}}, \mathbf{M}_{B_{21}}, \dots, \mathbf{M}_{B_{26}}, \mathbf{M}_{W_1}, \dots, \mathbf{M}_{W_3}, \mathbf{M}_B)$$

$$\mathcal{C} = \text{diag}(\mathbf{C}_O, \mathbf{C}_{B_{11}}, \dots, \mathbf{C}_{B_{16}}, \mathbf{C}_{B_{21}}, \dots, \mathbf{C}_{B_{26}}, \mathbf{C}_{W_1}, \dots, \mathbf{C}_{W_3}, \mathbf{C}_B)$$

$$\mathcal{G} = [\mathbf{G}_O^T, \mathbf{G}_{B_{11}}^T, \dots, \mathbf{G}_{B_{16}}^T, \mathbf{G}_{B_{21}}^T, \dots, \mathbf{G}_{B_{26}}^T, \mathbf{G}_{W_1}^T, \dots, \mathbf{G}_{W_3}^T, \mathbf{G}_B^T]^T$$

4 Conclusion

In this paper, a novel approach for deriving the dynamic models of a class of general constrained robots has been developed. With this approach, only separate dynamics of individual rigid bodies and two kinematics-based mapping matrices are needed to construct the dynamic equation of a robot. This feature makes the developed approach to be one of the most rigorous modeling approaches by far, because no approximation beyond rigid-body dynamics is made. Meanwhile, the derived dynamic equation possesses a closed form resembling to the well-known dynamics of the single-arm constrained robots.

Because the dynamics of rigid bodies are simple and standard, the unique information required for a particular application includes only the two kinematics-based mapping matrices. Thus, the dynamics issue of a complex robotic system is virtually converted into the kinematics issue plus the use of standard rigid-body dynamics. Five examples have been presented.

Nomenclature

- $\mathbf{q}$ = joint position coordinates of a robot
- $\dot{\mathbf{q}}$ = joint velocity coordinates of a robot
- $\ddot{\mathbf{q}}$ = joint acceleration coordinates of a robot
- $\mathbf{M}(\mathbf{q})$ = mass matrix in joint space
- $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ = Coriolis and centripetal matrix in joint space
- $\mathbf{G}(\mathbf{q})$ = gravitational vector of a robot
- $\boldsymbol{\tau}$ = joint control torques
- $\mathbf{v}$ = velocity coordinates in Cartesian space
- $\mathbf{F}$ = constraint force in Cartesian space
- $\mathbf{J}$ = Jacobian matrix from $\dot{\mathbf{q}}$ to $\mathbf{v}$
- n_1 = number of single-DOF joints
- n_{1a} = number of actuated single-DOF joints
- n_{1u} = number of unactuated single-DOF joints
- n_3 = number of three-DOF spherical joints
- n_{3a} = number of actuated three-DOF spherical joints
- n_{3u} = number of unactuated three-DOF spherical joints
- m = motion degrees of freedom
- n_c = number of overall constraints
- n = number of dimensions of a general constrained robot
- $\dot{\mathbf{q}}_a$ = velocity coordinates of all actuated joints

$\dot{\mathbf{q}}_{a1}$ = dependent velocity coordinates of actuated joints
 $\dot{\mathbf{q}}_{a2}$ = independent velocity coordinates of actuated joints
 $\dot{\mathbf{q}}_u$ = velocity coordinates of all unactuated joints
 $\dot{\mathbf{q}}_{u1}$ = dependent velocity coordinates of unactuated joints
 $\dot{\mathbf{q}}_{u2}$ = independent velocity coordinates of unactuated joints
 ξ_{a1} = joint frictional forces, torques, or moments corresponding to $\dot{\mathbf{q}}_{a1}$
 ξ_{a2} = joint frictional forces, torques, or moments corresponding to $\dot{\mathbf{q}}_{a2}$
 ξ_{u1} = joint frictional forces, torques, or moments corresponding to $\dot{\mathbf{q}}_{u1}$
 ξ_{u2} = joint frictional forces, torques, or moments corresponding to $\dot{\mathbf{q}}_{u2}$
 $\mathbf{R}_a$ = reordering matrix for $\dot{\mathbf{q}}_a$
 $\mathbf{R}_u$ = reordering matrix for $\dot{\mathbf{q}}_u$
 $\mathbf{J}_a$ = mapping matrix from $\dot{\mathbf{q}}_a$ to the n_c constraints
 $\mathbf{J}_{a1}$ = mapping matrix from $\dot{\mathbf{q}}_{a1}$ to the n_c constraints
 $\mathbf{J}_{a2}$ = mapping matrix from $\dot{\mathbf{q}}_{a2}$ to the n_c constraints
 $\mathbf{J}_u$ = mapping matrix from $\dot{\mathbf{q}}_u$ to the n_c constraints
 $\mathbf{J}_{u1}$ = mapping matrix from $\dot{\mathbf{q}}_{u1}$ to the n_c constraints
 $\mathbf{J}_{u2}$ = mapping matrix from $\dot{\mathbf{q}}_{u2}$ to the n_c constraints
 $\mathbf{J}_{11}$ = mapping matrix from $\dot{\mathbf{q}}_{a2}$ to $-\dot{\mathbf{q}}_{u1}$
 $\mathbf{J}_{12}$ = mapping matrix from $\dot{\mathbf{q}}_{u2}$ to $-\dot{\mathbf{q}}_{u1}$
 $\mathbf{J}_{21}$ = mapping matrix from $\dot{\mathbf{q}}_{a2}$ to $-\dot{\mathbf{q}}_{a1}$
 $\mathbf{J}_{22}$ = mapping matrix from $\dot{\mathbf{q}}_{u2}$ to $-\dot{\mathbf{q}}_{a1}$
 n_p = rank of $\mathbf{J}_u$ or $\mathbf{J}_{u1}$
 $\mathcal{A}$ = general joint space
 $\mathcal{O}$ = general task space
 $\mathcal{S}$ = extended subsystems space
 $\dot{\mathbf{q}}_A$ = joint velocity coordinates in space $\mathcal{A}$
 $\boldsymbol{\tau}_A$ = joint control torques in space $\mathcal{A}$
 $\boldsymbol{\tau}_a$ = control torque coordinates for actuated joints
 $\mathbf{v}_m$ = independent velocity coordinates in space $\mathcal{O}$
 $\boldsymbol{\eta}_f$ = constraint force coordinates in space $\mathcal{O}$
 $\boldsymbol{\eta}_m$ = dynamic force coordinates in space $\mathcal{O}$
 $\mathbf{T}_c$ = mapping matrix from $\dot{\mathbf{q}}_{a1}$ to where $\boldsymbol{\eta}_f$ is defined
 $\mathbf{J}_f$ = mapping matrix from $\dot{\mathbf{q}}_A$ to where $\boldsymbol{\eta}_f$ is defined
 $\mathbf{J}_m$ = mapping matrix from $\dot{\mathbf{q}}_A$ to $\mathbf{v}_m$
 $\mathbf{D}_m$ = dynamic force mapping matrix
 ${}^B\mathbf{V}$ = body frame $\{\mathbf{B}\}$ referenced and expressed linear and angular velocities
 $\mathbf{M}_B$ = mass matrix of a rigid body associated with frame $\{\mathbf{B}\}$
 $\mathbf{C}_B$ = skew-symmetric matrix associated with frame $\{\mathbf{B}\}$

$\mathbf{G}_B$ = gravitational force vector associated with frame $\{\mathbf{B}\}$
 ${}^B\mathbf{F}^*$ = net force and moment of a rigid body associated with frame $\{\mathbf{B}\}$
 Φ = set containing all frames each is attached to a corresponding rigid body
 Θ_1 = set containing sequential numbers of all single-DOF joints
 Θ_3 = set containing sequential numbers of all three-DOF spherical joints
 $\mathcal{V}$ = extended velocity coordinates of all rigid bodies in space $\mathcal{S}$
 $\mathcal{F}^*$ = extended net forces/moments of all rigid bodies in space $\mathcal{S}$
 $\mathcal{M}$ = block diagonal matrix with $\mathbf{M}_B$ as an element
 $\mathcal{C}$ = block diagonal matrix with $\mathbf{C}_B$ as an element
 $\mathcal{G}$ = extended vector with $\mathbf{G}_B$ as an element
 $\mathcal{M}_G$ = mass matrix of a general constrained robot
 $\mathcal{C}_G$ = skew-symmetric matrix of a general constrained robot
 $\mathcal{G}_G$ = vector of a general constrained robot
 $\mathbf{J}_O$ = mapping matrix connecting space $\mathcal{A}$ with space $\mathcal{O}$
 $\mathbf{J}_S$ = mapping matrix connecting space $\mathcal{A}$ with space $\mathcal{S}$

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Evaluation of the Interfacial Strength of Layered Structures by Indentation Method

The delamination of thin coating films from substrates is a critical issue for the reliability of micro- and nanoelectronic devices. Indentation methods have the potential to measure interfacial strength in micro- and nanofilm thickness coating films. In this paper, indentation tests of layered structures are simulated using the damage-based cohesive zone model. When the delamination initiates, the indentation load and depth curve tend to deviate from the indentation load and depth curve for the perfectly bonded case. When the interface is stiffer than the coating film, a brittlelike delamination occurs on the interface; when the stiffness of the interface is smaller than that of the coating layer, a ductilelike delamination occurs on the interface. The ratio of shear moduli, $\mu_{\text{int}}/\mu_{\text{pf}}$, characterizes the delamination behavior on the interface during indentation tests. Focusing on the discontinuous point during the indentation tests and introducing the balance of energy before and after the onset of delamination, the evaluation method of the interfacial strength is proposed. The proposed method can be used to estimate the interfacial strength when the ratio of hardness and the yield stress of the coating film is $3.5 < H_A/\sigma_y < 4.5$. [DOI: 10.1115/1.2839890]

Keywords: interface, thin film, indentation, finite element method, cohesive zone model, delamination, interfacial strength, plastic deformation, indentation hardness

1 Introduction

The delamination of thin coating films from substrates is a critical issue for the reliability of micro- and nanoelectronic devices, since interface failures may lead to failure of the total system. To measure the interfacial strength between a thin film and its substrate, several testing methods have been proposed, including double cantilever beam tests, four-point flexure beam tests, scratch tests, peel tests, and indentation tests [1–6]. More recently, several novel techniques have been proposed to measure the interfacial strength of thin films approaching nanoscale thickness [7–15]. However, these quantitative testing methods need a especially designed specimen and it is difficult to carry out in situ measurement of the interfacial strength for products themselves.

Micro- and nanoindentation techniques have been widely applied to evaluate the mechanical properties of thin films such as hardness, elastic modulus, and yield strength. A comprehensive survey of indentation methods can be found in Fischer-Cripps [16]. Indentation methods have also been applied to measure the adhesion strength between films and their substrates. One method for quantitative evaluation of the thin film adhesion strength can be found in Marshall and Evans [17]. The latter applied a linear fracture mechanics approach and proposed a method for the determination of the interfacial toughness. When the coating film is ductile, plastic deformation occurs largely under and near the indenter. In those cases, the linear fracture mechanics approach cannot be applied, since the plastic dissipation near the indenter is too large and thin film delamination would not occur. To solve these problems, so-called superlayer methods have been proposed by Gerberich et al. [18]. In the latter work, the authors used a hard

thin layer on a ductile coating layer and constrained the plastic deformation in the ductile layer in order to induce the delamination of the coating film.

During indentation tests, especially nanoindentation tests, the only information obtained is the indentation load (P) and indentation depth (h) relations (i.e., P - h curve) along with the indentation impression. Therefore, the evaluation of interfacial strength from those limited outputs is a challenging subject. Recently, cohesive zone models have been applied to indentation problems. Zhang et al. [19] have investigated the onset and propagation of delamination of interfaces in thin film systems during indentation by using a microwedge indenter. Li and Siegmund [20,21] have investigated the growth of delamination under indentation loading for strongly and weakly bonded ductile films. These evaluation methods for interfacial strength rely on the buckling-induced delamination of the thin film, and focus on the forces and moments acting on the delamination tip.

In this paper, we consider the energy dissipation during indentation tests for a ductile coating layer on an elastic substrate and propose an evaluation method for the interfacial strength. Indentation tests of a ductile coating layer on an elastic substrate are simulated by using the damage-based cohesive zone model [25,26] based on Ma and Kishimoto [27]. The effects of interfacial strength and film hardness on the evaluation results are also examined.

2 Damage-Based Cohesive Zone Model

Several cohesive laws have been proposed and successfully applied to bulk fracture problems [22–24]. To describe the damage and failure phenomena on the interface, damage-based cohesive zone models have been proposed [25–27]. An interfacial constitutive model has been developed, which takes into consideration thermodynamic restrictions. The details of the derivation of the cohesive zone model [26] are omitted here, and the interface constitutive relations are summarized.

When an interface deforms, as shown in Fig. 1, the opposite points across the interfacial zone, i.e., AB , will produce a relative

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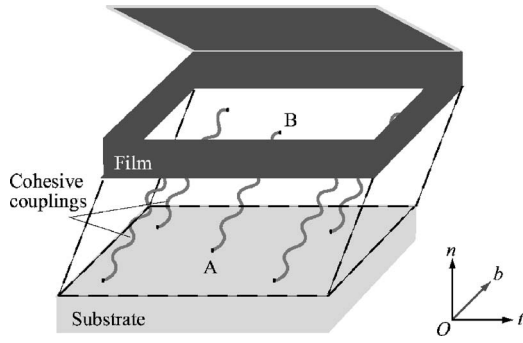


Fig. 1 Concept of interfacial cohesive coupling models

displacement, which defines an interfacial separation. The displacement can be scaled by displacement components, U_t , U_n , and U_b according to the right hand Cartesian coordinate system shown in Fig. 1. An equivalent interfacial separation is defined as

$$\lambda = \sqrt{\left(\frac{U_t}{\delta_t}\right)^2 + \left(\frac{\langle U_n \rangle_+}{\delta_n}\right)^2 + \left(\frac{U_b}{\delta_b}\right)^2} \quad (1)$$

where δ_t , δ_n , and δ_b are the critical interfacial separations of each direction. The symbol of $\langle \cdot \rangle_+$ means

$$\langle x \rangle_+ = \begin{cases} x & \text{in case } x \geq 0 \\ 0 & \text{in case } x < 0 \end{cases} \quad (2)$$

The equivalent interfacial separation indicates the interfacial bonding condition, that is,

$$\begin{aligned} 0 \leq \lambda \leq 1 &\Rightarrow \text{bonding} \\ 1 \leq \lambda &\Rightarrow \text{debonding} \end{aligned} \quad (3)$$

On the basis of damage mechanics, we introduce a damage parameter, $D(\lambda)$, to represent the damage process due to an interfacial deformation.

$$0 \leq D(\lambda) \leq 1 \quad (4)$$

where $D(\lambda)=0$ corresponds to a virgin element and $D(\lambda)=1$ to a fully damaged element, in other words, to the failure of the interface. The damage parameters are related to the equivalent interfacial separation, λ , through a damage dissipation potential [26].

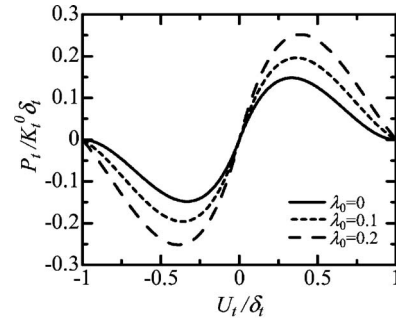
Apart from the Ma–Kishimoto model [27], in the derivation of the interfacial constitutive model, the Helmholtz free energy is taken as the total energy of nonlinear springs for each direction:

$$\begin{aligned} \Phi(U) = & K_t^0 U_t^2 \left(\frac{a}{2} - \frac{b}{3\delta_t} |U_t| + \frac{1}{4\delta_t^2} U_t^2 \right) \\ & + K_n^0 U_n^2 \left(\frac{a}{2} - \frac{b}{3\delta_n} U_n + \frac{1}{4\delta_n^2} U_n^2 \right) H(U_n) + \frac{1}{2} K_n U_n^2 H(-U_n) \\ & + K_b^0 U_b^2 \left(\frac{a}{2} - \frac{b}{3\delta_b} |U_b| + \frac{1}{4\delta_b^2} U_b^2 \right) \end{aligned} \quad (5)$$

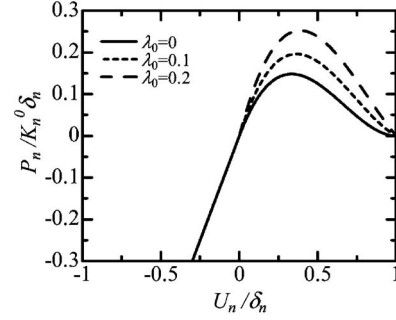
where $H(U_n)$ is the Heaviside function, K_t^0 , K_n^0 , K_b^0 are the initial values of interfacial rigidities in each direction, $a=1+b\lambda_0-\lambda_0^2$ is the interfacial parameter, and λ_0 is the damage threshold of equivalent interfacial separation. The interfacial constitutive equations can be obtained by partially differentiating the Helmholtz free energy with respect to the displacement in each direction.

With unloading-reloading condition, the interfacial constitutive law can be written as [26]

$$P_t = K_t^0 \{1 - b(\lambda_{\max} - \lambda_0) + (\lambda_{\max}^2 - \lambda_0^2)\} U_t$$



(a)



(b)

Fig. 2 Interfacial traction and separation relations: (a) Pure tangential deformation case; (b) pure normal deformation case

$$P_n = K_n^0 \{1 - b(\lambda_{\max} - \lambda_0) + (\lambda_{\max}^2 - \lambda_0^2)\} U_n H(U_n) + K_n^0 U_n H(-U_n) \quad (6)$$

$$P_b = K_b^0 \{1 - b(\lambda_{\max} - \lambda_0) + (\lambda_{\max}^2 - \lambda_0^2)\} U_b$$

where $\lambda_{\max}$ corresponds to the maximum interfacial separation during the loading-unloading history. Once $\lambda_{\max}$ reaches unity, the interface is separated freely except for the contact friction, and the interfacial traction and separation law becomes

$$\begin{aligned} P_t &= \mu_t K_t^0 U_n \text{sign}(U_t) H(-U_n) \\ P_n &= K_n^0 U_n H(-U_n) \end{aligned} \quad (7)$$

$$P_b = \mu_b K_b^0 U_n \text{sign}(U_b) H(-U_n)$$

where μ_t , μ_b denote the Coulomb friction coefficients for the tangential directions.

The mechanical behaviors of the above cohesive zone model in pure mode cases are shown in Fig. 2. Figure 2(a) corresponds to the pure tangential deformation. As the interfacial separation increases monotonically from zero in either + or - direction, the interfacial traction rises until it reaches the maximum value. After the traction reaches its maximum value, it begins to decrease and, finally, the traction disappears when $|U_t| = \delta_t$. The damage threshold of equivalent interfacial separation, λ_0 , affects the damage initiation and this results in the increase of the interfacial traction. Figure 2(b) corresponds to pure normal deformation. For the deformation in the + direction, as the interfacial separation increases monotonically from zero, the interfacial traction rises until it reaches the maximum value. After that point, the traction begins to decrease and, finally, disappears when $U_n = \delta_n$. The damage threshold of equivalent interfacial separation, λ_0 , affects the damage initiation and results in an increase in interfacial traction. These features are the same as the pure tangential deformation case. On the other hand, for the deformation in the - direction, no damage evolution occurs and this results in a linear relation between the interfacial traction and the interfacial separation.

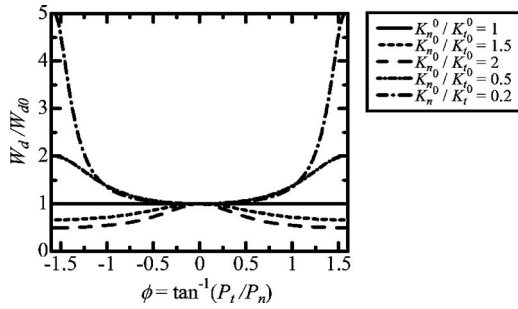


Fig. 3 The dependence of K_n^0/K_t^0 on the decohesion energy; $\delta_n^0/\delta_t^0=1$

The decohesion energy is defined by the total energy to break down one cohesive element and is obtained from the area of P - λ curve, that is,

$$W_d = \int_0^1 P d\lambda \quad (8)$$

where P is the total interfacial traction. From Eqs. (6) and (8), the decohesion energy is

$$W_d(\phi) = W_{d0} \left[\frac{1 + \frac{K_n^0}{K_t^0} \tan^2 \phi}{1 + \frac{\delta_n^0}{\delta_t^0} \left(\frac{K_n^0}{K_t^0} \right)^2 \tan^2 \phi} \right] \quad (9)$$

where ϕ is the phase angle, which is defined by the ratio of the normal to tangential traction, $\phi = \tan^{-1}(P_t/P_n)$. W_{d0} is the decohesion energy in the pure normal separation case for $\lambda_0=0$, i.e.,

$$W_{d0} = \frac{1}{12} K_n^0 \delta_n^2 \quad (10)$$

When one cohesive element breaks down, the interface separates and new surfaces are created. With A denoting the delamination area, the decohesion energy can be related to the interfacial energy release rate by

$$G(\phi) = \frac{W_d(\phi)}{A} \quad (11)$$

When $G(\phi)$ reaches the interfacial toughness, $\Gamma(\phi)$, interfacial debonding occurs. The decohesion energy is a function of K_n^0/K_t^0 and δ_n^0/δ_t^0 . Figure 3 shows the decohesion energy plotted against the phase angle in $\delta_n^0/\delta_t^0=1$ case. This figure suggests that the decohesion energy depends on the phase angle except for the isotropic interface case, $K_t^0=K_n^0$. To obtain a concave upward curve of the interfacial strength, the decohesion energy for the normal separation should be less than that for the tangential separation. Therefore, the initial rigidities should be $K_t^0 > K_n^0$. This tendency agrees with most experimental results for the interface toughness versus phase angle curves [3–5,28].

Figure 4 shows the decohesion energy versus the phase angle in $K_n^0/K_t^0=1$ case. The decohesion energy strongly depends on the ratio of the critical separations, δ_n^0/δ_t^0 , as compared with K_n^0/K_t^0 . To obtain a concave upward curve of the interfacial strength, the decohesion energy for the normal separation should be less than that for the tangential separation. Therefore, the interfacial critical separation should be $\delta_t^0 > \delta_n^0$.

3 Numerical Model of Indentation Test

A monolayer coating and substrate system is considered in an axisymmetric configuration. The shape of indentation is assumed to be a conical with a tip radius of R . The numerical model and boundary condition are shown in Fig. 5. The coating layer and

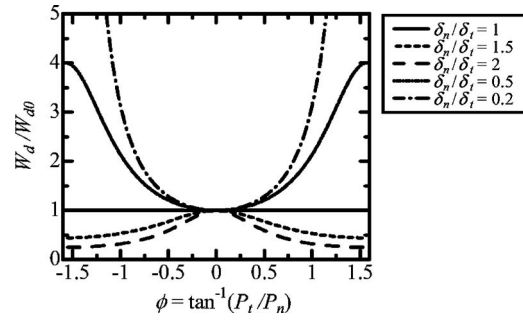


Fig. 4 The dependence of δ_n^0/δ_t^0 on the decohesion energy; $K_n^0/K_t^0=1$

substrate are assumed to be polyimide and silicon, respectively. The elastic modulus, Poisson's ratio, and yield stress are assumed to be $E_c=2.5$ GPa, $\nu_c=0.42$, $\sigma_y^c=100$ MPa for the polyimide layer; for silicon, these values are $E_s=190$ GPa, $\nu_s=0.23$, $\sigma_y^s=7000$ MPa, respectively. The stress-strain relation of polyimide and silicon are assumed to be an elastic-perfect plastic material for simplicity. The indenter is treated as rigid in this paper. The boundary conditions for the numerical simulation are that the bottom face of the substrate is fixed as $u_z(r)=0$, and the axisymmetric condition with $u_r(z)=0$ is applied at $r=0$. Loading is applied by enforced displacements of the indenter.

The isotropic cohesive zone models, $K_0=K_t^0=K_n^0$, $\delta_c=\delta_t=\delta_n$ in Eq. (6) are used between the film and the substrate. The interfacial parameters are taken to be $b=2$, $\lambda_0=0$. The other parameters K_0, δ_c are changed variously. The cohesive zone models are implemented into the finite element analysis program (ABAQUS Ver. 6.3) through the user-defined element (UEL) subroutine. It is noted that the numerical model is an axisymmetric configuration and, hence, the decohesion energy depends on the distance from the axisymmetric axis since the delamination area, A , is a function of r . To maintain a constant interfacial strength, the interfacial rigidities are taken as a function of r . Moreover, it is convenient to define the equivalent interfacial shear stiffness μ_{int} , which is obtained from the equilibrium condition of the shear force acting on the interface.

$$\mu_{int} = \frac{16d\Gamma(\pi/2)}{3\delta_c^2} \quad (12)$$

where $\Gamma(\pi/2)$ is the interfacial strength for pure tangential deformation, which is obtained from Eq. (11).

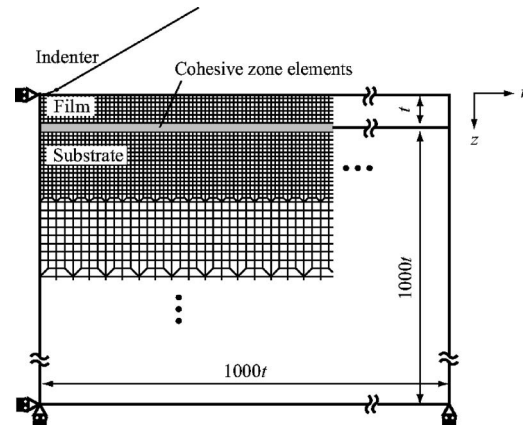


Fig. 5 Numerical model of indentation test on film and substrate system

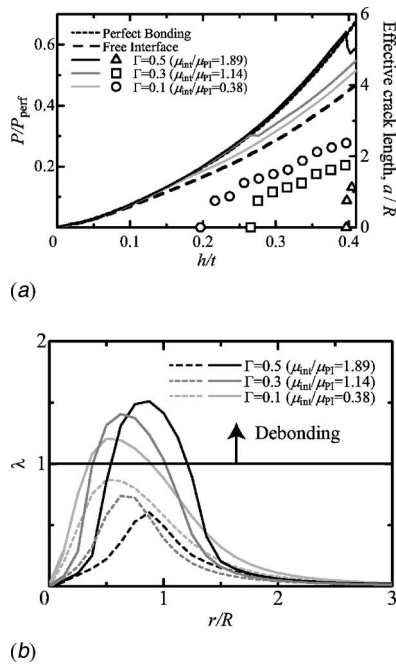


Fig. 6 (a) Indentation load and depth curve for $\delta_c=0.01$. Symbols show the effective crack length during the indentation. (b) Equivalent interfacial separation just before and after debonding.

4 Numerical Results

The interfacial parameters for isotropic interface model are the interfacial strength Γ and the interfacial critical displacement δ_c . Note that the interfacial strength does not depend on the phase angle in the isotropic model. In this section, the effects of these parameters on the indentation load and depth curve are discussed.

Figure 6(a) shows the loading stage of the P - h (indentation load-depth) curve for $\delta_c=0.01$. The upper (lower) dotted line is the case when the interface is perfectly bonding (debonding). The indentation load is normalized by the indentation load in the perfectly bonded case P_{perf} and indentation depth is normalized by the film thickness t . Symbols show the effective crack length a , which was calculated from the number of broken cohesive elements. As the interfacial strength Γ increases, the loading curves approach the perfectly bonded case. When $\Gamma=0.1$, the cohesive elements are gradually broken down and the indentation load-depth curve deviates from the perfect bonding curve around $h/t=0.15$. This point is slightly different from the onset of debonding, $h/t=0.2$ and this difference is due to the compliance effect of cohesive zone elements. On the contrary, pop-in behaviors (sudden drops) in the indentation load can be observed for the cases $\Gamma=0.3, 0.5$. These points correspond to the onset of debonding. At these points, a number of cohesive elements are simultaneously broken down as shown in Fig. 6(b) and, then, the indentation load suddenly decreases. After the onset of debonding, the length of interface crack increases at a constant rate as the indentation depth increases in all cases. The pop-in behaviors can be observed only in the case when $\mu_{\text{int}}/\mu_{\text{PI}} \geq 1$, where μ_{PI} is the shear modulus of the polyimide layer. This implies that when the interface is stiffer than the coating film, a brittlelike delamination occurs on the interface. On the contrary, when the stiffness of the interface is smaller than that of the coating film, a ductilelike delamination occurs on the interface. Therefore, the shear modulus, $\mu_{\text{int}}/\mu_{\text{PI}}$, characterizes the delamination behavior on the interface during the indentation tests.

Figure 7 shows the relation between the phase angle at the interface crack tip and the effective crack length a normalized by

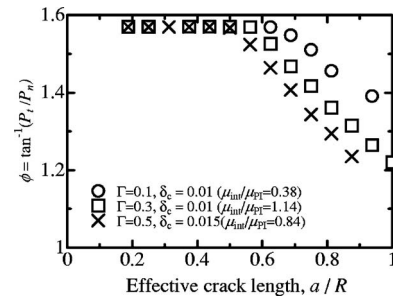


Fig. 7 The phase angle of interface crack tip and effective crack length

indentation tip radius R . At first, the interface crack propagates under shear dominant (Mode II) condition. Then, over $a/R=0.6$, the stress condition at the interface crack tip changes to mixed mode condition. The interface parameters only affect the transition point from Mode II to mixed mode in this study.

5 Evaluation of Interfacial Strength

5.1 Elastic Film Case. From the above results, the indentation load-depth curve tends to deviate from the perfectly bonded case when interfacial delamination occurs. In the following section, we focus on this transition phenomenon and evaluate the interfacial strength from the energy balance before and after the onset of the delamination.

When the coating film is assumed to be elastic, the typical indentation load-depth curve obtained is shown in Fig. 8(a). The corresponding equivalent interfacial separation is shown in Fig. 8(b). In these figures, “Point A” indicates conditions just before the onset of delamination, while “Point B” indicates those just

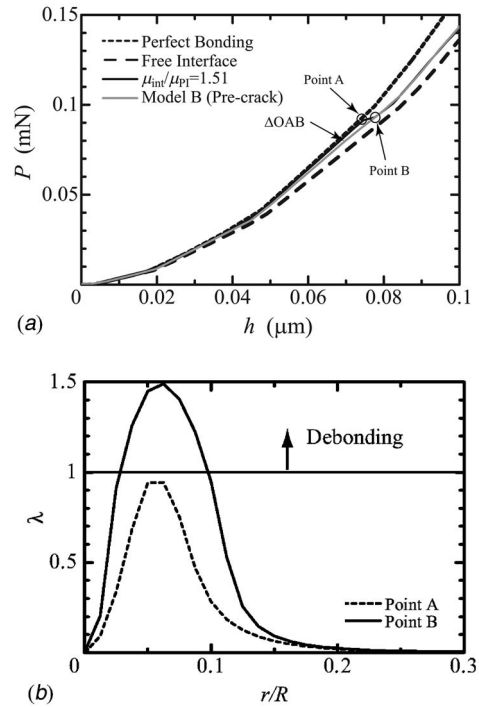


Fig. 8 (a) Indentation load and depth curve for the case of $\Gamma=0.1$, $\delta_c=0.005$, and $\mu_{\text{int}}/\mu_{\text{PI}}=1.51$. Polyimide layer is assumed to be elastic material. The indentation load-depth curve for Model B (precracked model) coincides with the original model at Point B. (b) Equivalent interfacial separation at Points A and B.

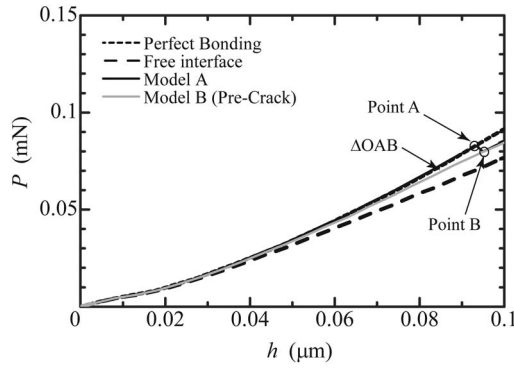


Fig. 9 Indentation load and depth curve for $\Gamma=0.1$, $\delta_c=0.005$, $\mu_{\text{int}}/\mu_{\text{pl}}=1.51$; elastic-plastic film case. Polyimide layer is assumed to be elastic perfectly plastic material. Compared to Fig. 8(a), the indentation load is lower due to the plastic deformation of the polyimide layer.

after delamination onset. To estimate the strain energy stored at Point B, we introduce another numerical model (Model B), in which the broken cohesive elements at Point B are eliminated. This precracked model describes the interface condition when the indentation loading stops and unloading starts at Point B. In indentation experiments, the crack length may be measured by optical observation or other methods. The indentation load-depth curve for “Model B” is shown in Fig. 8(a). This indentation curve coincides with the result of original model, “Model A,” on Point B. We then introduce the energy balance before and after the onset of delamination. At Point A, the work done by indentation is calculated from the area under the indentation load-depth curve, S_A . This energy is stored as the elastic strain energy of the film and its substrate (U_E^A) and the dissipation energy of the cohesive elements (U_I^A), i.e., $S_A = U_E^A + U_I^A$. At Point B in Model B, the work done by indentation can be obtained from the area under the indentation load-depth curve, S_B . This energy is stored in the elastic strain energy of the film and substrate (U_E^B) and the dissipation energy of the cohesive elements (U_I^B), i.e., $S_B = U_E^B + U_I^B$. The energy release rate during the delamination ($A \rightarrow B$) is described as

$$G = \frac{U_A + W_{AB} - U_B}{A_d} \quad (13)$$

where W_{AB} is the work done by the indenter and A_d is the delamination area during the $A \rightarrow B$ transition. If we assume that the damage condition of cohesive elements in Model A is the same as that in Model B, i.e., $U_I^A \approx U_I^B$, the energy release rate is approximately expressed as

$$G = \frac{U_E^A + W_{AB} - U_E^B}{A_d} = \frac{(S_A - U_I^A) + W_{AB} - (S_B - U_I^B)}{A_d} \approx \frac{S_A + W_{AB} - S_B}{A_d} \quad (14)$$

This numerator corresponds to the area of OAB as shown in Fig. 8(a). The ratio of the interfacial strength (defined by the cohesive elements) to the obtained energy release rate, G/Γ , is equal to 1.1. Thus, this method predicts the interfacial strength within 10% accuracy.

5.2 Elastic-Plastic Film Case. The above method is now applied to an elastic-plastic film on a substrate. The material properties of film and substrate have already been described in Sec. 3. The interfacial material properties are the same as for the previous cases. The indentation load and depth curves are shown in Fig. 9. Compared to Fig. 8(a), the indentation load is lower than in the

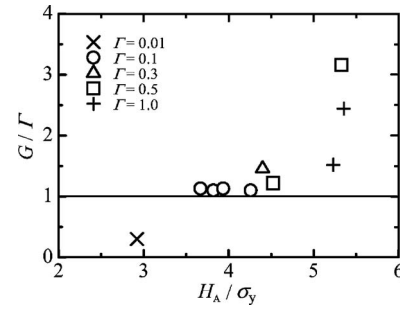


Fig. 10 The effects of interfacial strength on the evaluation results

elastic film case due to the plastic deformation of the polyimide layer. As with the elastic film case, we focus on the transition phenomenon to evaluate the interfacial strength.

At Point A in Fig. 9, the work done by indentation is calculated from the area under the indentation load-depth curve (S_A) and is stored in the elastic strain energy of the film and substrate (U_E^A), the plastic dissipation (U_P^A) in the film, and the dissipation energy in the cohesive elements (U_I^A), i.e., $S_A = U_E^A + U_P^A + U_I^A$. At Point B in Model B, the work done by indentation can be obtained from the area under the indentation load-depth curve (S_B) and is stored in the elastic strain energy of the film and substrate (U_E^B), the plastic dissipation (U_P^B) in the film, and the dissipation energy in the cohesive elements (U_I^B), i.e., $S_B = U_E^B + U_P^B + U_I^B$. The energy release rate during the delamination ($A \rightarrow B$) is given by

$$G = \frac{U_E^A + W_{AB} - U_E^B}{A_d} = \frac{W_{AB}}{A_d} + \frac{S_A - S_B}{A_d} - \frac{U_I^A - U_I^B}{A_d} - \frac{U_P^A - U_P^B}{A_d} \quad (15)$$

where W_{AB} is the work done by the indenter during the $A \rightarrow B$ transition. The first and second terms on the right side can be calculated from the indentation load-depth curve. If we assume that the damage condition of the cohesive elements and the plastic zone size in Model A is the same as those in Model B, i.e., $U_I^A \approx U_I^B$ and $U_P^A \approx U_P^B$, the energy release rate can be approximated by

$$G = \frac{W_{AB} + S_A - S_B}{A_d} \quad (16)$$

This numerator corresponds to the area of OAB, as shown in Fig. 9. The G/Γ ratio, defined as above, is 1.1. Thus, this method also predicts the interfacial strength within 10% accuracy under the above assumptions.

6 Discussion

To investigate the validity of the proposed method, numerical simulation under different interfacial parameters was carried out and the results are shown in Fig. 10. The vertical axis is the G/Γ ratio, as defined in Sec. 5.1. The horizontal axis is the ratio of the yield stress of the film to the hardness at the onset of the delamination, H_A , which is defined by $H_A = P_A/A_A$. A_A is the projected area when the indentation load is P_A . From this figure, the obtained results agree well with the given interfacial strength when $3.5 < H_A/\sigma_y < 4.5$. On the contrary, when H_A/σ_y is larger than 4.5, the obtained results overestimate the interfacial strength. This is because the plastic dissipation in the film is large enough that the assumption that the plastic dissipations are the same before and after the delamination is no longer valid. In those cases, it is necessary to evaluate the fourth term on the right side in Eq. (15) properly. When H_A/σ_y is less than 3.5, the obtained result underestimates the interfacial strength. This is because the interfacial

strength is so weak that delamination of the film occurs at an early stage of the indentation tests. In those cases, the plastic dissipations that accompany the film delamination need to be considered [21].

7 Conclusion

In this paper, indentation testing of layered structures was studied by using the damage-based cohesive zone model. From the indentation load and depth curve, an evaluation method for the interfacial strength is proposed. In this study, the following can be drawn.

1. The damage-based cohesive zone model was applied to the indentation simulation and the effects of cohesive parameters on the indentation response were examined. When the delamination initiated, the indentation load and depth curve were found to deviate from the perfect bonding curve.
2. When the interface was stiffer than the coating film, a brittlelike delamination occurred on the interface; when the stiffness of the interface was smaller than that of the coating layer, a ductilelike delamination occurred on the interface. The ratio of shear moduli, $\mu_{\text{int}}/\mu_{\text{pl}}$, characterizes the delamination behavior on the interface during indentation tests.
3. At first, the interface crack initiates and propagates under shear dominant (Mode II) condition. Then, over around $a/R=0.6$, the stress condition at the interface crack tip changes to mixed mode condition. The interface parameters only affect the transition point from Mode II to mixed mode condition.
4. Focusing on the discontinuous point during the indentation tests and introducing the energy balance before and after the onset of delamination, an evaluation method of the interfacial strength was proposed.
5. The proposed method can be used to estimate the interfacial strength when the ratio of the indentation hardness to the yield stress of the film is $3.5 < H_A/\sigma_y < 4.5$. The plastic deformation around the indenter and the delamination behavior of the film were found to influence the evaluation results.

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Transient Wave Propagation in Multilayered Viscoelastic Media: Theory, Numerical Computation, and Validation

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This paper extends the classical problem of transient wave propagation in multilayered solids to transient wave propagation in multilayered viscoelastic solids. Laplace and Hankel transforms and the transfer-matrix approach are used in the formulation together with the elastic-viscoelastic correspondence principle in linear viscoelasticity. The derived formula provides a theoretical basis to allow effective and efficient numerical algorithms to be developed. MATLAB is used to develop a computer program DYNALAYER^T that implements the theory developed. The numerical results are compared with the existing data available in literature and those obtained from finite element analysis using ANSYS. Excellent agreement has been observed from comprehensive comparisons, which verifies the validity of the theory, algorithm, and computer program developed in this study. The conclusion and findings of this study may result in a number of engineering applications, such as nondestructive evaluation of highway and airport pavements, petroleum exploration, countermine technology, geophysical inversion, structural health monitoring, and vehicle weigh-in-motion systems. [DOI: 10.1115/1.2839906]

Keywords: wave equation, viscoelasticity, elastodynamics, integral transform, finite element analysis

1 Introduction

Wave propagation, in a multilayered solid under dynamic loads, has long been a subject of great interest due to its relevance to a number of applications in the field of geophysics, seismology, petroleum exploration, and civil engineering [1–6]. This is witnessed by the well-known works of Thomson [7], Haskell [8,9], Dunkin [10], Harkruder [11], Apsel [12], Kausel and Roesset [13], Kausel and Peek [14], and Luco and Aspel [15]. Recent studies on this subject can be found in Stokoe et al. [3], Pak and Guzina [16], Roesset et al. [17], Sun [5,18,19], Sun and Deng [20,21], Diti [22], Sun and Greenberg [23], Liang and Zeng [24], and Sun et al. [6]. Methods for dealing with wave propagation in a multilayered solid primarily include ray expansion, finite element method, and transfer-matrix and stiffness-matrix approaches.

Ray expansion coupled with the Cagniard-de Hoop technique has been developed [25–28]. The technique based on surface wave synthesis and ray expansions is appealing owing to its physical interpretation that can be given to the different terms involved. However, they do not provide a complete representation of the response of the layered solid. The purely analytical treatment using Cagniard-de Hoop technique becomes too complex to be tractable when the number of layers is large [15,29].

Finite element methods (FEMs) offer a purely numerical alternative for tackling the problem [30]. FEMs differentiate the layered solid into a large number of meshes and solve a set of coupled ordinary differential equations in which the time integration can be evaluated using direct integration methods such as the Newmark method or the Wilson method [5,31]. The treatment of infinite boundary in FEM is implemented using either one of the two alternatives: artificial boundary or the boundless element. The benefit of FEM is that complex constitutive models, as well as

irregular geometric conditions, can be conveniently handled in a unified framework. However, the disadvantage of FEM is also apparent. On one hand, the use of an artificial boundary introduces false wave reflection, while the use of the boundless element introduces unpredictable accuracy reduction. On the other hand, solving a wave propagation problem involving a large number of meshes can be much more time consuming than its static counterpart. This makes FEM an inefficient alternative, less attractive than the analytical treatment of the subject.

Approaches that lead to a complete description of wave propagation in a multilayered solid while having a good efficiency are the transfer-matrix approach [7,8,10] and the stiffness-matrix approach [13,14,32]. The former relates a transformed response at the bottom of a layer, in the form of a transfer matrix, to a corresponding quantity at the top of a lower layer. The latter expands the transcendental functions in a stiffness matrix in terms of wave number and takes up to the second order of wave number as an approximation. Although both approaches are essentially the same, the stiffness-matrix approach requires each layer to be thin in order to achieve necessary accuracy.

Previous studies on wave propagation in multilayered solids only treat elastic materials [7,13,15,16,22,33]. What is missing in the literature is the consideration of viscoelastic materials in the study of wave propagation in multilayered solids. In reality viscosity, as reflected by the damping effect, exists in all physical systems. For instance, strong viscoelasticity of asphalt concrete pavement in summer should not be ignored in the modeling and analysis [11,36,43]. Viscoelastic waves differ significantly from elastic waves, as both material and geometric dispersions are possible [34]. This may result in completely distinct wave propagation patterns [20,21,35]. Recently, Benatar et al. [36] simplified the Pochhammer frequency equation and formulated corrections for geometric dispersion for the phase velocity and attenuation. Zhao and Gary [37] presented a three-dimensional solution of the longitudinal wave propagation in an infinite linear viscoelastic cylindrical bar.

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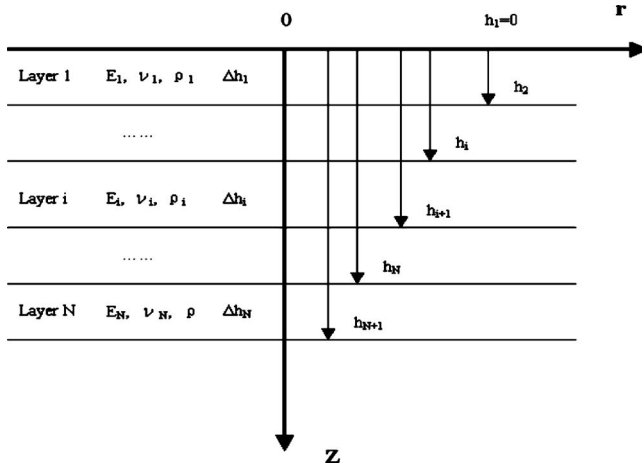


Fig. 1 A multilayered solid with a bedrock

An original contribution of this study is to incorporate various forms of viscosity in the analysis of wave propagation in multilayered solids. In addition to the consideration of viscoelastic materials, this paper deals with transient rather than steady-state wave propagation. Transient wave propagation has been a subject less frequently studied in literature due to its technical difficulties. Instead of using the Fourier transform valid for steady-state analyses, the Laplace transform is adopted here since it treats transient phenomena naturally by taking into account initial condition of the wave propagation problem. Furthermore, efficient fast algorithms are developed in this paper for numerically evaluating transient wave propagation with comprehensive numerical comparisons.

The remainder of this paper is organized as follows. Section 2 presents physical models and governing equations of multilayered elastic solids considered in this study. Section 3 applies various integral transforms and their inversions to solve the transient response of multilayered elastic solids using the transfer-matrix approach. Section 4 describes constitutive equations of viscoelastic models and transient solutions of multilayered viscoelastic solids obtained using the elastic-viscoelastic corresponding principles. Sections 6 and 7 introduce fast algorithms for implementing the Laplace transform and its inversion, and the Hankel transform and its inversion, respectively. Section 8 provides a comparison between static solution obtained here and that of FEM. Sections 9 and 10, respectively, use an elastic half space and a multilayered viscoelastic solid for validation. The result of using the derived formulation and that of using FEM are compared with each other. Section 11 draws conclusions.

2 Governing Equations of Multilayered Elastic Solids

An effective approach for solving a viscoelastic problem is to seek the solution of its elastic counterpart, and then apply the elastic-viscoelastic correspondence principle to update the elastic solution to obtain the viscoelastic solution [38]. In other words, transient wave propagation in multilayered elastic solids is the basis for the analysis of transient wave propagation in multilayered viscoelastic solids. In this section, we only consider a multilayered elastic solid.

The material of the layer is characterized as follows: Within each layer, the solid is isotropic and homogeneous, and has uniform thickness and identical material properties, while these properties vary for different layers. Each layer is assumed to extend to infinity horizontally. Figure 1 depicts a multilayered solid made of linear elastic material with Young's modulus E_i , Poisson's ratio ν_i , and density ρ_i for each layer. In this study, we consider an axisymmetric, distributed load. It is convenient to represent the governing equation in a cylindrical coordinate system (r, θ, Z) . Be-

sides the global cylindrical coordinate system (r, θ, Z) , local cylindrical coordinate systems (r_i, θ_i, z_i) are employed as well for the i th layer. The relationship between global and local coordinate systems is $r_i=r$, $\theta_i=\theta$, and $z_i=Z-h_i$, in which h_i are distances between the surface of the multilayered solid and the upper interface of the i th layer. For the purpose of convenience, unless otherwise necessary, the subscript i is omitted in the following derivation. Beneath the N th layer, two physical models can be used: a half space extended to infinity vertically and very stiff bedrock. Because a half space can be treated as a special case of bedrock model when the thickness of the N th layer is set to infinity, only the bedrock model is presented here.

The motion of a multilayered elastic solid is governed by Navier's equation [33]

$$\mu \nabla^2 \mathbf{F} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{F} + \rho \mathbf{f} = \rho \frac{\partial^2 \mathbf{F}}{\partial t^2} \quad (1a)$$

or equivalently

$$(\lambda + 2\mu) \nabla \nabla \cdot \mathbf{F} - \mu \nabla \times \nabla \times \mathbf{F} + \rho \mathbf{f} = \rho \frac{\partial^2 \mathbf{F}}{\partial t^2} \quad (1b)$$

where λ and μ are Lamé elastic constants, respectively, ρ is the mass density, $\mathbf{F}$ is the displacement vector, and $\mathbf{f}$ is the body force. Body force only needs to be considered to describe the earthquake or explosion source and for the very lowest frequency ($\ll 0.01$ Hz) seismic waves [39]. For the homogeneous equation of motion, the body force can be ignored, that is, $\mathbf{f}=0$. The two Lamé's constants λ and μ can also be represented in terms of Young's elastic modulus E and Poisson's ratio ν .

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad (2a)$$

$$\mu = \frac{E}{2(1+\nu)} \quad (2b)$$

Equation (1a) and (1b), when applied to the i th layer, yields two wave equations for each layer:

$$(c_d^2 - c_s^2) \frac{\partial \Delta}{\partial r} + c_s^2 \left(\nabla^2 u - \frac{u}{r^2} \right) - \ddot{u} = 0 \quad (3a)$$

$$(c_d^2 - c_s^2) \frac{\partial \Delta}{\partial z} + c_s^2 \nabla^2 w - \ddot{w} = 0 \quad (3b)$$

where $u=u(r, z, t)$ and $w=w(r, z, t)$ are displacements of the i th layer along r and z directions, respectively, $c_d=\sqrt{(\lambda+2\mu)/\rho}$ and $c_s=\sqrt{\mu/\rho}$ are dilatational and shear wave velocities, respectively, and one and two dots in (3a) and (3b) stand for first and second derivatives with respect to time t , respectively. Constitutive equations of linear elastic solids read

$$\tau_{rz} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \quad (4a)$$

$$\sigma_z = \lambda \Delta + 2\mu \frac{\partial w}{\partial z} \quad (4b)$$

Stresses and displacements of the solids need to satisfy boundary conditions at $z=z_b$,

$$\tau_{rz}(r, z, t) = \tau_{rz}(r, z_b, t) \quad (r, z) \in S_\sigma \quad (5a)$$

$$\sigma_z(r, z, t) = \sigma_z(r, z_b, t) \quad (r, z) \in S_\sigma \quad (5b)$$

$$u(r, z, t) = u(r, z_b, t) \quad (r, z) \in S_u \quad (5c)$$

$$w(r, z, t) = w(r, z_b, t) \quad (r, z) \in S_u \quad (5d)$$

For a transient problem, stresses and displacements also need to satisfy initial conditions

$$u(r, z, 0^+) = u_0(r, z) \quad (6a)$$

$$w(r, z, 0^+) = w_0(r, z) \quad (6b)$$

$$\dot{u}(r, z, 0^+) = \dot{u}_0(r, z) \quad (6c)$$

$$\dot{w}(r, z, 0^+) = \dot{w}_0(r, z) \quad (6d)$$

Equations (3a), (3b), (4a), (4b), (5a)–(5d), and (6a)–(6d) are linear partial differential equations, and constitute a complete mathematical description of elastodynamic wave equations for each layer of the multilayered media. Integral transformation is an effective method for solving this type of problems as they convert partial differential equations into differential equations and/or algebraic equations, of which the solution can be easily obtainable. This method also gives rise to meaningful physical interpretation of the constructed solution since the solution can be expressed in terms of a fundamental solution (the Green's function) [5,23].

3 Transient Response of a Multilayered Elastic Medium

Without loss of generality, in this study, it is assumed that the multilayered elastic medium be at rest initially, yielding the following initial condition:

$$u_0(r, z) = \dot{u}_0(r, z) = w_0(r, z) = \dot{w}_0(r, z) = 0 \quad (7)$$

Define the Laplace transform and its inversion as follows [40,41]:

$$\tilde{f}(q) = \int_0^\infty f(t) e^{-qt} dt \quad (8a)$$

$$f(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \tilde{f}(q) e^{qt} dq \quad (8b)$$

where q is the complex frequency corresponding to time t . When the Laplace transform is taken with respect to depth z , the corresponding vertical wave number is denoted as ξ . Define a ν th order Hankel transform of $f(r)$ and its inversion as follows [40,41]:

$$\bar{f}(s) = \int_0^\infty r f(r) J_\nu(sr) dr \quad (9a)$$

$$f(r) = \int_0^\infty s \bar{f}(s) J_\nu(sr) ds \quad (9b)$$

where J_ν is the Bessel function of the first kind of order ν , and s is the radial wave number corresponding to depth z .

For each single layer (e.g., the i th layer), apply the Laplace transform with respect to time t , the Hankel transform with respect to radius r , and the Laplace transform with respect to depth z , respectively. After these integral transforms, Eqs. (3a) and (3b) become two algebraic equations and can be solved with simple mathematical manipulation [24,33,42].

$$\tilde{u} = \frac{\left[c_s^2 s \tilde{u}_{z0} - (c_d^2 - 2c_s^2) \xi \tilde{w}_{z0} + \frac{1}{\rho} \tilde{\tau}_{z0} \right] (c_d^2 s^2 - q^2 - c_s^2 \xi^2) + (c_d^2 - c_s^2) \xi s \left[c_d^2 s \tilde{w}_{z0} + \left(c_d^2 - c_s^2 - \frac{\lambda}{\rho} \right) \xi \tilde{u}_{z0} + \frac{1}{\rho} \tilde{\sigma}_{z0} \right]}{(c_s^2 s^2 - q^2 - c_d^2 \xi^2)(c_d^2 s^2 - q^2 - c_s^2 \xi^2) + (c_d^2 - c_s^2)^2 \xi^2 s^2} \quad (10a)$$

$$\tilde{w} = \frac{(c_s^2 s^2 - q^2 - c_d^2 \xi^2) \left[c_d^2 s \tilde{w}_{z0} + \left(c_d^2 - c_s^2 - \frac{\lambda}{\rho} \right) \xi \tilde{u}_{z0} + \frac{1}{\rho} \tilde{\sigma}_{z0} \right] - \left[c_s^2 s \tilde{u}_{z0} - (c_d^2 - 2c_s^2) \xi \tilde{w}_{z0} + \frac{1}{\rho} \tilde{\tau}_{z0} \right] (c_d^2 - c_s^2) \xi s}{(c_s^2 s^2 - q^2 - c_d^2 \xi^2)(c_d^2 s^2 - q^2 - c_s^2 \xi^2) + (c_d^2 - c_s^2)^2 \xi^2 s^2} \quad (10b)$$

Here, $\tilde{u}(\xi, 0, q)$, $\tilde{w}(\xi, 0, q)$, $\tilde{\sigma}(\xi, 0, q)$, and $\tilde{\tau}_{rz}(\xi, 0, q)$ are denoted as $\tilde{u}_{z0}$, $\tilde{w}_{z0}$, $\tilde{\sigma}_{z0}$, and $\tilde{\tau}_{z0}$, respectively. The solution given by (10a) and (10b) is a displacement field in the transformed domain (i.e., the frequency-wave-number domain) and needs to be converted back to the original space-time domain using inverse integral transforms.

Equations (10a) and (10b) can be decomposed into additive terms of fractions, whose inverse Laplace transforms with respect to radial wave number s can be evaluated analytically. The process of partial fraction expansions and inverse Laplace transform is tedious and omitted. A MATLAB code containing symbolic operation of inverse Laplace transform is given in Appendix A for the verification of the result presented in this paper. Here, we only present the displacement field and the stress field after the inversion.

$$\begin{bmatrix} \tilde{u}(\xi, z, q) \\ \tilde{w}(\xi, z, q) \\ \tilde{\tau}_{rz}(\xi, z, q) \\ \tilde{\sigma}_z(\xi, z, q) \end{bmatrix} = [\Phi(z)] \begin{bmatrix} \tilde{u}(\xi, 0, q) \\ \tilde{w}(\xi, 0, q) \\ \tilde{\tau}_{rz}(\xi, 0, q) \\ \tilde{\sigma}_z(\xi, 0, q) \end{bmatrix} \quad (11)$$

in which transfer function matrix $[\Phi(z)]$ is given by

$$[\Phi(z)] = [\Phi(\xi, z, q)] = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_{31} & \phi_{32} & \phi_{33} & \phi_{34} \\ \phi_{41} & \phi_{42} & \phi_{43} & \phi_{44} \end{bmatrix} \quad (12)$$

The entries of the above matrix are given in Appendix B.

Since we omitted the subscript i in foregoing derivations, depth z in (11) should virtually be z_i . Note that the left-hand side of (11) is displacement and stress fields in the Hankel–Laplace domain at depth z_i , a vertical distance from a point of interest to the upper interface of the i th layer in a local coordinate system, while the right-hand side of (11) is displacement and stress fields in the Hankel–Laplace domain at the upper interface $z_i=0$ in the local coordinate system.

Matrix equation (11) implies that, in a local coordinate system of the i th layer, displacement and stress fields at any points in the i th layer can be represented in terms of their counterparts $\tilde{u}(\xi, 0, q)$, $\tilde{w}(\xi, 0, q)$, $\tilde{\tau}_{rz}(\xi, 0, q)$, and $\tilde{\sigma}_z(\xi, 0, q)$ at the upper interface of the i th layer. Instead of focusing on a single layer, now examine the entire model of a multilayered system, as shown in Fig. 1. The relationship between a local coordinate system (r_i, z_i, θ_i) of the i th layer and a global coordinate system (r, Z, θ) of the entire multilayered solid is

$$Z = z_i + h_i \quad (13)$$

As a result, displacement and stress fields at the upper interface of the i th layer $\tilde{u}(\xi, 0, q)$, $\tilde{w}(\xi, 0, q)$, $\tilde{\tau}_{rz}(\xi, 0, q)$, and $\tilde{\sigma}_z(\xi, 0, q)$ in a local coordinate system are indeed $\tilde{u}(\xi, h_i, q)$, $\tilde{w}(\xi, h_i, q)$, $\tilde{\tau}_{rz}(\xi, h_i, q)$, and $\tilde{\sigma}_z(\xi, h_i, q)$ in the global coordinate system, while the transfer function matrix $[\Phi(z_i)]$ in (11) should be replaced by $[\Phi(Z - h_i)]$ in the global coordinate system. As for the left-hand side of (11), we simply replace z by Z in displacement and stress fields as both variables z and Z refer to the same point of interest. Now, Eq. (11) in the global coordinate system reads

$$\begin{bmatrix} \tilde{u}(\xi, Z, q) \\ \tilde{w}(\xi, Z, q) \\ \tilde{\tau}_{rz}(\xi, Z, q) \\ \tilde{\sigma}_z(\xi, Z, q) \end{bmatrix} = [\Phi(Z - h_i)] \begin{bmatrix} \tilde{u}(\xi, h_i, q) \\ \tilde{w}(\xi, h_i, q) \\ \tilde{\tau}_{rz}(\xi, h_i, q) \\ \tilde{\sigma}_z(\xi, h_i, q) \end{bmatrix} \quad (14)$$

When a dynamic load is applied on the surface of the multilayered system, the boundary condition is typically given in terms of stress. As a result, one would like to express the displacement field in terms of the stress field. For this purpose, define displacement and stress fields as $[\tilde{U}(Z)] = [\tilde{u}(\xi, Z, q), \tilde{w}(\xi, Z, q)]$ and $[\tilde{T}(Z)] = [\tilde{\tau}_{rz}(\xi, Z, q), \tilde{\sigma}_z(\xi, Z, q)]$, respectively. Equation (14) can be rewritten as

$$[\tilde{U}(Z), \tilde{T}(Z)]^T = [\Phi(Z - h_i)][\tilde{U}(h_i), \tilde{T}(h_i)]^T \quad (15)$$

where T stands for transpose. For the interface between the i th layer and the $(i-1)$ th layer, define $Z = h_i^+$ the plane belonging to the $(i-1)$ th layer, and $Z = h_i^-$ the plane belonging to the i th layer. At plane $Z = h_{i+1}^+$, Eq. (15) becomes

$$[\tilde{U}(h_{i+1}^+), \tilde{T}(h_{i+1}^+)]^T = [\Phi(\Delta h_i)][\tilde{U}(h_i^-), \tilde{T}(h_i^-)]^T \quad (16)$$

where Δh_i is the thickness of the i th layer, $\Delta h_i = h_{i+1}^+ - h_i^-$. When a continuous interface condition is assumed, that is, $[\tilde{U}(h_i^-)] = [\tilde{U}(h_i^+)]$ and $[\tilde{T}(h_i^-)] = [\tilde{T}(h_i^+)]$, Eq. (16) can be expanded as follows:

$$\begin{aligned} [\tilde{U}(h_{i+1}^+), \tilde{T}(h_{i+1}^+)]^T &= [\Phi(\Delta h_i)][\tilde{U}(h_i^-), \tilde{T}(h_i^-)]^T \\ &= [\Phi(\Delta h_i)][\tilde{U}(h_i^+), \tilde{T}(h_i^+)]^T \\ &= [\Phi(\Delta h_i)][\Phi(\Delta h_{i-1})][\Phi(\Delta h_{i-2})] \cdots \\ &\quad \times [\Phi(\Delta h_2)][\Phi(\Delta h_1)][\tilde{U}(h_1), \tilde{T}(h_1)]^T \end{aligned} \quad (17)$$

So, Eq. (15) becomes

$$[\tilde{U}(Z), \tilde{T}(Z)]^T = [\Phi(Z - h_i)][\Phi(\Delta h_{i-1})] \cdots [\Phi(\Delta h_2)][\Phi(\Delta h_1)] \times [\tilde{U}(h_1), \tilde{T}(h_1)]^T \quad (18)$$

where $h_i \leq Z \leq h_{i+1}$. Equation (18) means that, for each layer, the transformed displacements and stresses at the bottom interface can be expressed in terms of their counterparts at the upper interface.

When a bedrock exists underneath the N th layer (the $(N+1)$ th layer), the displacement field is set to zero on the surface of bedrock at $Z = h_{N+1}$, that is, $u(r, Z, t)_{Z=h_{N+1}} = 0$ and $w(r, Z, t)_{Z=h_{N+1}} = 0$. When a half-space model is used as a foundation model, one may simply set $u(r, Z, t)_{Z=\infty} = 0$ and $w(r, Z, t)_{Z=\infty} = 0$. Since the latter can be treated as a special case of bedrock model of which the N th layer has a finite thickness, only the former is studied in this paper. Consequently, the transformed displacement field $\tilde{u}(\xi, h_{N+1}, q) = 0$ and $\tilde{w}(\xi, h_{N+1}, q) = 0$, or equivalently,

$$[\tilde{U}(h_{N+1})] = [\tilde{u}(\xi, h_{N+1}, q), \tilde{w}(\xi, h_{N+1}, q)] = [0, 0] \quad (19)$$

The transformed displacement field $[\tilde{U}(h_i)]^T$ can be represented as a function of the transformed stress field $[\tilde{T}(h_i)]$. So, the boundary condition, often given as a known stress distribution on the surface of the multilayered solid, can be eventually incorporated into the analysis. As a result, displacement and stress fields in the Laplace–Hankel domain can be ultimately represented in terms of stresses (in the Laplace–Hankel domain) on the surface of the multilayered solid, which is virtually the boundary condition [24,42].

$$[\tilde{U}(Z), \tilde{T}(Z)]_{4 \times 1}^T = [\Psi(Z)]_{4 \times 2} [\tilde{T}(h_1)]_{2 \times 1}^T \quad (20)$$

where

$$[\Psi(Z)] = [\Phi(Z - h_i)][\Phi(\Delta h_{i-1})] \cdots [\Phi(\Delta h_2)][\Phi(\Delta h_1)] \begin{bmatrix} [\mathbf{R}_1] \\ \mathbf{I} \end{bmatrix}$$

is a 4×2 matrix;

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a 2×2 unit matrix; for $1 \leq i \leq N$, $[\mathbf{R}_i] = -\mathbf{K}_{11}^{-1} \mathbf{K}_{12}$ is a 2×2 matrix and

$$[\mathbf{R}_{N+1}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

in which

$$[\mathbf{K}] = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} = [\Phi(\Delta h_N)][\Phi(\Delta h_{N-1})] \cdots [\Phi(\Delta h_{i+1})][\Phi(\Delta h_i)]$$

in which $[\mathbf{K}]$ is a 4×4 matrix, and $[\mathbf{K}_{11}]$, $[\mathbf{K}_{12}]$, $[\mathbf{K}_{21}]$, and $[\mathbf{K}_{22}]$ are 2×2 block matrices each;

$$[\Phi(\Delta h_i)] = \begin{bmatrix} \Theta_{11}^i & \Theta_{12}^i \\ \Theta_{21}^i & \Theta_{22}^i \end{bmatrix}$$

where

$$[\Theta_{11}^i] = \begin{bmatrix} \phi_{11}(\Delta h_i) & \phi_{12}(\Delta h_i) \\ \phi_{21}(\Delta h_i) & \phi_{22}(\Delta h_i) \end{bmatrix}, \quad [\Theta_{12}^i] = \begin{bmatrix} \phi_{13}(\Delta h_i) & \phi_{14}(\Delta h_i) \\ \phi_{23}(\Delta h_i) & \phi_{24}(\Delta h_i) \end{bmatrix}$$

$$[\Theta_{21}^i] = \begin{bmatrix} \phi_{31}(\Delta h_i) & \phi_{32}(\Delta h_i) \\ \phi_{41}(\Delta h_i) & \phi_{42}(\Delta h_i) \end{bmatrix}$$

and

$$[\Theta_{22}^i] = \begin{bmatrix} \phi_{33}(\Delta h_i) & \phi_{34}(\Delta h_i) \\ \phi_{43}(\Delta h_i) & \phi_{44}(\Delta h_i) \end{bmatrix}$$

Apply inverse Laplace transform and inverse Hankel transform of order 1 to $\tilde{u}(\xi, Z, q)$ and $\tilde{\tau}_{rz}(\xi, Z, q)$, and of order zero to $\tilde{w}(\xi, Z, q)$ and $\tilde{\sigma}_z(\xi, Z, q)$, Eq. (20) produces

$$\begin{Bmatrix} u(r, Z, t) \\ w(r, Z, t) \\ \tau_{rz}(r, Z, t) \\ \sigma_z(r, Z, t) \end{Bmatrix} = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \int_0^\infty \xi [\mathbf{J}(\xi r)] [\Psi(Z)] [\tilde{T}(h_1)]^T d\xi e^{qt} dq \quad (21)$$

where

$$[\mathbf{J}(\xi r)] = \begin{bmatrix} \mathbf{J}_1(\xi r) & 0 & 0 & 0 \\ 0 & \mathbf{J}_0(\xi r) & 0 & 0 \\ 0 & 0 & \mathbf{J}_1(\xi r) & 0 \\ 0 & 0 & 0 & \mathbf{J}_0(\xi r) \end{bmatrix}$$

and

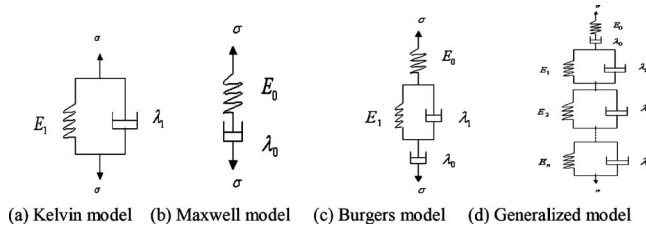


Fig. 2 A schematic plot of four viscoelastic models

$$[\Psi(Z)] = \begin{bmatrix} \psi_1(Z) \\ \psi_2(Z) \\ \psi_3(Z) \\ \psi_4(Z) \end{bmatrix} = \begin{bmatrix} \psi_{11}(Z) & \psi_{12}(Z) \\ \psi_{21}(Z) & \psi_{22}(Z) \\ \psi_{31}(Z) & \psi_{32}(Z) \\ \psi_{41}(Z) & \psi_{42}(Z) \end{bmatrix}$$

in which $[\psi_1(Z)]$, $[\psi_2(Z)]$, $[\psi_3(Z)]$, and $[\psi_4(Z)]$ are 1×2 matrices, respectively. With these notations, Eq. (21) can be decomposed into

$$u(r, Z, t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \int_0^\infty \xi J_1(\xi r) [\psi_1(Z)] [\bar{\tau}_{rz}(h_1), \bar{\sigma}_z(h_1)]^T d\xi e^{qt} dq \quad (22a)$$

$$w(r, Z, t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \int_0^\infty \xi J_0(\xi r) [\psi_2(Z)] [\bar{\tau}_{rz}(h_1), \bar{\sigma}_z(h_1)]^T d\xi e^{qt} dq \quad (22b)$$

$$\tau_{rz}(r, Z, t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \int_0^\infty \xi J_1(\xi r) [\psi_3(Z)] [\bar{\tau}_{rz}(h_1), \bar{\sigma}_z(h_1)]^T d\xi e^{qt} dq \quad (22c)$$

$$\sigma_z(r, Z, t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \int_0^\infty \xi J_0(\xi r) [\psi_4(Z)] [\bar{\tau}_{rz}(h_1), \bar{\sigma}_z(h_1)]^T d\xi e^{qt} dq \quad (22d)$$

The integrands of (21) and (22a)–(22d) play the role of Green's function as appearing in the analysis of linear systems [3,18].

4 Transient Response of a Multilayered Viscoelastic Medium

4.1 Elastic-Viscoelastic Correspondence Principle. Stress-strain relationships of viscoelastic materials can be described in two ways: mechanical models and creep-compliance curves [43]. If one description is available, the other can be determined uniquely as well. Here, the mechanical model is adopted. In this study, we considered four types of viscoelastic models: Kelvin model, Maxwell model, Burgers model, and the generalized model [43,44]. The generalized viscoelastic model contains three other models as special cases. Figure 2 depicts these viscoelastic models using springs and dashpots. In Fig. 2, σ is the stress, $E_0, E_1, \dots, E_n$ are Young's elastic moduli corresponding to spring constants, while $T_0, T_1, \dots, T_n$ are retardation times corresponding to dashpot damping coefficients $\lambda_0, \lambda_1, \dots, \lambda_n$ through the relationship $T_j = \lambda_j / E_j$.

Transient responses of multilayered viscoelastic solids can be obtained by applying the elastic-viscoelastic correspondence principle to transient response of multilayered elastic solids, which has been obtained in previous sections. Specifically, according to the correspondence principle of elastic-viscoelastic theory [38], the viscoelastic solution of a linear system can be obtained as indicated in Fig. 3.

The equivalent modulus E_e for a specific viscoelastic model is

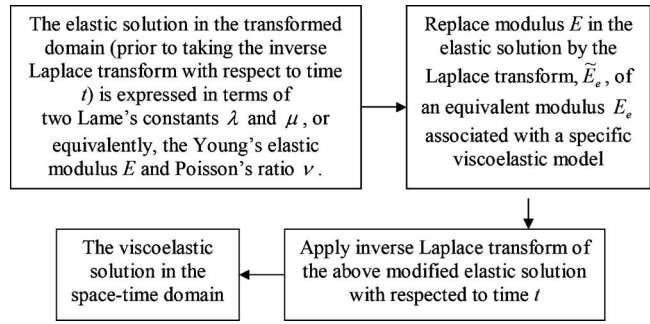


Fig. 3 Procedures of applying the elastic-viscoelastic corresponding principles

defined through stress-strain relationship $E_e = \sigma / \varepsilon$. Therefore, E_e is no longer a constant but a time dependent function. Coefficients that involve modulus E are ϕ_{ij} ($i, j = 1, \dots, 4$), $\dot{\phi}_{ij}$ ($i = 1, 2$, and $j = 1, \dots, 4$), α^2 , β^2 , p^2 , c_d , and c_s . Before taking inverse Laplace transform with respect to time t , Young's modulus E needs to be replaced by $\tilde{E}_e$. The rest of this section dedicates to identify $\tilde{E}_e$ corresponding to a specific viscoelastic model.

4.2 Kelvin Model. A Kelvin model is a combination of spring and dashpot in parallel, as depicted in Fig. 2(a). Both the spring and the dashpot experience identical strain ε , and the total stress is the sum of the two stresses

$$\sigma = E_1 \varepsilon + \lambda_1 \frac{\partial \varepsilon}{\partial t} = E_1 \left(\varepsilon + T_1 \frac{\partial \varepsilon}{\partial t} \right) \quad (23)$$

where $T_1 = \lambda_1 / E_1$ is the retardation time parameter. Define differential operator $D = \partial / \partial t$. Equation (23) can be expressed as

$$\sigma = E_e \varepsilon \quad (24)$$

where equivalent modulus $E_e = E_1 + \lambda_1 D = E_1 (1 + T_1 D)$. For each layer, equivalent elastic modulus E_e in the Laplace transform domain becomes

$$\tilde{E}_e = E_1 + \lambda_1 q = E_1 (1 + T_1 q) \quad (25)$$

where q is the complex frequency in the Laplace transform domain corresponding to time t .

4.3 Maxwell Model. A Maxwell model is a combination of spring and dashpot in series, as indicated in Fig. 2(b). Both the spring and the dashpot experience identical stress and the total strain is the sum of two strains

$$\varepsilon = \frac{\sigma}{E_0} + \frac{\sigma}{\lambda_0 \frac{\partial}{\partial t}} = \frac{\sigma}{E_0} + \frac{\sigma}{\lambda_0 D} = \frac{\sigma}{E_0} + \frac{\sigma}{T_0 E_0 D} \quad (26)$$

where $T_0 = \lambda_0 / E_0$. The equivalent elastic modulus derived from (26) is

$$E_e = \frac{\sigma}{\varepsilon} = \frac{E_0 T_0 D}{T_0 D + 1} \quad (27)$$

In the Laplace transform domain, the equivalent elastic modulus becomes

$$\tilde{E}_e = \frac{E_0 T_0 q}{T_0 q + 1} \quad (28)$$

4.4 Burgers Model. A Burgers model is a combination of Maxwell and Kelvin models in series, as indicated in Fig. 2(c). The total strain is composed of three parts

$$\varepsilon = \frac{\sigma}{E_0} + \frac{\sigma}{E_1 + \lambda_1 \frac{\partial}{\partial t}} + \frac{\sigma}{\lambda_0 \frac{\partial}{\partial t}} = \sigma \left(\frac{T_0 D + 1}{E_0 T_0 D} + \frac{1}{E_1 (T_1 D + 1)} \right) \quad (29)$$

The equivalent elastic modulus derived from (29) is

$$E^* = \frac{\sigma}{\varepsilon} = \left[\frac{T_0 D + 1}{E_0 T_0 D} + \frac{1}{E_1 (T_1 D + 1)} \right]^{-1} \quad (30)$$

In the Laplace transform domain, the equivalent elastic modulus becomes

$$\tilde{E}^* = \left[\frac{T_0 q + 1}{E_0 T_0 q} + \frac{1}{E_1 (T_1 q + 1)} \right]^{-1} \quad (31)$$

4.5 Generalized Model. Figure 2(d) shows a generalized model that can be used to characterize a wide variety of viscoelastic material. It is a combination of one Maxwell model and n Kelvin models in series. The total strain is composed of $(n+1)$ parts

$$\varepsilon = \sigma \frac{T_0 D + 1}{E_0 T_0 D} + \sigma \sum_{i=1}^n \frac{1}{E_i (T_i D + 1)} \quad (32)$$

The equivalent elastic modulus derived from (32) is

$$E^* = \frac{\sigma}{\varepsilon} = \left[\frac{T_0 D + 1}{E_0 T_0 D} + \sum_{i=1}^n \frac{1}{E_i (T_i D + 1)} \right]^{-1} \quad (33)$$

In the Laplace transform domain, the equivalent elastic modulus becomes

$$\tilde{E}^* = \left[\frac{T_0 q + 1}{E_0 T_0 q} + \sum_{i=1}^n \frac{1}{E_i (T_i q + 1)} \right]^{-1} \quad (34)$$

5 Fast Evaluation of Laplace Transform and Its Inversion

Due to the involvement of viscoelastic materials and the existence of multiple layers in the considered model, the evaluation of the solution is too complicated to permit analytical computation. Therefore, numerical computation has to be developed. In addition, the derived formula must be examined by comparing the existing solution with results obtained using other approaches. Because the evaluation of Eq. (21) involves intensive numerical computation of Laplace and Hankel transforms and their inversions, fast algorithms of these integral transforms are used in order to achieve high computational efficiency.

The Laplace transform of an arbitrary dynamic load exerted on the surface of a multilayered solid needs to be numerically evaluated. The evaluation of Laplace transform as defined in Eq. (8a) can be implemented using numerical integration (e.g., the Simpson and Lobatto method), provided that the transient dynamic load is truncated at a certain point in time. Specifically, in this study, a truncated form of Laplace transform is evaluated using recursive adaptive Lobatto quadrature in a MATLAB environment. Numerical evaluation of inverse Laplace transform as given in (8b) and of inverse Hankel transform is more time consuming than the evaluation of Laplace transform. To improve computational efficiency, an efficient algorithm given by Brancik [45] that takes advantage of the fast Fourier transform (FFT) algorithm is adopted here for implementing numerical evaluation of Laplace transform. The accuracy of the method can be improved by using ε algorithm.

To start, a discrete Laplace transform can be approximated by

$$\tilde{f}^k = C^k \left\{ 2 \operatorname{Re} \left[\sum_{n=0}^{\infty} F_n E_n^k \right] - F_0 \right\} \quad (35)$$

$$\tilde{f}^k = \tilde{f}(kT), \quad C_k = \frac{\Omega}{2\pi} e^{ckT}, \quad F_n = F(c - jn\Omega), \quad E_n^k = e^{-jkTn\Omega} \quad (36)$$

where $k=0, 1, \dots, N-1$, and T and $\Omega=2\pi/(NT)$ are sampling periods in original and transform domains, respectively. Equation (35) corresponds to a Fourier series approximation of the original signal $f(t)$ when the error can theoretically be controlled on the interval $t \in (0; NT)$. Practically, to suppress an increased error at the end of this interval, the required maximum time is supposed to be $t_m = (M-1)T$, with $M=N/2$ as the number of resultant computed points, which leads to the condition of choosing $\Omega = \pi(1 - 1/M)/t_m$. The coefficient c can be determined from

$$c \approx \alpha - \frac{\Omega}{2\pi} \ln E_r \quad (37)$$

where E_r denotes a desired relative error.

To minimize the error toward this theoretical value, the infinite sum in (35) must be evaluated as much accurately as possible. The solution then consists of three steps. First, this sum is evaluated using only first N terms when a FFT algorithm can be applied (i.e., $N=2^m$, where m is an integer). Second, after truncating the result of the FFT operation to have only a length M , the ε algorithm is applied to give a precision to the resultant sum. The ε algorithm uses only a few additional terms above those N used by the FFT algorithm; however, the sum becomes as if it were evaluated using greatly more terms. Finally, the result of the ε -algorithm application is substituted into (35) to finish the computation. Expressing these operations in vector form, (35) can be written as

$$\tilde{f}^M = C^M \circ \{ 2 \operatorname{Re}[E\{\operatorname{FFT}(\mathbf{F}^N)\}] - \mathbf{F}_0^M \} \quad (38)$$

where particular vectors of upper indexed lengths are created according to (36), namely, for $k=0, 1, \dots, M-1$, $n=0, 1, \dots, N-1$. Especially, $\mathbf{F}_0^M$ is the M -element constant vector of value c . $E\{\cdot\}$ designates an operator of the ε algorithm (including the operation of $N \rightarrow M$ vector length reduction). The symbol $\circ$ means Hadamard product of matrices (also called element-by-element product in MATLAB environment).

The principle of the ε algorithm can be explained by means of a lozenge diagram in Fig. 4. The first column is formed with $\varepsilon_{-1}^{(s)} = \mathbf{0}^M$, $s=1, 2, 3, \dots$, where $\mathbf{0}^M$ means an M -element zero vector. The second column represents partial sums computed recurrently as

$$\varepsilon_0^{(s+1)} = \varepsilon_0^{(s)} + \mathbf{F}_{N+s} \mathbf{F}_{N+s}^M, \quad s=0, 1, 2, \dots \quad (39)$$

where $\mathbf{F}_n^M$ is an M -element vector created according to the exponential term E_n^k , for $k=0, 1, \dots, M-1$ and a given n , and the initial $\varepsilon_0^{(0)}$ term is the result of the FFT operation truncated to have the length M . The leftover columns are computed using the formula

$$\varepsilon_{r+1}^{(s)} = \varepsilon_{r-1}^{(s+1)} + [\varepsilon_r^{(s+1)} - \varepsilon_r^{(s)}]^{-1}, \quad r, s=0, 1, 2, \dots \quad (40)$$

when the inversion operation is supposed to run componentwise. Thus, the sequence of successive approximations $\varepsilon_0^{(0)}, \varepsilon_2^{(0)}, \varepsilon_4^{(0)}, \dots$ usually converges much more quickly than the original sequence of partial sums. To start a computation using $2P+1$ partial sums, the $\varepsilon_{2P}^{(0)}$ term is the required result of the ε algorithm. This algorithm could be numerically instable if P is chosen too large. Based on a number of trials, it is found that $P=2$ or $P=3$ is a good choice.

6 Fast Evaluation of Hankel Transform and Its Inversion

In Eq. (21), high oscillation patterns of integral kernels due to the presence of Bessel functions impose a computational challenge. A significant amount of computation time would be re-

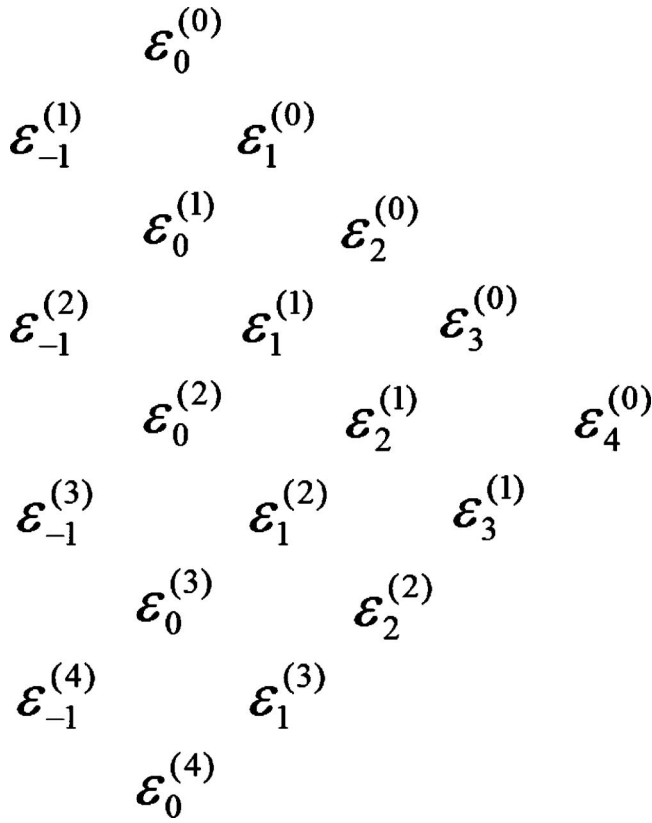


Fig. 4 The ε -algorithm lozenge diagram

quired if this integral were to be evaluated using conventional methods, such as the Simpson and the Gaussian algorithms, in order to achieve a specified accuracy. Candel [46] proposed a fast algorithm for numerically evaluating Hankel transforms, which has the same order of operations comparable to that of a two-dimensional FFT.

Hankel transform of function $f(x)$ with order k is defined by

$$F_k(r) = \int_0^\infty \xi f(\xi) J_k(\xi r) d\xi \quad (41)$$

Taking an asymptotic expansion

$$e^{1/2z(t-1/t)} = \sum_{k=-\infty}^{+\infty} t^k J_k(z) \quad (42)$$

let $t = e^{i\theta}$ and $z = \xi r$. Multiply both sides of (42) by $\xi f(\xi)$, and integrate it from 0 to ∞ ,

$$\int_0^\infty e^{i\xi r \sin \theta} f(\xi) \xi d\xi = \sum_{k=-\infty}^{+\infty} e^{ik\theta} F_k(r) \quad (43)$$

Based on the theory of Fourier series expansion, (43) implies that

$$F_r(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(r \sin \theta) e^{-ik\theta} d\theta \quad (44)$$

where function φ is defined as

$$\varphi(\eta) = \int_0^\infty e^{i\xi\eta} \xi f(\xi) d\xi \quad (45)$$

which can now be evaluated efficiently using FFT algorithm. As a result, instead of using (41) directly, the Hankel inverse transform $F_r(r)$ can be equivalently evaluated as the FFT of (45) and the

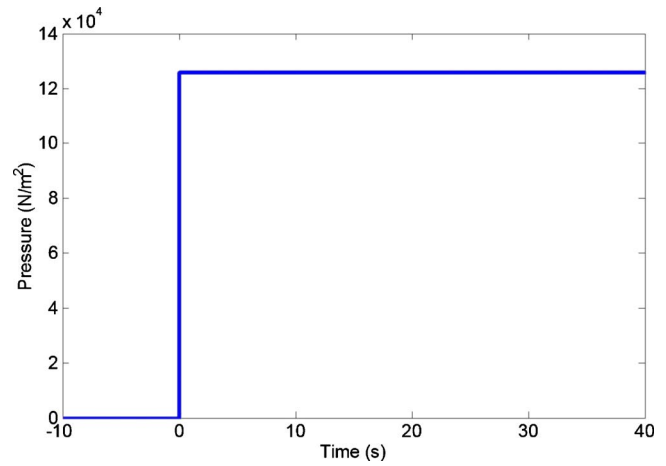


Fig. 5 Heaviside step function type of dynamic load used in static analysis

integration of (44) from 0 to 2π . Based on previous analyses, a computer program DYNALAYER^T is developed in the MATLAB environment to implement the analysis of transient wave propagation in multilayered viscoelastic solids.

7 Validation Through Static Solution of a Multilayered Elastic Solid

When the external dynamic load exerted on the surface of the multilayered solid is a Heaviside step function as shown in Fig. 5, transient wave propagation dissipated after a sufficiently long duration. The transient dynamic response of the solid should eventually die out and degenerate to a static solution of situation when a static load with the same magnitude as that in Fig. 5 is applied on the surface of the solid. Static responses of a multilayered solid can be viewed as an asymptotic solution of transient wave propagation. The static solution obtained using the derived formula is compared with that of FEM. For the purpose of comparison, the dynamic load specified in Fig. 5 and a four-layer elastic solid on top of a bedrock with properties given in Table 1 are used, in which parameter a is the radius of exerted circular load. ANSYS, a widely used software package for finite element analysis, is adopted as a tool for computing static displacement using FEM.

Since the bedrock is a rigid body and does not allow any displacement, it is implemented in ANSYS by setting displacements at the bottom of the fourth layer to be zero along the x , y , and z directions. In theory, the multilayered solid extends horizontally to infinity, which is implemented in ANSYS by choosing a cylinder of finite size along the radial dimension. This unavoidably will cause additional computational error. To mitigate such an error, the horizontal size of the cylinder should be much larger relative to the size of the exerted circular load. However, the larger the cylinder, the longer the computation time will take as it requires more elements and computational resource, if the same level of precision is to be maintained. After comparing a number of cylinders with different radii, it was found that, for load size of radius $a = 0.15$ m, when the radius of the cylinder exceeds 8 m, the results of finite element analysis become stable and no apparent changes can be found. In addition, displacements of the side surface of the finite-size cylinder along the x and y directions are set to be zero in ANSYS in order to mimic the infinite horizontal boundary condition of the multilayered solid.

Figure 6 shows the finite element model used in ANSYS. The upper plot in Fig. 6 shows a side view of the model. To increase numerical precision while maintaining computational burden, along the vertical z direction, the upper layers use dense elements while the lower layers use sparse elements. Along the radial r direction, the inner part of the cylinder uses dense elements while

Table 1 Parameters of a four-layer elastic solid

Parameters	Values	Parameters	Values
E_1	$0.1794 \times 10^{10} \text{ N/m}^2$ (0.26 Gpsi)	E_2	$0.138 \times 10^{10} \text{ N/m}^2$ (0.2 Gpsi)
E_3	$0.0138 \times 10^{10} \text{ N/m}^2$ (20 kpsi)	E_4	$0.0055 \times 10^{10} \text{ N/m}^2$ (8 kpsi)
ν_1	0.35	ν_3	0.35
ν_3	0.35	ν_3	0.3
ρ_1	140 pcf (2242.6 kg/m ³)	ρ_2	125 pcf (2002.3 kg/m ³)
ρ_3	125 pcf (2002.3 kg/m ³)	ρ_4	110 pcf (1762 kg/m ³)
Δh_1	0.1524 m (6 in.)	Δh_2	0.2032 m (8 in.)
Δh_3	0.3048 m (12 in.)	Δh_4	0.762 m (30 in.)
σ_0	$1.259 \times 10^5 \text{ N/m}^2$ ($P=2000 \text{ lb}$)	a	0.15 m

the outer part of the cylinder uses spare elements. The lower plot in Fig. 6 shows an overview of the model in which it can be seen that the horizontal boundary condition of the finite-size cylinder is completely restricted to simulate zero horizontal displacement of an infinite multilayered solid. Figure 7 shows surface deflection (vertical displacement) of the solid as a function of distance from the center of the load. In this figure, a continuous curve is obtained theoretically using a degraded dynamic solution (i.e., static solution) of this paper, while discrete dots are obtained from ANSYS. These results match very well with each other, indicating the correctness of the analytical derivation under static load, which verifies, in part, the theory and formula developed in this study.

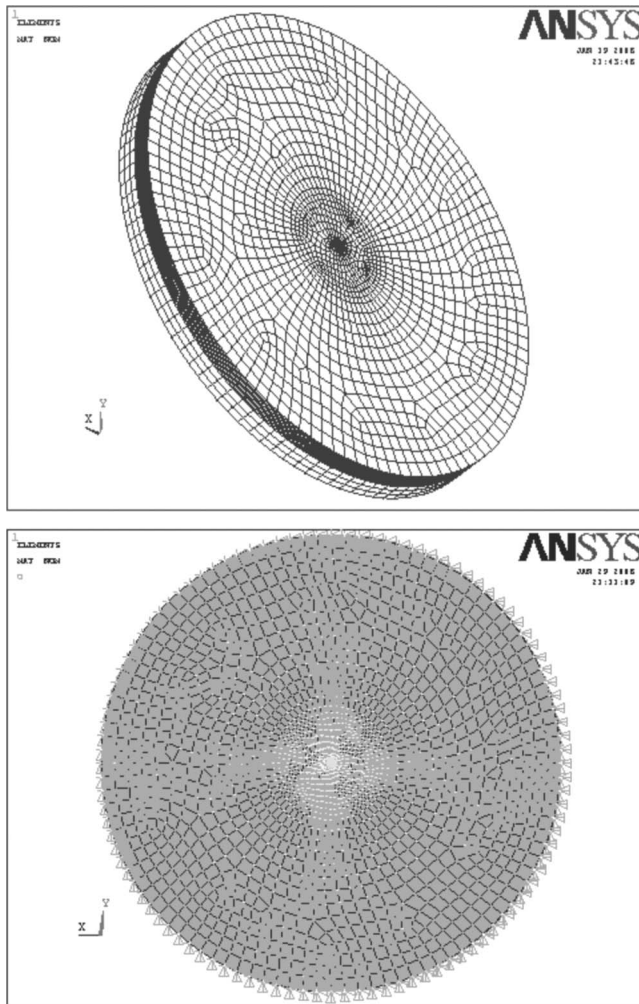


Fig. 6 The finite element model used in ANSYS for static analysis

8 Validation Through Transient Wave Propagation in a Half-Space Elastic Solid

An elastic half space is a special case of an elastic multilayered solid when material properties of each layer are identical. When the exerted transient load is a Heaviside function and has an infinite radius as its spatial distribution, dynamics of multilayered solids becomes a two-dimensional transient problem involving only the depth and the time. In this loading scenario, the analytical transient solution for an elastic half space is available in the literature [33], which is used to compare with solutions of this paper and the finite element analysis. Because the model used in the latter two approaches is associated with a bedrock foundation, the bedrock should be set to be deep enough, so that the effect of this rigid layer on dynamic surface displacement is negligible. In this study, the layer thickness on top of bedrock is set to be 1 m to mimic a half space, which has been shown to be able to provide an adequate precision.

The transient dynamic load used here is a Heaviside function given in Fig. 8. The pressure on the top of the cylinder is set to be zero at time $t=0$ s and then it instantly jumps to 600 kPa and remains at the same magnitude. Table 2 gives parameters used for transient loading and the elastic half space. In Table 2, h_0 , $h_{0.25}$, $h_{0.75}$, and $h_{0.75}$ correspond to four different depths from the surface, and $t=0.0137$ s represents the time duration within which the dynamic response is computed.

Figure 9 shows the finite element model used for numerical computation, in which the upper figure illustrates a side view of the finite cylinder, while the lower figure gives an overview of the cylinder where the divided elements can be seen. While performing finite element analysis, element SOLID 45 in ANSYS is used [47]. The analysis object is a cylinder of radius 0.1 m and depth 1 m. The length of element edge, a parameter used in specifying the element in ANSYS, is set to be 0.01 m. Divisions of elements are automatically calculated and rounded up to the next integer from line lengths. For the boundary conditions, displacements of the bottom of the layer along the x , y , and z directions are all set to zero, and the horizontal displacements of the side surface of the cylinder along x and y directions are set to zero.

Figure 10 shows the displacements at four different depths h_0 , $h_{0.25}$, $h_{0.75}$, and $h_{0.75}$ obtained from three methods, respectively. In this figure, solid lines represent theoretical results given by Erigen and Suhubi [33]. Discrete dots represent results derived from the theory of this paper. Discrete dots with cross represent numerical results obtained from finite element analysis using ANSYS. The derived analytical results in this paper match theoretical results surprisingly well, while numerical results of finite element analysis are close to those of this paper and theoretical results for depths h_0 , $h_{0.5}$, and $h_{0.75}$. For depth $h_{0.25}$, the numerical results deviate a little from results obtained using other two approaches within time duration [0.005 s, 0.01 s], which can be caused by numerical computation error, limited size of the cylinder as well as the number of elements in finite element analysis. In general, it can be stated that results obtained from different approaches are

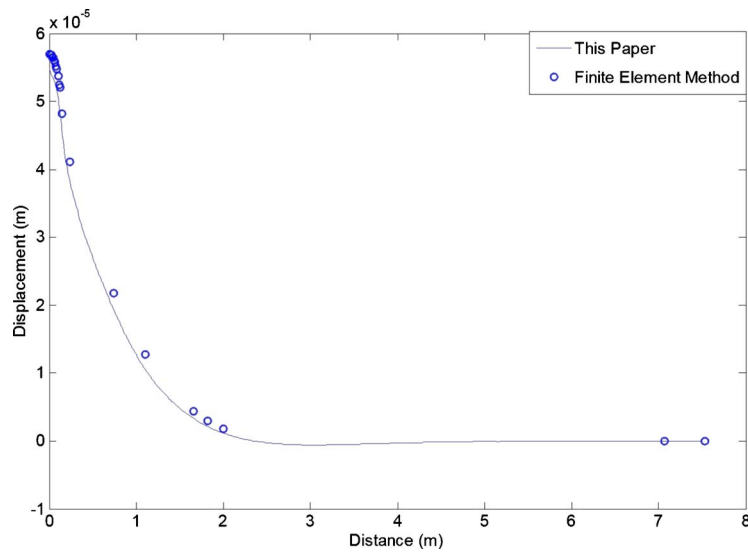


Fig. 7 Comparison between theoretical and numerical solutions of static loading

very close to each other, verifying the correctness and accuracy of the theory and derivation in this study.

9 Verification of Transient Wave Propagation in Multilayered Viscoelastic Solids

The previous comparisons validate the derivation and computation of this paper for multilayered elastic solids. A more extensive comparison is needed to examine the analysis of multilayered viscoelastic solids, of which no analytical results have been given in the literature. As such, only the result obtained from finite element analysis of multilayered viscoelastic solids can be used as a reference for comparison with the result obtained using DYNALAYER^T.

9.1 Specifying Constitutive Models of Viscoelastic Materials in ANSYS. First of all, various constitutive models of viscoelastic material used in DYNALAYER^T have to be modeled properly and consistently in ANSYS in order to make a meaningful comparison. Three methods are used in ANSYS to represent a viscoelastic ma-

terial [47]: the generalized Maxwell model, the Prony series, and the user defined model. The generalized Maxwell model is adopted here because all four viscoelastic models in Fig. 2 can be treated as a special case of the generalized Maxwell models.

In ANSYS, the stress function of a viscoelastic material using the generalized Maxwell model representation is given in an integral form. The constitutive equation for an isotropic viscoelastic material in the context of small strain theory can be written as [47]

$$\sigma = \int_0^t 2G(t-\tau) \frac{de}{d\tau} + \mathbf{I} \int_0^t K(t-\tau) \frac{d\Delta}{d\tau} d\tau \quad (46)$$

where σ is the Cauchy stress; e and Δ are the deviated and volumetric parts of the strains; $G(t)$ and $K(t)$ are the shear and bulk relaxation kernel functions, respectively; t is the present time and τ is the time argument; and $\mathbf{I}$ represents the unit tensor. The material model is available in viscoelastic elements VISCO88, and VISCO89 for small deformation viscoelasticity and elements LINK180, SHELL181, PLANE182, PLANE183, SOLID185,

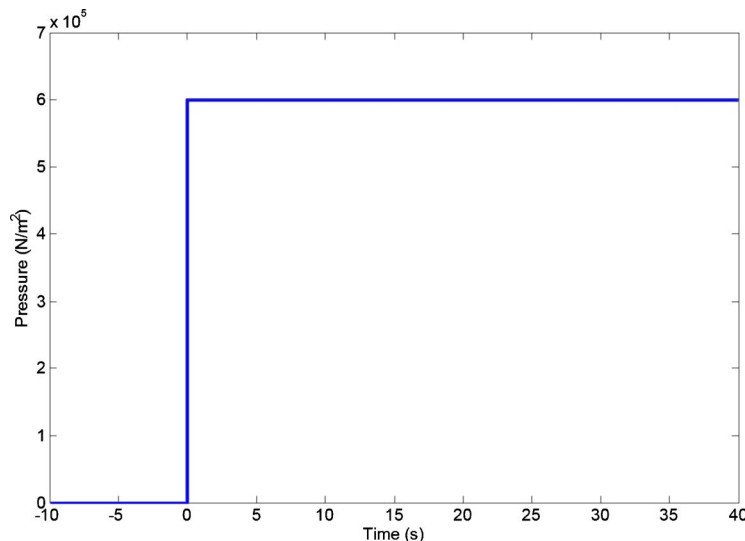


Fig. 8 Heaviside step function type of dynamic load used in half-space analysis

Table 2 Parameters of an elastic half space

Parameters	Values	Parameters	Values
E	2×10^8 Pa	ν	0.25
H	1 m	ρ	1800 kg/m ³
σ	6×10^5 Pa	r	10 m
h_0	0 m	$h_{0.25}$	0.25 m
$h_{0.5}$	0.5 m	$h_{0.75}$	0.75 m
t	0.0137 s		

SOLID186, SOLID187, BEAM188, and BEAM189 for small and large deformation viscoelasticity (ANSYS 2003).

For viscoelastic elements VISCO88 and VISCO89, the material properties are expressed in integral form (46) using the following representation of the kernel function (ANSYS 2003):

$$G(\zeta) = G_\infty + \sum_{i=1}^{n_G} G_i e^{(-\zeta/\lambda_i^G)} \quad (47a)$$

$$K(\zeta) = K_\infty + \sum_{i=1}^{n_K} K_i e^{(-\zeta/\lambda_i^K)} \quad (47b)$$

$$G_i = C_i(G_0 - G_\infty) \quad (48a)$$

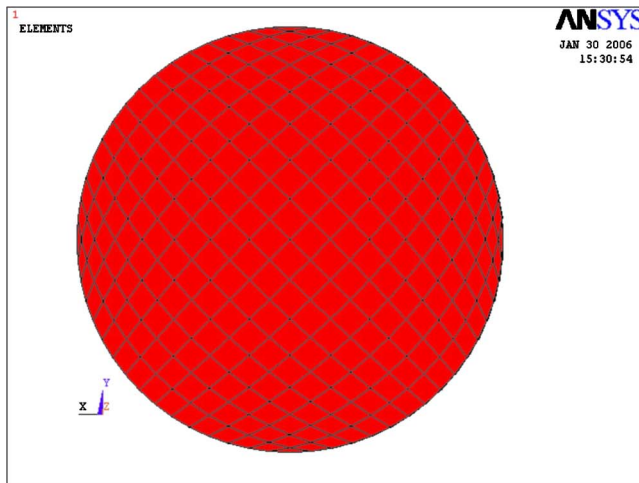
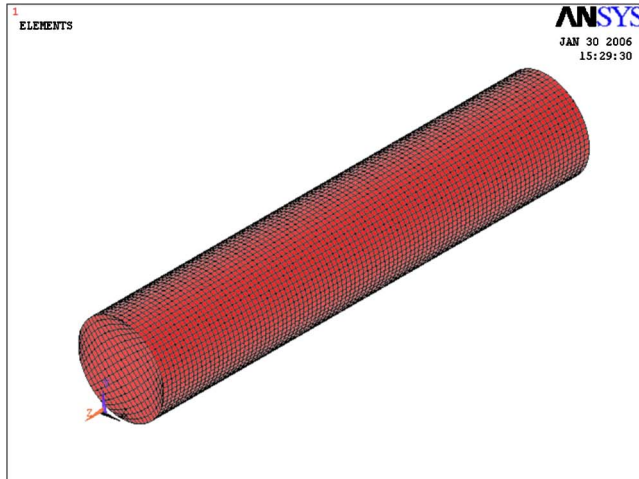


Fig. 9 The finite element model used in ANSYS for mimicking an elastic half space

$$K_i = D_i(K_0 - K_\infty) \quad (48b)$$

where ζ is the reduced or pseudotime; n_G and n_K are the numbers of Maxwell elements used to approximate the shear relaxation kernel and the bulk relaxation kernel; C_i and D_i are coefficients associated with the instantaneous response for shear behavior and bulk behavior; G_0 and K_0 represent the initial shear modulus and the initial bulk modulus; and G_∞ and K_∞ represent the final shear modulus and the final bulk modulus; and λ_i^G and λ_i^K are coefficients associated with a discrete relaxation spectrum in shear and in bulk, respectively. The number of Maxwell elements used for volumetric behavior n_K and the number of Maxwell elements used for shear behavior n_G need not to be the same, nor do coefficients λ_i^G and λ_i^K [47]. They are chosen to be the same because in this study, we assume identical viscoelastic behavior for the shear and bulk responses. Since VISCO88 is a two-dimensional element and VISCO89 is a three-dimensional element, the latter is employed here.

According to viscoelasticity theory, a viscoelastic material is often described by one of the following three equivalent integral forms:

$$\sigma(t) = \varepsilon_0 E(t) + \int_0^t E(t-t') \frac{d\varepsilon(t')}{dt'} dt' \quad (49a)$$

$$\sigma(t) = \varepsilon(t) E(0) + \int_0^t \varepsilon(t') \frac{dE(t-t')}{d(t-t')} dt \quad (49b)$$

$$\sigma(t) = \int_{t'=-\infty}^{t'=+\infty} E(t-t') d\varepsilon(t') \quad (49c)$$

If no initial strain exists in the material, the time dependent stress response is given by the so-called Boltzmann superposition integral

$$\sigma(t) = \int_0^t E(t-t') \frac{d\varepsilon(t')}{dt'} dt' \quad (50)$$

In order to use ANSYS to simulate the viscoelastic material and compare with the result derived from previous sections, it is necessary to represent the known relaxation modulus in Boltzmann superposition integral in terms of shear and bulk moduli used in ANSYS. In ANSYS, a generalized Maxwell model and (47a) and (47b) is specified by up to 95 coefficients. The first 45 coefficients are related to thermal effect and therefore are set to zero. The remaining 50 coefficients are related to viscoelasticity and can be determined by deriving (47a) and (47b) from the relaxation modulus of a specific viscoelastic model (e.g., Maxwell model) in Boltzmann superposition integral form.

For a Maxwell viscoelastic model, the relaxation modulus is given by $E(t) = E_0 e^{-t/T_0}$ [44]. Thus,

$$G(t) = \frac{E(t)}{2(1+\nu)} = \frac{E_0}{2(1+\nu)} e^{-t/T_0} \quad (51a)$$

$$K(t) = \frac{E(t)}{3(1-2\nu)} = \frac{E_0}{3(1-2\nu)} e^{-t/T_0} \quad (51b)$$

where ν is the Poisson ratio and assumed to be constant. Comparing (51a) and (51b) with (47a) and (47b), the following results are found:

$$G_\infty = 0, \quad C_1 = 1, \quad G_0 = \frac{E_0}{2(1+\nu)}, \quad \lambda_1^G = T_0, \quad n_G = 1$$

$$K_\infty = 0, \quad D_1 = 1, \quad K_0 = \frac{E_0}{3(1-2\nu)}, \quad \lambda_1^K = T_0, \quad n_K = 1$$

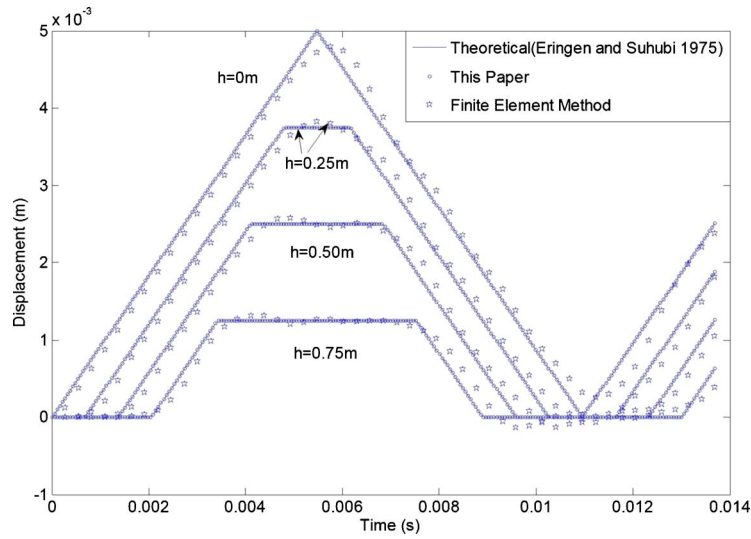


Fig. 10 Comparisons of transient displacement responses of an elastic half space to a two-dimensional Heaviside dynamic load using three different methods

For a Kelvin viscoelastic model, the relaxation modulus $E(t) = E_1 + \lambda_1 \delta(t)$, where $\delta(t)$ is the Dirac delta function. Following the relaxation modulus representation in ANSYS [47], in this study, $E'(t) = E_1 + (E_0 - E_1)e^{-t/T} = E_1(1 + (E_0/E_1 - 1)e^{-t/T})$ is used to approximate the relaxation modulus involving the Dirac delta function. Letting $\int_0^\infty E'(t)dt = \int_0^\infty E(t)dt$, it follows that $(E_0/E_1 - 1)T = \lambda_1/E_1 = T_1$. The ratio E_0/E_1 should be larger enough to provide an accurate approximation of the real relaxation modulus. In this study, we set $E_0/E_1 = 8.000000001 \times 10^9$. Therefore, we have $8 \times 10^9 T = T_1$ and $E'(t) = E_1 + (8.000000001 \times 10^9 E_1 - E_1)e^{-t/(T_1/8 \times 10^9)}$. Now,

$$G(t) = \frac{E'(t)}{2(1+\nu)} = \frac{E_1}{2(1+\nu)} + \left(\frac{8.000000001 \times 10^9 E_1}{2(1+\nu)} - \frac{E_1}{2(1+\nu)} \right) e^{-t/(T_1/8 \times 10^9)} \quad (52a)$$

$$K(t) = \frac{E(t)}{3(1-2\nu)} = \frac{E_1}{3(1-2\nu)} + \left(\frac{8.000000001 \times 10^9 E_1}{3(1-2\nu)} - \frac{E_1}{3(1-2\nu)} \right) e^{-t/(T_1/8 \times 10^9)} \quad (52b)$$

Comparing (52a) and (52b) with (47a) and (47b), the following results are obtained:

$$G_\infty = \frac{E_1}{2(1+\nu)}, \quad C_1 = 1, \quad G_0 = \frac{8.000000001 \times 10^9 E_1}{2(1+\nu)}$$

$$\lambda_1^G = \frac{T_1}{8 \times 10^9}, \quad n_G = 1$$

$$K_\infty = \frac{E_1}{3(1-2\nu)}, \quad D_1 = 1, \quad K_0 = \frac{8.000000001 \times 10^9 E_1}{3(1-2\nu)}$$

$$\lambda_1^K = \frac{T_1}{8 \times 10^9}, \quad n_K = 1$$

For a Burgers viscoelastic model, the relaxation modulus $E(t) = 1 / \sqrt{p_1^2 - 4p_2} [(q_1 - \alpha q_2)e^{-\alpha t} - (q_1 - \beta q_2)e^{-\beta t}]$, where $\alpha = 1/2p_2(p_1 - \sqrt{p_1^2 - 4p_2})$ and $\beta = 1/2p_2(p_1 + \sqrt{p_1^2 - 4p_2})$ (note that $\beta > \alpha$), $p_1 = (\lambda_0/E_0) + (\lambda_1/E_1) + (\lambda_0/E_1)$, $p_2 = (\lambda_0/E_0)(\lambda_1/E_1)$, $q_1 = \lambda_0$, $q_2 = \lambda_0 \lambda_1 / E_1$. Therefore, we have

$$G(t) = \frac{E(t)}{2(1+\nu)} = \frac{1}{2(1+\nu)\sqrt{p_1^2 - 4p_2}} [(q_1 - \alpha q_2)e^{-\alpha t} - (q_1 - \beta q_2)e^{-\beta t}] \quad (53a)$$

$$K(t) = \frac{E(t)}{3(1-2\nu)} = \frac{1}{3(1-2\nu)\sqrt{p_1^2 - 4p_2}} [(q_1 - \alpha q_2)e^{-\alpha t} - (q_1 - \beta q_2)e^{-\beta t}] \quad (53b)$$

Comparing (53a) and (53b) with (47a) and (47b), the following results are found:

$$G_\infty = 0, \quad C_1 = \frac{q_1 - \alpha q_2}{(\beta - \alpha)q_2}, \quad C_2 = \frac{\beta q_2 - q_1}{(\beta - \alpha)q_2}$$

$$G_0 = \frac{(\beta - \alpha)q_2}{2(1+\nu)\sqrt{p_1^2 - 4p_2}}$$

$$\lambda_1^G = \frac{1}{\alpha}, \quad \lambda_2^G = \frac{1}{\beta}, \quad n_G = 2$$

$$K_\infty = 0, \quad D_1 = \frac{q_1 - \alpha q_2}{(\beta - \alpha)q_2}, \quad D_2 = \frac{\beta q_2 - q_1}{(\beta - \alpha)q_2}$$

$$K_0 = \frac{(\beta - \alpha)q_2}{3(1-2\nu)\sqrt{p_1^2 - 4p_2}}$$

$$\lambda_1^K = \frac{1}{\alpha}, \quad \lambda_2^K = \frac{1}{\beta}, \quad n_K = 2$$

To summarize, Table 3 lists coefficients corresponding to the generalized Maxwell model representation in ANSYS for specifying

Table 3 Coefficients of the generalized Maxwell model in ANSYS [47]

Coefficient No. in ANSYS	Maxwell model	Kelvin model	Burgers model
46	$E_0/[2(1+\nu)]$	$8 \times 10^9 E_1/[2(1+\nu)]$	$[(\beta-\alpha)q_2]/[2(1+\nu)\sqrt{p_1^2-4p_2}]$
47	0	$E_1/[2(1+\nu)]$	0
48	$E_0/[3(1-2\nu)]$	$8 \times 10^9 E_1/[3(1-2\nu)]$	$[(\beta-\alpha)q_2]/[3(1-2\nu)\sqrt{p_1^2-4p_2}]$
49	0	$E_1/[3(1-2\nu)]$	0
50	1	1	2
51	1	1	$(q_1-\alpha q_2)/[(\beta-\alpha)q_2]$
52	0	0	$(\beta q_2-q_1)/[(\beta-\alpha)q_2]$
61	T_0	$T_1/(8 \times 10^9)$	$1/\alpha$
62	0	0	$1/\beta$
71	1	1	2
76	1	1	$(q_1-\alpha q_2)/[(\beta-\alpha)q_2]$
77	0	0	$(\beta q_2-q_1)/[(\beta-\alpha)q_2]$
86	T_0	$T_1/(8 \times 10^9)$	$1/\alpha$
87	0	0	$1/\beta$

ing Maxwell model, Kelvin model, and Burgers model in Fig. 2. Coefficients of coefficient numbers not listed in Table 3 are all zeros.

9.2 Numerical Comparison and Verification. In this comparison, a four-layer solid identical to that of the previous section is used, except that in the former, the material property is vis-

coelastic. In ANSYS [47], a finite-size cylinder of radius 8 m and of depth 1.4224 m is used to model the four-layer viscoelastic solid, which is large enough relative to the loading area. The mesh grid of the cylinder is shown in Fig. 11. Displacements at the bottom of the layer along the x , y , and z directions are set to zero, and the horizontal displacements of the side surface of the cylinder along x and y directions are set to zero. In this comparison, the used transient load is a circular load of radius 0.15 m. As plotted in Fig. 12(a), it takes an impulse function $\sigma = \alpha t e^{-\beta t}$ as its magnitude, in which coefficients $\alpha = 2.4491 \times 10^8$ and $\beta = 150$. In three viscoelastic models (Kelvin, Maxwell, and Burgers), the parameters are set to $T_0 = T_1 = 0.02$ s. Moreover, in Burgers model, the moduli of two springs are set to be equal to half of the elastic modulus of the same layer.

Transient dynamic displacements of two Points A and B on the surface of the multilayered solid are evaluated using DYNALAYER^T and ANSYS, respectively. Point A is located exactly at the center of the exerted impact load, while Point B is apart from the center of the load at a distance of 0.91 m. It is found that the number of iterations has a higher effect on computational accuracy than that of elements, especially for the Kelvin model. In this comparison, the total analysis duration is set to 0.12 s. Two numbers of iterations are used: 128 iterations for Maxwell model and Burgers model, and 256 iterations for Kelvin model, respectively.

Figure 12(b), 8(c), and 8(d) show the comparison using DYNALAYER^T and ANSYS. Almost perfect consistency between the result of this paper and that of finite element analysis is observed for all three viscoelastic models. This further proves the validity of the theory and the software developed in this paper. In addition, it should be stressed that computational efficiency of DYNALAYER^T is much higher than that of ANSYS. For instance, it took DYNALAYER^T and ANSYS (with 2000 elements) 8 min and 686 min to run this case, respectively. Even when reducing the number of finite elements to 920, ANSYS still requires 65 min of execution time in order to achieve the same level of accuracy.

10 Conclusions

Based on the theory and formula developed, numerical algorithms are designed and a computer program DYNALAYER^T is developed in MATLAB environment to implement efficient numerical computation. The results obtained in this paper have been validated by comparing with results available in the literature and results obtained from the finite element analysis using ANSYS. The following concluding remarks can be made.

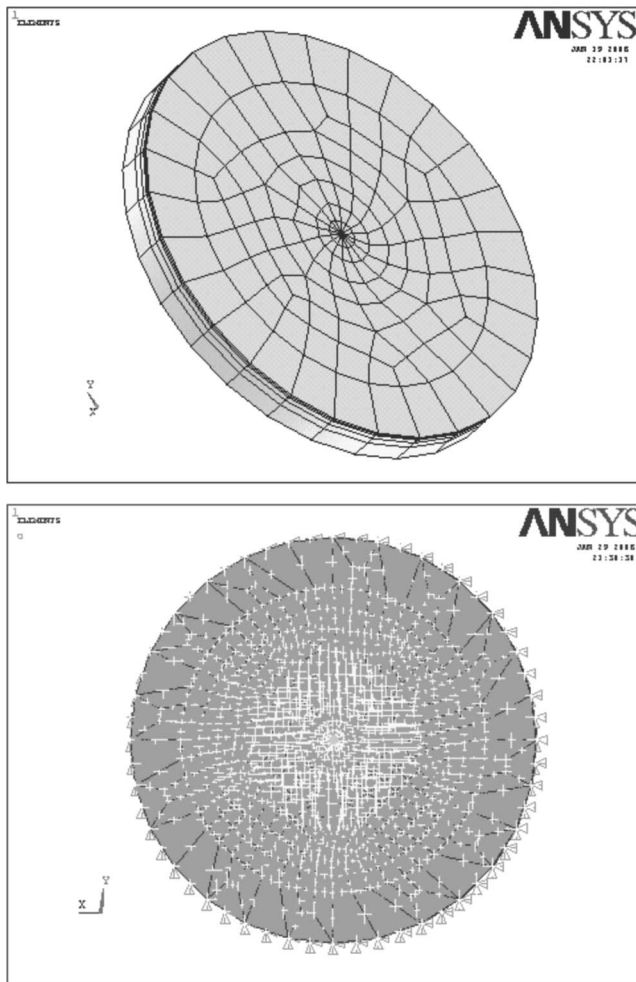


Fig. 11 The finite element model for viscoelastic multilayered solid analysis

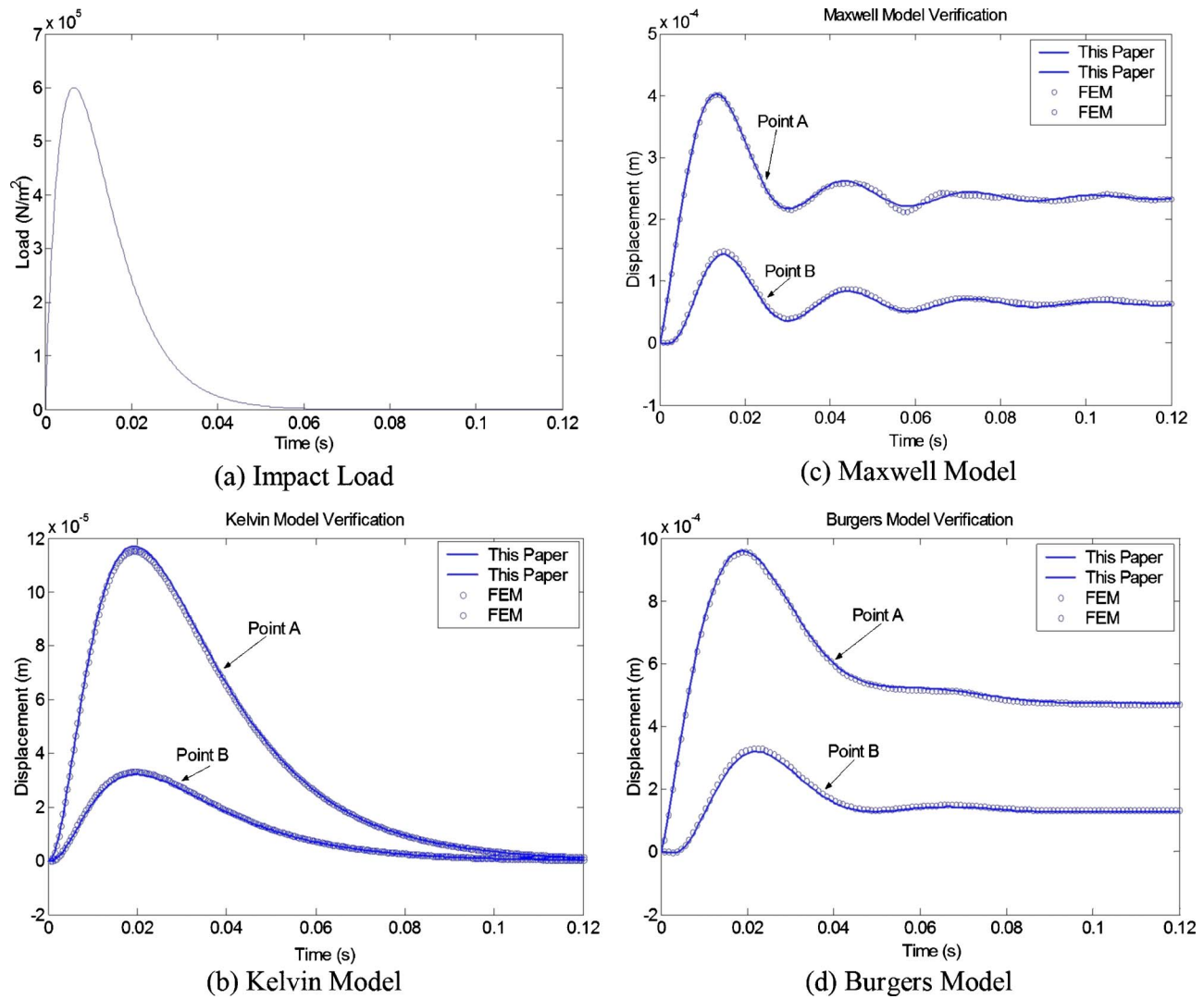


Fig. 12 Comparison of a four-layer viscoelastic solid

- The derived formulation and developed computer program theory can be used easily to construct and compute Green's function (fundamental solution) of multilayered viscoelastic solid, which is of paramount importance to many applications.
- Because we use Laplace transform rather than Fourier transform in the analysis as well as the adoption of fast algorithm for evaluating Laplace and Hankel transforms, transient wave propagation in viscoelastic solids with any specific initial condition and arbitrary dynamic load can be effectively and efficiently analyzed.
- Different types of viscoelastic models considered here allow any complicated viscoelastic constitutive relationship to be captured or approximated.
- The result of this study can result in a number of engineering applications, such as nondestructive evaluation of highway and airport pavements, petroleum exploration, counter-

mine technology, geophysical inversion, structural health monitoring, and vehicle weigh-in-motion systems.

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Appendix A

A MATLAB code for evaluating inverse Laplace transform is given below, in which C_s , C_d , S , ξ , Q , ρ , $\bar{u}_{2o}$, $\bar{w}_{2o}$, $\bar{\tau}_{2o}$, $\bar{\sigma}_{2o}$, μ , λ , z , $\bar{\mu}$, $\bar{w}$, $\bar{u}$, and $\bar{w}$, respectively.

syms Cs Cd S Xi Q Rho Uzo Wzo Tzo Dzo Mu Lam Z

$$\begin{aligned} \text{Ubar} = & ((\text{Cs}^2 \cdot \text{S}^2 \cdot \text{Uzo} - (\text{Cd}^2 - 2 \cdot \text{Cs}^2) \cdot \text{Xi} \cdot \text{Wzo} + 1/\text{Rho} \cdot \text{Tzo}) \cdot (\text{Cd}^2 \cdot \text{S}^2 - \text{Q}^2 - \text{Cs}^2 \cdot \text{Xi}^2) \dots \\ & + (\text{Cd}^2 - \text{Cs}^2) \cdot \text{Xi} \cdot \text{S} \cdot (\text{Cd}^2 \cdot \text{S} \cdot \text{Wzo} + (\text{Cd}^2 - \text{Cs}^2 - \text{Lam}/\text{Rho}) \cdot \text{Xi} \cdot \text{Uzo} + 1/\text{Rho} \cdot \text{Dzo})) / \dots \\ & ((\text{Cs}^2 \cdot \text{S}^2 - \text{Q}^2 - \text{Cd}^2 \cdot \text{Xi}^2) \cdot (\text{Cd}^2 \cdot \text{S}^2 - \text{Q}^2 - \text{Cs}^2 \cdot \text{Xi}^2) + \dots \\ & (\text{Cd}^2 - \text{Cs}^2)^2 \cdot \text{Xi}^2 \cdot \text{S}^2); \end{aligned}$$

$$\begin{aligned} \text{Wbar} = & ((\text{Cs}^2 \cdot \text{S}^2 - \text{Q}^2 - \text{Cd}^2 \cdot \text{Xi}^2) \cdot (\text{Cd}^2 \cdot \text{S} \cdot \text{Wzo} + (\text{Cd}^2 - \text{Cs}^2 - \\ & \text{Lam}/\text{Rho}) \cdot \text{Xi} \cdot \text{Uzo} + 1/\text{Rho} \cdot \text{Dzo})) \dots \\ & - (\text{Cs}^2 \cdot \text{S} \cdot \text{Uzo} - (\text{Cd}^2 - 2 \cdot \text{Cs}^2) \cdot \text{Xi} \cdot \text{Wzo} + 1/\text{Rho} \cdot \text{Tzo}) \cdot (\text{Cd}^2 - \text{Cs}^2) \cdot \text{Xi} \cdot \text{S} / \dots \\ & ((\text{Cs}^2 \cdot \text{S}^2 - \text{Q}^2 - \text{Cd}^2 \cdot \text{Xi}^2) \cdot (\text{Cd}^2 \cdot \text{S}^2 - \text{Q}^2 - \text{Cs}^2 \cdot \text{Xi}^2) + \dots \\ & (\text{Cd}^2 - \text{Cs}^2)^2 \cdot \text{Xi}^2 \cdot \text{S}^2); \end{aligned}$$

$$\text{U} = \text{ilaplace}(\text{Ubar}, \text{S}, \text{Z})$$

$$\text{W} = \text{ilaplace}(\text{Wbar}, \text{S}, \text{Z})$$

Appendix B

$$\phi_{11} = \frac{\alpha^2 + \xi^2}{p^2} \cosh(\alpha z) - \frac{2\xi^2}{p^2} \cosh(\beta z) \quad (\text{B1a})$$

$$\phi_{12} = \frac{2\xi\alpha}{p^2} \sinh(\alpha z) - \frac{\xi(\alpha^2 + \xi^2)}{\beta p^2} \sinh(\beta z) \quad (\text{B1b})$$

$$\phi_{13} = \frac{1}{\mu} \left[\frac{\alpha}{p^2} \sinh(\alpha z) - \frac{\xi^2}{\beta p^2} \sinh(\beta z) \right] \quad (\text{B1c})$$

$$\phi_{14} = \frac{1}{\mu} \left[\frac{\xi}{p^2} \cosh(\alpha z) - \frac{\xi}{p^2} \cosh(\beta z) \right] \quad (\text{B1d})$$

$$\phi_{21} = \frac{2\xi\beta}{p^2} \sinh(\beta z) - \frac{\xi(\alpha^2 + \xi^2)}{\alpha p^2} \sinh(\alpha z) \quad (\text{B2a})$$

$$\phi_{22} = \frac{\alpha^2 + \xi^2}{p^2} \cosh(\beta z) - \frac{2\xi^2}{p^2} \cosh(\alpha z) \quad (\text{B2b})$$

$$\phi_{23} = \frac{1}{\mu} \left[\frac{\xi}{p^2} \cosh(\beta z) - \frac{\xi}{p^2} \cosh(\alpha z) \right] \quad (\text{B2c})$$

$$\phi_{24} = \frac{1}{\mu} \left[\frac{\beta}{p^2} \sinh(\beta z) - \frac{\xi^2}{\alpha p^2} \sinh(\alpha z) \right] \quad (\text{B2d})$$

$$\alpha^2 = \xi^2 + \frac{q^2}{c_s^2} \quad (\text{B3a})$$

$$\beta^2 = \xi^2 + \frac{q^2}{c_d^2} \quad (\text{B3b})$$

$$p^2 = \frac{q^2}{c_s^2} \quad (\text{B3c})$$

$$\phi_{31} = \mu \left[\frac{(\alpha^2 + \xi^2)^2}{\alpha p^2} \sinh(\alpha z) - \frac{4\xi^2\beta}{p^2} \sinh(\beta z) \right] \quad (\text{B4a})$$

$$\phi_{32} = \mu \frac{2\xi(\alpha^2 + \xi^2)}{p^2} [\cosh(\alpha z) - \cosh(\beta z)] \quad (\text{B4b})$$

$$\phi_{33} = \frac{\alpha^2 + \xi^2}{p^2} \cosh(\alpha z) - \frac{2\xi^2}{p^2} \cosh(\beta z) \quad (\text{B4c})$$

$$\phi_{34} = \frac{\xi(\alpha^2 + \xi^2)}{\alpha p^2} \sinh(\alpha z) - \frac{2\xi\beta}{p^2} \sinh(\beta z) \quad (\text{B4d})$$

$$\phi_{41} = \mu \frac{2\xi(\alpha^2 + \xi^2)}{p^2} [\cosh(\beta z) - \cosh(\alpha z)] \quad (\text{B5a})$$

$$\phi_{42} = \mu \left[\frac{(\alpha^2 + \xi^2)^2}{\beta p^2} \sinh(\beta z) - \frac{4\xi^2\alpha}{p^2} \sinh(\alpha z) \right] \quad (\text{B5b})$$

$$\phi_{43} = \frac{\xi(\alpha^2 + \xi^2)}{\beta p^2} \sinh(\beta z) - \frac{2\xi\alpha}{p^2} \sinh(\alpha z) \quad (\text{B5c})$$

$$\phi_{44} = \frac{\alpha^2 + \xi^2}{p^2} \cosh(\beta z) - \frac{2\xi^2}{p^2} \cosh(\alpha z) \quad (\text{B5d})$$

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Modeling of a One-Sided Bonded and Rigid Constraint Using Beam Theory

In beam theory, constraints can be classified as fixed/pinned depending on whether the rotational stiffness of the support is much greater/less than the rotational stiffness of the freestanding portion. For intermediate values of the rotational stiffness of the support, the boundary conditions must account for the finite rotational stiffness of the constraint. In many applications, particularly in microelectromechanical systems and nanomechanics, the constraints exist only on one side of the beam. In such cases, it may appear at first that the same conditions on the constraint stiffness hold. However, it is the purpose of this paper to demonstrate that even if the beam is perfectly bonded on one side only to a completely rigid constraining surface, the proper model for the boundary conditions for the beam still needs to account for beam deformation in the bonded region. The use of a modified beam theory, which accounts for bending, shear, and extensional deformation in the bonded region, is required in order to model this behavior. Examples are given for cantilever, bridge, and guided structures subjected to either transverse loads or residual stresses. The results show significant differences from the ideal bond case. Comparisons made to a three-dimensional finite element analysis show a good agreement.
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1 Introduction

If a beam is attached at one or more ends to a support, the nature of the boundary condition at the support depends on the relative stiffness of the beam to that of the support. In particular, if the rotational stiffness of the support is much greater/less than the rotational bending stiffness of the beam, then the constraint is modeled as fixed/pinned. An example of such a classical fixed constraint is shown in Fig. 1(a). For intermediate values of rotational stiffness, the appropriate boundary condition must include the rotational compliance of the support.

However, there are many applications in which the beam is constrained only on one side (Figs. 1(b) and 1(c)). Typically, such configurations occur in microelectromechanical systems (MEMS) and nanomechanics applications. These beams are fabricated on a substrate after which a release process is used where a portion of the substrate material is removed by selective etching. The resulting cantilever (Fig. 1(b)) or “fixed-fixed” beam (i.e., a bridge, Fig. 1(c)) is constrained on its lower surface only. If the rotational stiffness of the substrate is much greater than that of the beam, it would be tempting to model these constraints as fixed. However, the one-sided nature of the constraint introduces some rotational compliance into the structure, which can affect the modeling of the constraint.

It will be shown here that a bending moment M_0 applied to the fixed (anchor) portion of the beam will produce a finite angle of rotation even for a perfect bond to an ideally rigid substrate (Fig. 2(a)). This rotation angle cannot be predicted by classical beam theory in which cross sections are forced to remain perpendicular to the beam centroidal axis. Thus, in this paper, we implement a modified beam theory that accounts for deformations due to bending, shear, and extension in the anchor region and is therefore able to model the rotational stiffness in the anchor region. The correction factors that account for displacements due to rotational compliance are determined and found to depend on the ratio of the

length of the suspended section of the beam to its thickness. These corrections are typically comparable to or greater than the additional corrections due to shear deformation in the suspended portion of the beam. The results obtained compare favorably with a three-dimensional finite element analysis.

A similar mechanism occurs due to residual stresses (Fang and Wickert [1]; Greek and Chitica [2]). Suppose that a uniform tensile prestress is present in a thin film that has been deposited on a substrate (Fig. 2(b)). After patterning and release of a cantilever, the bottom surface of the film in the anchor region is constrained against horizontal displacement, but the top surface is not. Thus, the top surface contracts, whereas the bottom surface is constrained, resulting in what has been appropriately called a “take-off” angle (θ_0) [1], as illustrated in Fig. 2(c). In the freestanding portion, the beam simply contracts uniformly with little consequence. In Ref. [1], plane strain finite element modeling was used to develop curve-fit equations for the take-off angle. We will use a modified beam theory for a cantilever and show that it gives similar results to the finite element analysis of Ref. [1]. We then apply our model to determine deflections due to a residual stress in a bridge structure.

2 Beam Theory With Bending, Shear, and Extensional Deformation

An element through the thickness of the beam of differential length dx is shown in Fig. 3. By using the equilibrium of the differential element (Fig. 3(a)),

$$\frac{dQ}{dx} = -p, \quad \frac{dM}{dx} = Q - T \frac{dw}{dx} + \frac{qh}{2}, \quad \frac{dT}{dx} = -q \quad (1)$$

is obtained. Then, using the deformation relations (e.g., Refs. [3,4]), the following are obtained:

$$Q = \kappa^2 GA \left(\frac{dw}{dx} + \psi \right), \quad M = EI \frac{d\psi}{dx}, \quad T = EA \frac{du}{dx} \quad (2)$$

In Eqs. (1) and (2), T is the axial force, Q is the internal shear force, M is the internal bending moment, p and q are the applied vertical and horizontal loads per unit length, respectively, u (Fig.

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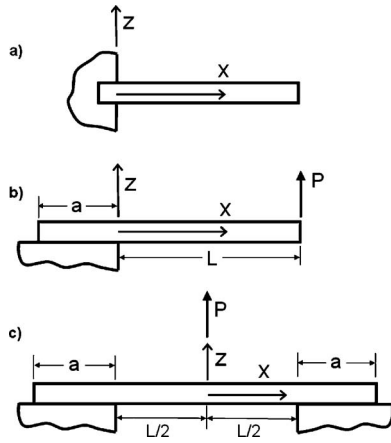


Fig. 1 (a) A beam with a classical fixed support. (b) A cantilever beam with a one-sided fixed support. (c) A beam with one-sided fixed supports at each end, i.e., a bridge structure.

3(b)) and w are the axial and transverse displacements of the beam axis in the x and z directions, respectively, ψ (Fig. 3(c)) is the rotation angle of the cross section, E is the Young's modulus, κ^2 is the shear correction factor [3], G is the shear modulus, which is related to the Young's modulus and Poisson's ratio by $E=2G(1+\nu)$, and $I=bh^3/12$ is the second moment of the rectangular cross-sectional area A of height h and width b .

Combining Eqs. (1) and (2) gives

$$\frac{d}{dx} \left[\kappa^2 GA \left(\frac{dw}{dx} + \psi \right) \right] + p = 0 \quad (3)$$

$$\frac{d}{dx} \left(EI \frac{d\psi}{dx} \right) - \kappa^2 GA \left(\frac{dw}{dx} + \psi \right) + T \frac{dw}{dx} = \frac{qh}{2} \quad (4)$$

$$\frac{d}{dx} \left(EA \frac{du}{dx} \right) = -q \quad (5)$$

Note that the modified beam theory given by Eqs. (3)–(5) includes *deformations* due to bending, extension, and shear, all three of which will be needed to describe the deformation in the anchor region.

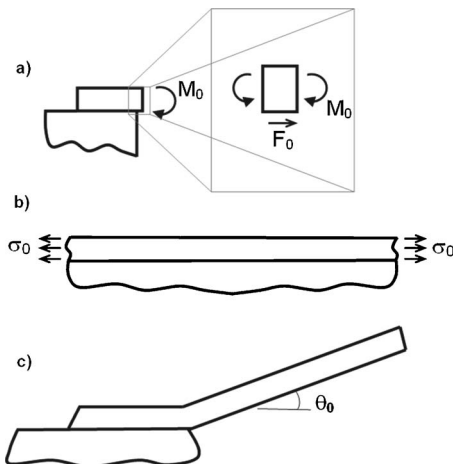


Fig. 2 (a) A one-sided support subjected to an applied moment M_0 . (b) A thin film under a uniform tensile prestress before release. (c) The anchor and the free portions of a cantilever beam after release from the substrate.

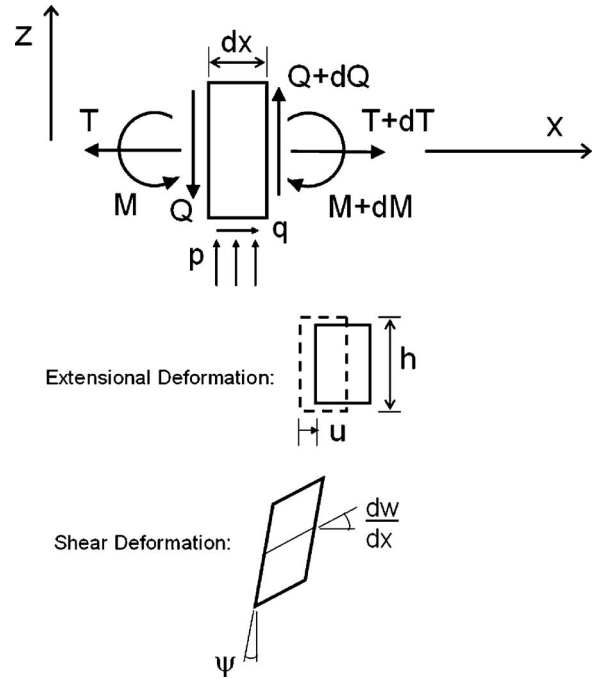


Fig. 3 Differential elements for a modified beam theory that includes extensional, shear, and bending deformations

3 Anchor Region

We now consider the anchor region, i.e., the portion of the beam which is bonded on one side only to the rigid surface (Figs. 1(b) and 1(c)). The constraint conditions for the anchor region are

$$w = 0, \quad u - \frac{h}{2}\psi = 0, \quad x < 0 \quad (6)$$

i.e., there is zero transverse displacement, and points on the bottom surface of the beam do not move horizontally. Combining Eqs. (4)–(6) for a uniform rectangular cross-section beam gives

$$\frac{EAh^2}{3} \frac{d^2\psi}{dx^2} - \kappa^2 GA \psi = 0 \quad (7)$$

Solving for ψ yields

$$\psi(x) = C_1 \sinh \alpha x/h + C_2 \cosh \alpha x/h \quad (8)$$

where the constants of integrations are yet to be determined and

$$\alpha = \sqrt{\frac{3\kappa^2}{2(1+\nu)}} \quad (9)$$

The effect of a bending moment applied to the anchor region (Fig. 2(a)) can now be determined. However, if a bending moment $M=EI(d\psi/dx)$ at $x=0^+$ is specified, then by the third equation in Eq. (2) and the second in Eq. (6), an unknown reactive axial force F_0 at the anchor corner ($x=0^-$) also exists. Thus, applying

$$EA \frac{du}{dx} \Big|_{x=0^-} = F_0, \quad M|_{x=0} = EI \frac{d\psi}{dx} \Big|_{x=0^-} = M_0 - F_0 h/2 \quad (10)$$

and

$$EA \frac{du}{dx} \Big|_{x=-a} = 0, \quad M|_{x=-a} = EI \frac{d\psi}{dx} \Big|_{x=-a} = 0 \quad (11)$$

leads to

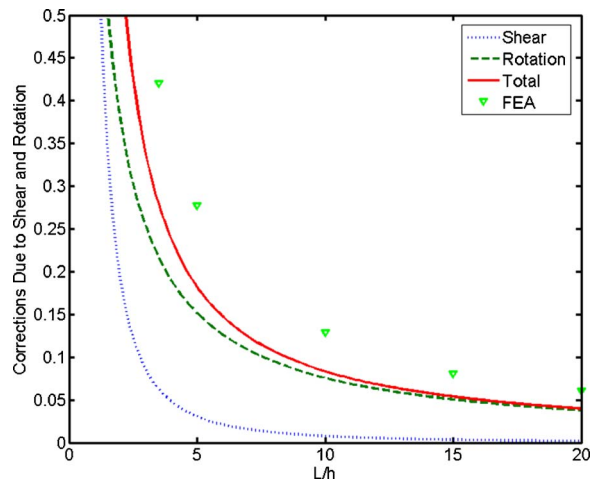


Fig. 4 Correction factors due to shear deformation and rotational compliance for a cantilever beam with an end load

$$C_1 = 3M_0/EAh^2\alpha, \quad C_2 = C_1/\tanh(\alpha a/h) \quad (12)$$

Thus, a beam that is perfectly bonded on one side only to an ideally rigid surface behaves as a rotational spring with finite stiffness given by

$$K_r = \frac{M_0}{\psi(0)} = \frac{EAh\alpha}{3} \tanh \frac{\alpha a}{h} \quad (13)$$

4 Examples for Structures With Transverse Loads

4.1 Cantilever Beam. Consider first a cantilever beam with a concentrated load at its end (Fig. 1(b)). The rotation of the fixed end will be equal to $\psi_0 = PL/K_r$, and thus the end deflection will increase by $\psi_0 L = PL^2/K_r$. Hence,

$$w(L) = \frac{PL^3}{3EI} \left(1 + \frac{(1+\nu)}{2\kappa^2} \left(\frac{h}{L} \right)^2 + \frac{3}{2} \gamma \right) \quad (14)$$

where

$$\gamma = \frac{h/L}{2\alpha \tanh(\alpha a/h)} \quad (15)$$

In Eq. (14), the first term is due to cantilever bending, the second term is the usual correction due to shear deformation of the free portion of the cantilever [3], and the third term is due to the rotational compliance of the support. Note that the classical shear correction term varies as the square of h/L , whereas the rotational stiffness term varies directly as h/L .

In Fig. 4, the importance of shear deformation and rotational compliance (for a long bond length, i.e., $a \gg h$) is shown, each compared to the end deflection due to bending, as functions of the length-to-thickness ratio. Also shown is the total correction due to the combined effects of shear deformation and rotational compliance. Thus, these results are the second (for shear) and third (for rotation) terms of Eq. (14) for a Poisson's ratio of $\nu=0.3$, which leads to $\kappa^2=0.850$ [3] and $\alpha \approx 0.990$. For a completely fixed end, the effect of shear deformation is only about 3.1% for $L/h=5$. However, the rotational compliance of the constraint has a 15.1% effect compared with bending for a total correction of 18.2%.

Also shown in Fig. 4 are numerical results of a finite element simulation using the three-dimensional 20-node solid 95 elements of ANSYS®. These elements were distributed uniformly along the length, width, and thickness directions, with an element length no greater than one-fifth the height of the square cross section. The overlap length a was taken to be equal to the suspended length of the beam L . Convergence was assured by approximately doubling the number of elements until the deflection changed by less than

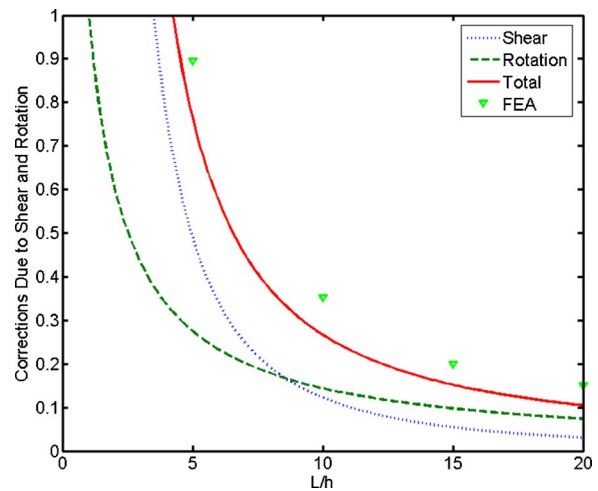


Fig. 5 Correction factors due to shear deformation and rotational compliance for a bridge structure with a load at its midpoint

1%, although, on average, the deflection changed by about 0.3% upon convergence. This procedure resulted in a minimum of 9662 elements (longer beams required more elements). As can be seen, there is a good agreement with the finite element analysis, which displays somewhat greater displacements. It is emphasized that the curves shown in Fig. 4 are *corrections* to the classical beam theory. A comparison of the end deflections for $L/h=20, 15, 10, 5$, and 3.5 gives differences between the finite element analysis and the modified beam theory of 2.0%, 2.5%, 4.0%, 7.4%, and 9.9%, respectively. It is noted that the modified beam theory does not include thickness deformation, which may contribute to the small discrepancy with the finite element analysis.

Although the effect of rotational compliance is greater for shorter bond lengths, that dependence is not strong for reasonable values of a/h . For example, when $a/h=2$, the rotational stiffness decreases by only 3.7%. It is noted that for $a \gg h$, Eq. (15) becomes $\gamma = h/2\alpha L$.

4.2 Bridge Structure. A similar procedure can be used for the bridge structure (Fig. 1(c)), which, for a classical two-sided constraint, corresponds to a fixed-fixed beam. The algebra is more complicated and is omitted for brevity. The result is

$$w(0) = \frac{PL^3}{192EI} \left(1 + \frac{8(1+\nu)}{\kappa^2} \left(\frac{h}{L} \right)^2 + \frac{3\gamma}{1+\gamma} \right) \quad (16)$$

where, again, the first term is due to bending, the second is due to shear, and the third is due to rotational compliance. It is also noted that because the rotational compliance of the support is included, the bridge configuration does *not* correspond to a combination of four cantilever beams each of one-quarter the bridge length.

Numerical results are shown in Fig. 5, also for a Poisson's ratio of 0.3 and for infinitely long one-sided bonds. For a classical fixed-fixed beam, the effect of shear deformation is about 12.2% for $L/h=10$. However, for $L/h=10$, the rotational stiffness of the constraint has a 14.4% effect for a total correction of 26.6%. Also, because the effect of rotation compliance varies approximately as h/L (Eqs. (15) and (16), with $a \gg h$), its relative effect is more important than classical shear deformation, which varies as $(h/L)^2$ as L/h increases. Also shown in Fig. 5 are results of the finite element simulation, which was performed in the same manner as for the cantilever. Convergence required a minimum of 28,773 elements; on average, the deflection changed by about 0.5% upon convergence. Again, there is a good agreement between the modified beam theory and the finite element analysis, which displays somewhat greater displacements. A comparison of the end deflec-

tions for $L/h=20, 15, 10, 5$, and 3.5 gives differences between the finite element analysis and the modified beam theory of 4.0%, 4.0%, 6.4%, 6.9%, and 11.4%, respectively.

4.3 Guided Support. In a guided support, one end of the beam is fixed whereas the other is rigidly attached to a substrate, which is allowed to move vertically without rotation. It is noted that the deflection at the end of this configuration of length L is twice that of a cantilever of length $L/2$. Thus, the correction terms for this configuration are also given by Fig. 4, but with L representing one-half the length of the guided beam.

5 Structures With Residual Stress

Let σ_0 be the uniform prestress acting in the axial direction before the structure is released. After release of the structure, the bottom surface of the film in the anchor region is constrained against horizontal displacement, but the top surface is unconstrained. The result is a residual stress, the effect of which we investigate in this section. Thus, for a tensile/compressive prestress, the top surface contracts/expands whereas the bottom surface is prevented from contracting or expanding, resulting in a take-off angle [1]. Both cantilever and bridge structures are considered.

5.1 Cantilever Beam. Note that in the suspended part of the cantilever (Fig. 2(c)), the uniform tensile/compressive prestress is relieved by an inconsequential contraction/expansion during release. However, the residual stress in the anchor region leads to the development of a take-off angle [1]. Similarly, a stress gradient (if it exists) is also completely relieved in the suspended section, producing a residual curl as well as a take-off angle, as described in Ref. [1]. In particular, a prestress gradient that varies from tension/compression on the top of the film to compression/tension at the bottom of the film will, upon release, produce an upward/downward residual curl. The deflection due to that residual curl typically dominates the deflection due to the take-off angle [1]. That curl is described in Ref. [5] and does not depend on shear deformation. Thus, in this work, we only consider the take-off angles due to the uniform part of the prestress.

Now, consider the anchor region of the film that is bonded to the substrate (Fig. 2(c)). It is noted that in the classical beam theory, shear deformation is neglected and, consequently, the rotation of a cross section always remains perpendicular to the centroidal axis. Hence, the classical beam theory treats the bonded region as a fixed support, and there is a zero slope and a zero rotation angle at the ends of the anchor region and at the beginning of the suspended section. Again, it can be shown that the inclusion of shear deformation alone will not overcome this deficiency. Thus, in the following analysis, we continue to include both shear and extensional deformations in the anchor region.

The conditions that at the bonded interface, the displacements due to the prestresses are unaffected by the release process are

$$w = 0, \quad u - \frac{h}{2}\psi = \frac{\sigma_0}{E}x + C_0 \quad (17)$$

where $C_0=0$ is arbitrary. In Eq. (17), the displacements are measured from the unstressed configuration. By combining Eqs. (4), (5), and (17), the rotation angle can still be expressed by Eqs. (7) and (8). Similarly, Eqs. (10) and (11) are replaced by

$$M|_{x=0,-a} = EI \left. \frac{d\psi}{dx} \right|_{x=0,-a} = -F_0 \frac{h}{2}, \quad T|_{x=0,-a} = EA \left. \frac{du}{dx} \right|_{x=0,-a} = F_0 \quad (18)$$

Here, there exist concentrated reaction forces (F_0) exerted by the substrate on the bonded part of the film at both ends of the bond. Solving Eqs. (8) and (17), subject to conditions (18) results in a take-off angle θ_0 given by

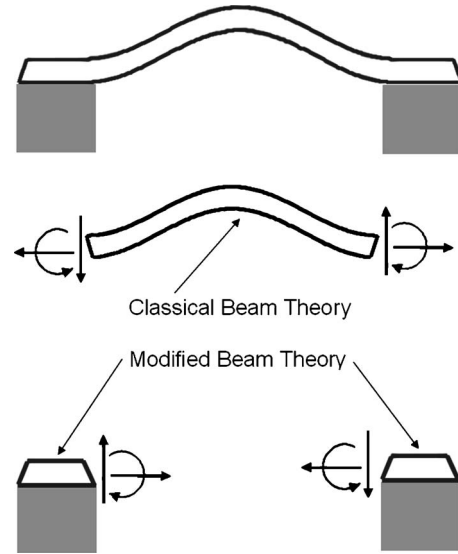


Fig. 6 Deformation of different regions of a bridge structure due to prestress

$$\theta_0 = -\psi|_{x=0} = \frac{3\sigma_0}{2E\alpha} \tanh\left(\frac{\alpha a}{2h}\right) \quad (19)$$

which for a long anchor region ($a \gg h$) becomes

$$\theta_0 = \frac{3\sigma_0}{2E\alpha} \quad (20)$$

Note that for a Poisson's ratio of $\nu=0.3$ and a rectangular cross section $\kappa^2=0.850$ [3], Eq. (20) gives $\theta_0=1.515(\sigma_0/E)$.

The result obtained by Fang and Wickert [1] by a curve fit to their plane strain finite element solution for a uniform prestress is

$$\theta_0 \approx \frac{\sigma_0}{E} (1.33 + 0.45\nu)(-0.014h + 1.022) \quad (21)$$

which is valid provided $a \gg h$, in the range $0.1 \leq \nu \leq 0.4$, for $0.5 \leq h \leq 3.0 \mu\text{m}$. A comparison of the results for θ_0 obtained using finite elements [1] and beam theory has been made in which E in Eq. (20) is replaced by the plane strain modulus $E/(1-\nu^2)$ in order to make the comparison valid. The average absolute value of the differences between the results of the modified beam theory and the finite element analysis in the above range of h and ν is 1.95% for θ_0 . The maximum absolute value of the difference is 6.01%.

5.2 Bridge Structure Deflection. Consider now a bridge structure that, prior to release, has a uniform normal stress. After release from the substrate, the constraints do not allow these prestresses to be completely relieved in either the anchors or in the suspended region. The classical beam theory predicts that the bridge will remain straight if the compressive force is less than the buckling load ($4\pi^2 EI/L^2$) for this configuration. Referring to Fig. 6, it is necessary to use our modified beam theory in the anchor regions. However, the suspended regions act as simple beams with end constraints and can be modeled using the ordinary classical beam theory with bending and extensional stiffness, but without extensional or shear deformations. Note that the upward deflection portrayed in Fig. 6 corresponds to a tensile prestress, whereas a downward deflection would occur for a compressive prestress.

In the suspended region, the differential equation becomes

$$EI \frac{d^2 w}{dx^2} - Tw = -M_0 \quad (22)$$

where M_0 is the bending moments acting at each end, and by symmetry the shear force at each end is zero. The solution of Eq. (22) is

$$w(x) = C_1 \sinh \beta x + C_2 \cosh \beta x - \frac{M_0}{T}, \quad \beta = \sqrt{\frac{T}{EI}} \quad (23)$$

By applying the boundary conditions that the displacement vanishes at each end (Fig. 1(c)), a relation between the applied moment and the end slope is obtained, i.e.,

$$\theta_0 = \left. \frac{dw}{dx} \right|_{x=(L/2)^-} = \frac{\beta M_0}{T} \tanh \frac{\beta L}{2} \quad (24)$$

The tension in the suspended section is changed due to its transverse deflection (which induces centroidal axis strains) and due to the rotation angles at its ends, i.e.,

$$T = \sigma_0 A + \frac{EA \Delta}{L} - \frac{EA h}{L} \left. \frac{dw}{dx} \right|_{x=L/2} \quad (25)$$

The change in length (Δ) of the centroidal axis of the suspended section due to its transverse displacement can be found from

$$\Delta = \frac{1}{2} \int_{-L/2}^{L/2} \left(\frac{dw}{dx} \right)^2 dx = \frac{\beta M_0^2}{4T} (\sinh \beta L - \beta L) / \cosh^2(\beta L/2) \quad (26)$$

Because the rotational stiffness of the suspended section is much greater than that of the anchor region, the suspended section deforms due to the rotation angle θ_0 given by Eqs. (19) and (20).

Combining Eqs. (11), (12), (19), (25), and (26) results in the following nonlinear equation:

$$\left(\frac{h}{L} \right)^2 \frac{(\beta L)^2}{12} = \frac{\sigma_0}{E} + \left(\frac{h}{L} \right) \frac{3\sigma_0}{2\alpha E} + \left(\frac{3\sigma_0}{4\alpha E} \right)^2 \frac{\sinh \beta L - \beta L}{\beta L \sinh^2(\beta L/2)} \quad (27)$$

which can be solved for β using standard methods. After determining β from Eq. (27), the midpoint deflection can be found from

$$\frac{w(0)}{h} = \left(\frac{3\sigma_0}{2\alpha E} \right) \frac{\cosh(\beta L/2) - 1}{(h/L)\beta L \tanh(\beta L/2)} \quad (28)$$

It is noted that the above analysis presupposes a tensile prestress. However, for compression, T is negative and β is purely imaginary. The same analysis can be used with

$$\beta = i\bar{\beta}, \quad \cosh \beta L = \cos \bar{\beta} L, \quad \sinh \beta L = i \sin \bar{\beta} L \quad (29)$$

where $i = \sqrt{-1}$ is the imaginary number. Hence, $\bar{\beta} = \sqrt{-T/EI}$ is real, and Eqs. (27) and (28) become

$$-\left(\frac{h}{L} \right)^2 \frac{(\bar{\beta} L)^2}{12} = \frac{\sigma_0}{E} + \left(\frac{h}{L} \right) \frac{3\sigma_0}{2\alpha E} + \left(\frac{3\sigma_0}{4\alpha E} \right)^2 \frac{\bar{\beta} L - \sin \bar{\beta} L}{\bar{\beta} L \sin^2(\bar{\beta} L/2)} \quad (30)$$

and

$$\frac{w}{h} = \left(\frac{3\sigma_0}{2\alpha E} \right) \frac{1 - \cos(\bar{\beta} L/2)}{(h/L)\bar{\beta} L \tan(\bar{\beta} L/2)} \quad (31)$$

respectively.

Results are shown in Figs. 7 and 8 for midpoint deflection versus prestress for different values of L/h . Figure 7 is for a compressive prestress, which also includes the buckling values in dashed vertical lines. In all cases, the midpoint deflects considerably before the fixed-fixed Euler buckling load is reached. In that

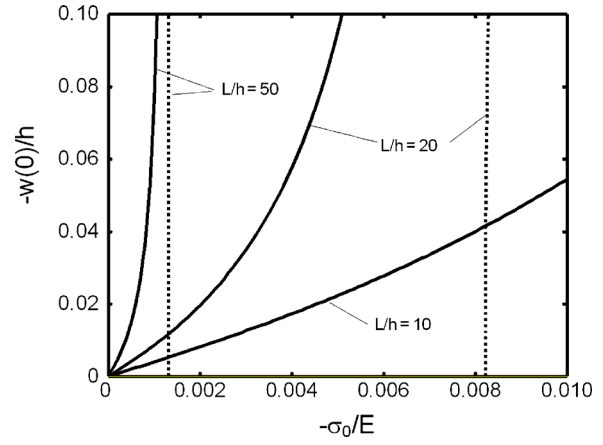


Fig. 7 Bridge structure midpoint deflection versus compressive prestress for different L/h . The dashed lines represent the arbitrary buckling deflections given by the classical beam theory.

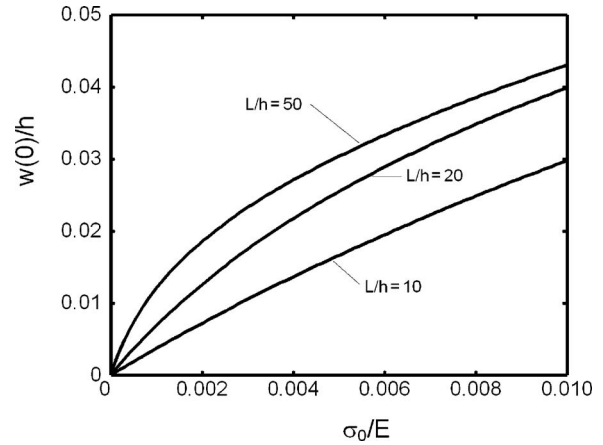


Fig. 8 Bridge structure midpoint deflection versus tensile prestress for different L/h

sense, the one-sided constraint acts in a similar way as a geometric imperfection [6]. However, even when the prestress is tensile, the midpoint deflects (Fig. 8) although, as expected, not as much as it does for a compressive prestress. For both tensile and compressive prestresses, the longer beams deflect more than the shorter ones.

6 Conclusions

In many applications, particularly in MEMS and in nanomechanics, a constraint exists only on one side of a cantilever or bridge structure. In such cases, it is shown here that the proper model for the beam accounts for bending, shear, and extensional deformations in the bonded region, resulting in a finite rotational compliance. Examples are given for cantilever, bridge, and guided structures. For transverse loading, the correction, due to this effect of rotational compliance, is often greater than the correction due to the shear deformation of the freestanding portion. For bridge structures, both tensile and compressive prestresses lead to transverse deflections, with the deflections becoming large at the fixed-fixed Euler buckling load. The results for transverse loading as well as for prestress agree well with finite element modeling.

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A Micromechanics-Based Elastoplastic Model for Amorphous Composites With Nanoparticle Interactions

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A constitutive model is proposed to investigate the strengthening mechanism and the relationship between nanostructures and effective mechanical properties of the aluminum-based amorphous nanocomposites. A continuum micromechanics-based, three-phase composite model comprises of Al particles, rare-earth enriched interlayers, and the amorphous aluminum matrix. The local stress field and deformation are formulated based on the concept of eigenstrain and equivalent inclusion method with consideration of both the particle-interlayer-matrix interaction and the particle-particle interaction. An ensemble-volume averaging technique is conducted to obtain the overall elastoplastic constitutive behavior for amorphous nanocomposites with randomly distributed spherical nanoparticles. Explicit expressions of the effective elastic stiffness and yield function in terms of the constituent properties and nanostructures are obtained. The effective elastoplastic stress-strain curves for uniaxial loading and the initial yield surfaces for axisymmetric loading are calculated. Simulations are conducted to investigate the effects of the particle size and pairwise particle interaction on the effective mechanical properties. [DOI: 10.1115/1.2839899]

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1 Introduction

Aluminum-based amorphous nanocomposites are attracting more interest because of their remarkable mechanical strength and reasonable ductility [1]. It has been reported that the yield strength of amorphous aluminum alloys is as high as 800 MPa in the amorphous state, and can be increased to 1.5 GPa by nanoparticle strengthening via partial crystallization [2–4]. The typical composition of Al-based amorphous nanocomposites is Al-TM-RE, where TM is a transition metal (TM) such as Ni, Fe, Co, Cr, and RE is a rare-earth (RE) element such as Y, La, Ce, Nd [5]. The partial crystallization process produces nanometer-scale α -Al fcc particles dispersed in the amorphous aluminum matrix. The nanoparticle size and interparticle distance are in the ranges of 5–50 nm and 7–100 nm, respectively [1,6]. The total particle volume fraction is preferably in the range of 10–30% to preserve ductility. Experiments showed that a higher annealing temperature and longer annealing time resulted in larger crystallized particle size and total particle volume fraction [7]. High-resolution electron microscopy examinations revealed that α -Al fcc nanoparticles exhibit a nearly spherical or an ellipsoidal morphology with no internal defects observed inside the nanoparticles [1]. Zhong et al. [5] and Hono et al. [6] observed that the RE components amassed around the Al nanoparticles to form an interlayer during the partial crystallization process due to the slow diffusivity of RE atoms. During the crystallization process, both TM and RE atoms are rejected from the α -Al phase. The atomic radius of the RE is usually much larger than the other solution elements, which may then result in a RE diffusivity that is slower than that of either Al

or TM by orders of magnitude. Hence, during the growth of α -Al nanoparticles, the rejected RE atoms are enriched at the interface and form a heterogeneous interlayer.

Many experimental results have demonstrated a remarkable increase in mechanical properties of Al-based amorphous nanocomposites. Inoue [1] measured the hardness and Young's modulus of $\text{Al}_{88}\text{Ni}_9\text{Ce}_2\text{Fe}_1$ nanocomposites. The results showed that Young's modulus and hardness increased monotonously proportional to the particle volume fraction, achieving values of about 73 GPa and 420 HV, respectively, at the volume fraction of 30%. Choi et al. [8] conducted experiments on $\text{Al}_{88}\text{Ni}_{10}\text{Nd}_2$ nanocomposites where the results exhibited an almost linear increase in micro-Vickers hardness from 220 for amorphous single phase alloys to 400 for amorphous nanocomposites with particle volume fraction of 32%. Gogebakan [9] also tested $\text{Al}_{85}\text{Y}_{10}\text{Ni}_5$ amorphous nanocomposites, resulting in a dramatic increase in the composite hardness.

Although the extraordinarily high strength of nanocomposites has been experimentally observed and qualitatively explained by the existence of nanoparticles and nanostructures, the underlying strengthening mechanisms remain to be quantitatively revealed. Kim et al. [10] proposed a phenomenological composite model that applied the mixture rule based on the volume fraction of each constituent to estimate the hardness of Al–Ni–Y nanocomposites. Kim and Hong [11] further extend their mixture rule model to a three-phase composite model comprising of Al particles, a RE enriched interlayer, and an amorphous aluminum matrix. While these models provide a straightforward means to estimate the mechanical properties of the nanocomposites, no detailed local stress field and plastic deformation were considered and the local matrix-particle interaction was neglected. Recently, a multiscale approach was proposed to model the overall elastoplastic behavior of amorphous nanocomposites [12], in which a three-phase nanostructure was considered and both the particle-matrix interaction and the particle-interlayer interaction were formulated based on Eshelby's eigenstrain method [13,14].

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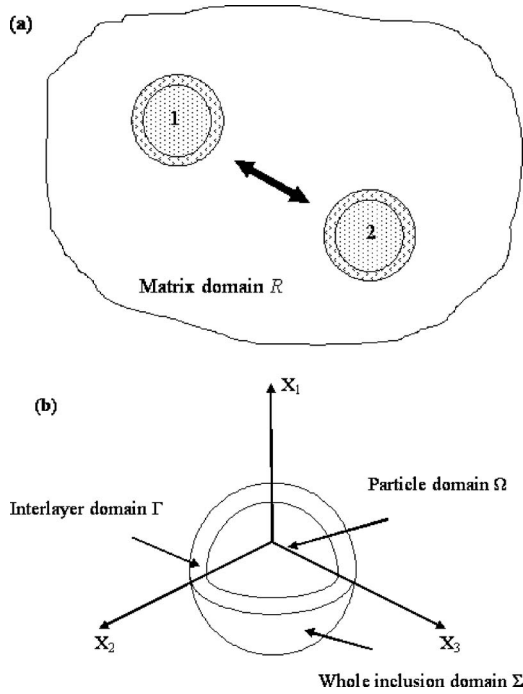


Fig. 1 (a) Schematic representation of the nanostructure of Al-based amorphous nanocomposites with pairwise particle interaction; (b) sketch of a spherical Al nanoparticle domain Ω and RE element enriched interlayer domain Γ embedded in the amorphous matrix domain R

With the increase of the particle volume fraction, the average distance between particles decreases. Therefore, the particle-particle interaction will have a significant effect on the overall mechanical properties of the nanocomposites with large volume fraction. The current study aims at addressing particularly the particle-particle interaction in addition to the particle-matrix and particle-interlayer interactions. The proposed framework is capable of simulating the mechanical response of nanocomposites with large particle volume fraction. The three distinguishing phases are modeled as the defect-free α -Al fcc nanoparticles of spherical shape with a radius of 5–50 nm, the amorphous aluminum matrix, and the RE enriched interlayer surrounding the nanoparticles. The overall mechanical properties of amorphous nanocomposites in terms of elastic stiffness and yield strength are derived via a homogenization averaging process.

2 Equivalent Inclusion Method

To characterize the interaction between particles, we will consider two spherical particles embedded in the infinite amorphous matrix domain R , as shown in Fig. 1. Each particle domain Ω is surrounded by a RE enriched interlayer domain Γ . The interlayer is perfectly bonded to both the particle and the matrix with no interfacial failure considered in the model. Furthermore, let $\Sigma = \Omega + \Gamma$ denote the generalized inclusion domain that contains both the particle and its corresponding interlayer.

2.1 Local Stress Field and Volume Averaging. When subject to a far-field external loading σ_0 , the heterogeneous local stress field can be expressed as

$$\sigma(\mathbf{x}) = \begin{cases} \mathbf{C}^0 : [\epsilon_0 + \epsilon'(\mathbf{x})] & \mathbf{x} \in R \\ \mathbf{C}^\Omega : [\epsilon_0 + \epsilon'(\mathbf{x})] & \mathbf{x} \in \Omega_i \\ \mathbf{C}^\Gamma : [\epsilon_0 + \epsilon'(\mathbf{x})] & \mathbf{x} \in \Gamma_i \end{cases} \quad (1)$$

where the subscript $i=1,2$ representing that the particle or the interlayer is belonging to the two inclusion domains, respectively.

The double dot symbol “:” indicates tensor contractions between a fourth-rank tensor and a second-rank tensor. Furthermore, ϵ_0 is the far-field strain corresponding to the far-field stress σ_0 with $\sigma_0 = \mathbf{C}^0 : \epsilon_0$, $\mathbf{C}^0$ is the elastic stiffness tensor of the amorphous matrix, $\epsilon'(\mathbf{x})$ represents the disturbance strain due to the heterogeneities, and $\mathbf{C}^\Omega$ and $\mathbf{C}^\Gamma$ denote the elastic stiffness tensors of Al nanoparticles and the RE enriched interlayer, respectively. For isotropic materials, these fourth-rank elastic stiffness tensors can be expressed as

$$\mathbf{C}_{ijkl}^\eta = \lambda^\eta \delta_{ij} \delta_{kl} + \mu^\eta (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad \eta = 0, \Omega, \Gamma \quad (2)$$

where λ^η and μ^η are Lamé’s constants for the corresponding phases.

Eshelby [13] and Mura [14] proposed the concept of eigenstrain (stress-free strain) and the equivalent inclusion method, in which a homogeneous material with equivalent eigenstrains in corresponding domains is used as a substitute for the heterogeneous material. For the problem described in Eq. (1), eigenstrain $\epsilon^*(\mathbf{x})$ is assumed to be inside the domains Ω_i and Γ_i , and the local stress field can be rewritten as

$$\sigma(\mathbf{x}) = \begin{cases} \mathbf{C}^0 : [\epsilon_0 + \epsilon'(\mathbf{x})] & \mathbf{x} \in R \\ \mathbf{C}^\Omega : [\epsilon_0 + \epsilon'(\mathbf{x})] = \mathbf{C}^0 : [\epsilon_0 + \epsilon'(\mathbf{x}) - \epsilon^*(\mathbf{x})] & \mathbf{x} \in \Omega_i \\ \mathbf{C}^\Gamma : [\epsilon_0 + \epsilon'(\mathbf{x})] = \mathbf{C}^0 : [\epsilon_0 + \epsilon'(\mathbf{x}) - \epsilon^*(\mathbf{x})] & \mathbf{x} \in \Gamma_i \end{cases} \quad (3)$$

Rearranging the terms in the last two equations, we have

$$\epsilon_0 + \epsilon'(\mathbf{x}) + \mathbf{A}^\Omega : \epsilon^*(\mathbf{x}) = \mathbf{0} \quad \mathbf{x} \in \Omega_i \quad (4)$$

$$\epsilon_0 + \epsilon'(\mathbf{x}) + \mathbf{A}^\Gamma : \epsilon^*(\mathbf{x}) = \mathbf{0} \quad \mathbf{x} \in \Gamma_i$$

where $\mathbf{A}^\Omega$ and $\mathbf{A}^\Gamma$ are the mismatch material property tensors for domains Ω and Γ , which can be expressed as

$$\begin{aligned} \mathbf{A}^\Omega &= (\mathbf{C}^\Omega - \mathbf{C}^0)^{-1} \cdot \mathbf{C}^0 \\ \mathbf{A}^\Gamma &= (\mathbf{C}^\Gamma - \mathbf{C}^0)^{-1} \cdot \mathbf{C}^0 \end{aligned} \quad (5)$$

respectively. The single dot symbol “ $\cdot$ ” indicates tensor contractions between two fourth-rank tensors. Generally speaking, eigenstrain is a function of the material properties of the constituents, nanostructures, domain shape, and far-field loading conditions. Eshelby showed that, for spheroidal particles, the eigenstrain is constant and can be expressed explicitly [13]. However, for the current problem, the eigenstrain is not constant due to the influence of the existing interlayer and particle interaction. Since we are interested in the overall properties instead of the local solutions, a volume averaging is conducted for Eq. (4), namely,

$$\frac{1}{\Omega_i} \int_{\Omega_i} [\epsilon_0 + \epsilon'(\mathbf{x}) + \mathbf{A}^\Omega : \epsilon^*(\mathbf{x})] d\mathbf{x} = \mathbf{0} \quad \mathbf{x} \in \Omega_i \quad (6)$$

$$\frac{1}{\Gamma_i} \int_{\Gamma_i} [\epsilon_0 + \epsilon'(\mathbf{x}) + \mathbf{A}^\Gamma : \epsilon^*(\mathbf{x})] d\mathbf{x} = \mathbf{0} \quad \mathbf{x} \in \Gamma_i$$

Since the two inclusions are embedded in an infinite domain and there is no difference between these two inclusion domains, therefore, we have

$$\frac{1}{\Omega_1} \int_{\Omega_1} \epsilon'(\mathbf{x}) d\mathbf{x} = \frac{1}{\Omega_2} \int_{\Omega_2} \epsilon'(\mathbf{x}) d\mathbf{x} = \bar{\epsilon}'_\Omega \quad (7)$$

$$\frac{1}{\Gamma_1} \int_{\Gamma_1} \epsilon'(\mathbf{x}) d\mathbf{x} = \frac{1}{\Gamma_2} \int_{\Gamma_2} \epsilon'(\mathbf{x}) d\mathbf{x} = \bar{\epsilon}'_\Gamma$$

and

$$\frac{1}{\Omega_1} \int_{\Omega_1} \boldsymbol{\varepsilon}^*(\mathbf{x}) d\mathbf{x} = \frac{1}{\Omega_2} \int_{\Omega_2} \boldsymbol{\varepsilon}^*(\mathbf{x}) d\mathbf{x} = \bar{\boldsymbol{\varepsilon}}_\Omega^*$$

(8)

$$\frac{1}{\Gamma_1} \int_{\Gamma_1} \boldsymbol{\varepsilon}^*(\mathbf{x}) d\mathbf{x} = \frac{1}{\Gamma_2} \int_{\Gamma_2} \boldsymbol{\varepsilon}^*(\mathbf{x}) d\mathbf{x} = \bar{\boldsymbol{\varepsilon}}_\Gamma^*$$

Here, we use $\bar{\boldsymbol{\varepsilon}}_\Omega^*$ and $\bar{\boldsymbol{\varepsilon}}_\Gamma^*$ to represent the volume-averaged eigenstrains in domains Ω and Γ , respectively. $\bar{\boldsymbol{\varepsilon}}_\Omega'$ and $\bar{\boldsymbol{\varepsilon}}_\Gamma'$ denote the volume-averaged disturbance strains in domains Ω and Γ , respectively. Thus, Eq. (6) becomes

$$\begin{aligned} \boldsymbol{\varepsilon}_0 + \bar{\boldsymbol{\varepsilon}}_\Omega' + \mathbf{A}^{\Omega} : \bar{\boldsymbol{\varepsilon}}_\Omega^* &= \mathbf{0} \\ \boldsymbol{\varepsilon}_0 + \bar{\boldsymbol{\varepsilon}}_\Gamma' + \mathbf{A}^{\Gamma} : \bar{\boldsymbol{\varepsilon}}_\Gamma^* &= \mathbf{0} \end{aligned} \quad (9)$$

2.2 Averaged Disturbance Strains and Eigenstrains. In general form, the disturbance strain $\boldsymbol{\varepsilon}'(\mathbf{x})$ due to an eigenstrain $\bar{\boldsymbol{\varepsilon}}^*$ in a domain V can be calculated as

$$\boldsymbol{\varepsilon}'(\mathbf{x}) = \int_V \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') : \bar{\boldsymbol{\varepsilon}}^* d\mathbf{x}' \quad (10)$$

Here, $\boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}')$ denotes the Green function that represents the disturbance strain at spot $\mathbf{x}$ due to a unit eigenstrain at $\mathbf{x}'$, which can be expressed explicitly for an infinite domain [14]. Furthermore, the volume-averaged disturbance strain over a domain W can be expressed as

$$\bar{\boldsymbol{\varepsilon}}_W' = \frac{1}{W} \int_W \int_V \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') : \bar{\boldsymbol{\varepsilon}}^* d\mathbf{x}' d\mathbf{x} \quad (11)$$

In the current problem, the volume-averaged disturbance strain in Domain Ω_1 (denoted by $\bar{\boldsymbol{\varepsilon}}_{\Omega_1}'$) is due to the eigenstrains in both inclusion domains (domains 1 and 2, including the particle and interlayer) and therefore can be decomposed into two parts:

$$\bar{\boldsymbol{\varepsilon}}_{\Omega_1}' = (\bar{\boldsymbol{\varepsilon}}_{\Omega_1}')_1 + (\bar{\boldsymbol{\varepsilon}}_{\Omega_1}')_2 \quad (12)$$

The inclusion domain 1 contains two subdomains: the particle domain Ω_1 and the interlayer domain Γ_1 with eigenstrains $\bar{\boldsymbol{\varepsilon}}_\Omega^*$ and $\bar{\boldsymbol{\varepsilon}}_\Gamma^*$, respectively. From Eq. (11), $(\bar{\boldsymbol{\varepsilon}}_{\Omega_1}')_1$ can be calculated as

$$\begin{aligned} (\bar{\boldsymbol{\varepsilon}}_{\Omega_1}')_1 &= \frac{1}{\Omega_1} \int_{\Omega_1} \left[\int_{\Omega_1} \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') : \bar{\boldsymbol{\varepsilon}}_\Omega^* d\mathbf{x}' \right. \\ &\quad \left. + \int_{\Gamma_1} \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') : \bar{\boldsymbol{\varepsilon}}_\Gamma^* d\mathbf{x}' \right] d\mathbf{x} \\ &= \left[\frac{1}{\Omega_1} \int_{\Omega_1} \int_{\Omega_1} \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') d\mathbf{x}' d\mathbf{x} \right] : \bar{\boldsymbol{\varepsilon}}_\Omega^* \\ &\quad + \left\{ \frac{1}{\Omega_1} \int_{\Omega_1} \left[\int_{\Sigma_1} \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') d\mathbf{x}' \right. \right. \\ &\quad \left. \left. - \int_{\Omega_1} \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') d\mathbf{x}' \right] d\mathbf{x} \right\} : \bar{\boldsymbol{\varepsilon}}_\Gamma^* \\ &= \mathbf{S}^{\Omega_1} : \bar{\boldsymbol{\varepsilon}}_\Omega^* + (\mathbf{S}^{\Sigma_1} - \mathbf{S}^{\Omega_1}) : \bar{\boldsymbol{\varepsilon}}_\Gamma^* \end{aligned} \quad (13)$$

In the above equations, $\mathbf{S}^{\Omega_1}$ and $\mathbf{S}^{\Sigma_1}$ are the Eshelby tensors for domains Ω_1 and Σ_1 , respectively. Because both Ω and Σ are spherical domain and the Eshelby tensors are related only to the domain shape, we have

$$\mathbf{S}^{\Sigma_1} = \mathbf{S}^{\Omega_1} = \mathbf{S} \quad (14)$$

where $\mathbf{S}$ is the Eshelby tensor and can be expressed explicitly for spherical domains as

$$S_{ijkl} = \frac{1}{15(1 - \nu_0)} [(5\nu_0 - 1)\delta_{ij}\delta_{kl} + (4 - 5\nu_0)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})] \quad (15)$$

with ν_0 Poisson's ratio of the matrix and δ_{ij} the Kronecker delta. Thus, Eq. (13) can be simplified as

$$(\bar{\boldsymbol{\varepsilon}}_{\Omega_1}')_1 = \mathbf{S} : \bar{\boldsymbol{\varepsilon}}_\Omega^* \quad (16)$$

It is noticed from Eq. (16) that the eigenstrain in the interlayer domain has no effect on the disturbance strain in the particle domain because of the symmetry of the spherical domain shape.

Similarly, the disturbance strain in particle domain Ω_1 due to the eigenstrain in the inclusion domain 2 can be written as

$$\begin{aligned} (\bar{\boldsymbol{\varepsilon}}_{\Omega_1}')_2 &= \frac{1}{\Omega_1} \int_{\Omega_1} \int_{\Omega_2} \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') : \bar{\boldsymbol{\varepsilon}}_\Omega^* d\mathbf{x}' d\mathbf{x} \\ &\quad + \frac{1}{\Omega_1} \int_{\Omega_1} \int_{\Gamma_2} \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') : \bar{\boldsymbol{\varepsilon}}_\Gamma^* d\mathbf{x}' d\mathbf{x} \\ &= \left[\frac{1}{\Omega_1} \int_{\Omega_1} \int_{\Omega_2} \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') d\mathbf{x}' d\mathbf{x} \right] : \bar{\boldsymbol{\varepsilon}}_\Omega^* \\ &\quad + \left\{ \frac{1}{\Omega_1} \int_{\Omega_1} \left[\int_{\Sigma_2} \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') d\mathbf{x}' \right. \right. \\ &\quad \left. \left. - \int_{\Omega_2} \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') d\mathbf{x}' \right] d\mathbf{x} \right\} : \bar{\boldsymbol{\varepsilon}}_\Gamma^* \\ &= \mathbf{T}(\Omega_1, \Omega_2) : \bar{\boldsymbol{\varepsilon}}_\Omega^* + [\mathbf{T}(\Omega_1, \Sigma_2) - \mathbf{T}(\Omega_1, \Omega_2)] : \bar{\boldsymbol{\varepsilon}}_\Gamma^* \end{aligned} \quad (17)$$

Here, the interaction integral is defined as $\mathbf{T}(V, W) = (1/V) \int_V \int_W \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') d\mathbf{x}' d\mathbf{x}$ and the explicit form can be obtained for spherical domains [15]. Due to the symmetry, the volume-averaged disturbance strain in domains Ω_1 and Ω_2 are identical and can be expressed as [Eqs. (12), (16), and (17)]

$$\bar{\boldsymbol{\varepsilon}}_\Omega' = [\mathbf{S} + \mathbf{T}(\Omega, \Omega)] : \bar{\boldsymbol{\varepsilon}}_\Omega^* + [\mathbf{T}(\Omega, \Sigma) - \mathbf{T}(\Omega, \Omega)] : \bar{\boldsymbol{\varepsilon}}_\Gamma^* \quad (18)$$

Here, we drop the subscript 1 or 2 for the domain symbols since the interaction integral $\mathbf{T}(V, W)$ is only related to the domain shape, size, and the distance between these two domains. It should be mentioned that the domains V and W are two separated domains without any common part.

The volume-averaged disturbance strain in domain Σ_1 can be similarly derived as

$$\bar{\boldsymbol{\varepsilon}}_{\Sigma_1}' = (\bar{\boldsymbol{\varepsilon}}_{\Sigma_1}')_1 + (\bar{\boldsymbol{\varepsilon}}_{\Sigma_1}')_2 \quad (19)$$

with

$$\begin{aligned} (\bar{\boldsymbol{\varepsilon}}_{\Sigma_1}')_1 &= \frac{1}{\Sigma_1} \int_{\Sigma_1} \int_{\Omega_1} \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') : \bar{\boldsymbol{\varepsilon}}_\Omega^* d\mathbf{x}' d\mathbf{x} \\ &\quad + \frac{1}{\Sigma_1} \int_{\Sigma_1} \int_{\Gamma_1} \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') : \bar{\boldsymbol{\varepsilon}}_\Gamma^* d\mathbf{x}' d\mathbf{x} \\ &= \left[\frac{1}{\Sigma_1} \int_{\Omega_1} \int_{\Sigma_1} \boldsymbol{\Gamma}(\mathbf{x}' - \mathbf{x}) d\mathbf{x} d\mathbf{x}' \right] : \bar{\boldsymbol{\varepsilon}}_\Omega^* \\ &\quad + \left[\frac{1}{\Sigma_1} \int_{\Gamma_1} \int_{\Sigma_1} \boldsymbol{\Gamma}(\mathbf{x}' - \mathbf{x}) d\mathbf{x} d\mathbf{x}' \right] : \bar{\boldsymbol{\varepsilon}}_\Gamma^* \\ &= \mathbf{S}^{\Sigma_1} : \bar{\boldsymbol{\varepsilon}}_\Omega^* + \mathbf{S}^{\Sigma_1} : (1 - f) \bar{\boldsymbol{\varepsilon}}_\Gamma^* = \mathbf{S} : [f \bar{\boldsymbol{\varepsilon}}_\Omega^* + (1 - f) \bar{\boldsymbol{\varepsilon}}_\Gamma^*] \end{aligned} \quad (20)$$

and

$$\begin{aligned}
(\bar{\epsilon}'_{\Sigma_1})_2 &= \frac{1}{\Sigma_1} \int_{\Sigma_1} \int_{\Omega_2} \Gamma(\mathbf{x} - \mathbf{x}') : \bar{\epsilon}_{\Omega}^* d\mathbf{x}' d\mathbf{x} \\
&+ \frac{1}{\Sigma_1} \int_{\Sigma_1} \int_{\Gamma_2} \Gamma(\mathbf{x} - \mathbf{x}') : \bar{\epsilon}_{\Gamma}^* d\mathbf{x}' d\mathbf{x} \\
&= \left[\frac{1}{\Sigma_1} \int_{\Sigma_1} \int_{\Omega_2} \Gamma(\mathbf{x} - \mathbf{x}') d\mathbf{x}' d\mathbf{x} \right] : \bar{\epsilon}_{\Omega}^* \\
&+ \left\{ \frac{1}{\Sigma_1} \int_{\Sigma_1} \left[\int_{\Sigma_2} \Gamma(\mathbf{x} - \mathbf{x}') d\mathbf{x}' \right. \right. \\
&\quad \left. \left. - \int_{\Omega_2} \Gamma(\mathbf{x} - \mathbf{x}') d\mathbf{x}' \right] d\mathbf{x} \right\} : \bar{\epsilon}_{\Gamma}^* \\
&= \mathbf{T}(\Sigma_1, \Omega_2) : \bar{\epsilon}_{\Omega}^* + [\mathbf{T}(\Sigma_1, \Sigma_2) - \mathbf{T}(\Sigma_1, \Omega_2)] : \bar{\epsilon}_{\Gamma}^* \quad (21)
\end{aligned}$$

Again, we drop the subscripts for the domain symbols and the volume-averaged disturbance strain in domain Σ can be obtained from Eqs. (19)–(21) as

$$\bar{\epsilon}'_{\Sigma} = [f\mathbf{S} + \mathbf{T}(\Sigma, \Omega)] : \bar{\epsilon}_{\Omega}^* + [(1-f)\mathbf{S} + \mathbf{T}(\Sigma, \Sigma) - \mathbf{T}(\Sigma, \Omega)] : \bar{\epsilon}_{\Gamma}^* \quad (22)$$

In the above equations, the relationship $\Gamma(\mathbf{x} - \mathbf{x}') = \Gamma(\mathbf{x}' - \mathbf{x})$ has been used. The volume ratio of double inclusion is defined as $f = \Omega/\Sigma$.

From the mixture rule, namely,

$$\bar{\epsilon}'_{\Sigma} = f\bar{\epsilon}'_{\Omega} + (1-f)\bar{\epsilon}'_{\Gamma} \quad (23)$$

the volume-averaged disturbance strain in domain Γ can be calculated as

$$\begin{aligned}
\bar{\epsilon}'_{\Gamma} &= \frac{1}{1-f} \bar{\epsilon}'_{\Sigma} - \frac{f}{1-f} \bar{\epsilon}'_{\Omega} = \frac{1}{1-f} [\mathbf{T}(\Sigma, \Omega) - f\mathbf{T}(\Omega, \Omega)] : \bar{\epsilon}_{\Omega}^* \\
&+ \frac{1}{1-f} [(1-f)\mathbf{S} + \mathbf{T}(\Sigma, \Sigma) + f\mathbf{T}(\Omega, \Omega) - 2\mathbf{T}(\Sigma, \Omega)] : \bar{\epsilon}_{\Gamma}^* \quad (24)
\end{aligned}$$

Combining Eqs. (18) and (24), we have

$$\begin{aligned}
\bar{\epsilon}'_{\Omega} &= \mathbf{E} : \bar{\epsilon}_{\Omega}^* + \mathbf{F} : \bar{\epsilon}_{\Gamma}^* \\
\bar{\epsilon}'_{\Gamma} &= \mathbf{G} : \bar{\epsilon}_{\Omega}^* + \mathbf{H} : \bar{\epsilon}_{\Gamma}^*
\end{aligned} \quad (25)$$

with

$$\begin{aligned}
\mathbf{E} &= \mathbf{S} + \mathbf{T}(\Omega, \Omega) \\
\mathbf{F} &= \frac{1}{f} \mathbf{T}(\Sigma, \Omega) - \mathbf{T}(\Omega, \Omega) \\
\mathbf{G} &= \frac{1}{1-f} [\mathbf{T}(\Sigma, \Omega) - f\mathbf{T}(\Omega, \Omega)] \\
\mathbf{H} &= \frac{1}{1-f} [(1-f)\mathbf{S} + \mathbf{T}(\Sigma, \Sigma) + f\mathbf{T}(\Omega, \Omega) - 2\mathbf{T}(\Sigma, \Omega)]
\end{aligned} \quad (26)$$

Here, the relationship $\mathbf{T}(\Omega, \Sigma) = (1/f)\mathbf{T}(\Sigma, \Omega)$ has been used. With the substitution of Eq. (25) into Eq. (9), the eigenstrains can be expressed as

$$\begin{aligned}
\bar{\epsilon}_{\Omega}^* &= [(\mathbf{E} + \mathbf{A}^{\Omega}) \cdot (\mathbf{H} + \mathbf{A}^{\Gamma}) - \mathbf{G} \cdot \mathbf{F}]^{-1} \cdot [\mathbf{F} - (\mathbf{H} + \mathbf{A}^{\Gamma})] : \epsilon^0 \\
\bar{\epsilon}_{\Gamma}^* &= [(\mathbf{E} + \mathbf{A}^{\Omega}) \cdot (\mathbf{H} + \mathbf{A}^{\Gamma}) - \mathbf{G} \cdot \mathbf{F}]^{-1} \cdot [\mathbf{G} - (\mathbf{E} + \mathbf{A}^{\Omega})] : \epsilon^0 \quad (27)
\end{aligned}$$

The equivalent eigenstrain for the whole inclusion domain (particle and interlayer) can be therefore obtained as

$$\begin{aligned}
\bar{\epsilon}_{\Sigma}^* &= f\bar{\epsilon}_{\Omega}^* + (1-f)\bar{\epsilon}_{\Gamma}^* \\
&= -[(\mathbf{E} + \mathbf{A}^{\Omega}) \cdot (\mathbf{H} + \mathbf{A}^{\Gamma}) - \mathbf{G} \cdot \mathbf{F}]^{-1} \cdot [f(\mathbf{H} - \mathbf{F} + \mathbf{A}^{\Gamma}) \\
&\quad + (1-f)(\mathbf{E} - \mathbf{G} + \mathbf{A}^{\Omega})] : \epsilon_0 \quad (28)
\end{aligned}$$

2.3 Discussion. When the second inclusion does not exist, $\mathbf{T}(\Sigma, \Omega) = \mathbf{T}(\Omega, \Omega) = \mathbf{0}$. The coefficients in Eq. (26) become

$$\mathbf{E} = \mathbf{S}, \quad \mathbf{F} = \mathbf{0}, \quad \mathbf{G} = \mathbf{0}, \quad \mathbf{H} = \mathbf{S} \quad (29)$$

and Eqs. (27) and (28) become

$$\begin{aligned}
\bar{\epsilon}_{\Omega}^* &= -(\mathbf{S} + \mathbf{A}^{\Omega})^{-1} : \epsilon_0 \\
\bar{\epsilon}_{\Gamma}^* &= -(\mathbf{S} + \mathbf{A}^{\Gamma})^{-1} : \epsilon_0 \quad (30)
\end{aligned}$$

and

$$\bar{\epsilon}_{\Sigma}^{*0} = -[f(\mathbf{S} + \mathbf{A}^{\Omega})^{-1} + (1-f)(\mathbf{S} + \mathbf{A}^{\Gamma})^{-1}] : \epsilon_0 \quad (31)$$

Here, we use $\bar{\epsilon}_{\Sigma}^{*0}$ to represent the equivalent eigenstrain for the double inclusion without considering particle interaction [16].

If the interlayer thickness is small compared with the distance between inclusions, we have $\mathbf{T}(\Sigma, \Omega) = \mathbf{T}(\Sigma, \Sigma) = \mathbf{T}(\Omega, \Omega)$ and Eq. (26) reduces to

$$\begin{aligned}
\bar{\epsilon}_{\Omega}^* &= -[\mathbf{S} + \mathbf{T}(\Omega, \Omega) + \mathbf{A}^{\Omega}]^{-1} : \epsilon_0 \\
\bar{\epsilon}_{\Gamma}^* &= -[\mathbf{S} - \mathbf{T}(\Omega, \Omega) + \mathbf{A}^{\Gamma}]^{-1} : \epsilon_0 \quad (32)
\end{aligned}$$

and correspondingly, Eq. (28) becomes

$$\bar{\epsilon}_{\Sigma}^* = -\{f[\mathbf{S} + \mathbf{T}(\Omega, \Omega) + \mathbf{A}^{\Omega}]^{-1} + (1-f)[\mathbf{S} - \mathbf{T}(\Omega, \Omega) + \mathbf{A}^{\Gamma}]^{-1}\} : \epsilon_0 \quad (33)$$

3 Constitutive Modeling

To obtain the effective material properties, we consider a representative volume element (RVE), in which spherical Al nanoparticles surrounded by RE enriched interlayer are randomly dispersed in amorphous Al matrix. The interlayer thickness is assumed to be small compared with the average distance between particles to simplify the derivation.

3.1 Pairwise Interaction. For an inclusion located at $\mathbf{x}_1$, the interaction effect of an inclusion located at $\mathbf{x}_2$ on the equivalent eigenstrain can be expressed as [17–20]

$$\bar{\mathbf{d}}_{\Sigma}^*(\mathbf{x}_1 - \mathbf{x}_2) = \bar{\epsilon}_{\Sigma}^* - \bar{\epsilon}_{\Sigma}^{*0} \quad (34)$$

where $\bar{\epsilon}_{\Sigma}^*$ is the equivalent eigenstrain considering particle-particle interaction, which is given by Eq. (28), and $\bar{\epsilon}_{\Sigma}^{*0}$ is the equivalent eigenstrain of double inclusion (Eq. (31)). Therefore, the interaction effect of all the other inclusions can be calculated by integrating $\bar{\mathbf{d}}_{\Sigma}^*(\mathbf{x}_1 - \mathbf{x}_2)$ over all possible positions ($\mathbf{x}_2$) of the second inclusions within the RVE. The ensemble-average process can be expressed as [17]

$$\langle \bar{\mathbf{d}}_{\Sigma}^* \rangle(\mathbf{x}_1) = \int_{V-\Sigma_1} \bar{\mathbf{d}}_{\Sigma}^*(\mathbf{x}_1 - \mathbf{x}_2) P(\mathbf{x}_2 | \mathbf{x}_1) d\mathbf{x}_2 \quad (35)$$

where $P(\mathbf{x}_2 | \mathbf{x}_1)$ is the conditional probability function for finding the second inclusion centered at $\mathbf{x}_2$ given the first inclusion centered at $\mathbf{x}_1$. In addition, angular brackets denote the ensemble-averaging operator and V is the integration domain determined by the inclusion shape. Therefore, the ensemble-averaged equivalent eigenstrain can be expressed as

$$\langle \bar{\epsilon}_{\Sigma}^* \rangle = \bar{\epsilon}_{\Sigma}^{*0} + \langle \bar{\mathbf{d}}_{\Sigma}^* \rangle \quad (36)$$

Since nanoparticles are uniformly distributed in the amorphous matrix, $P(\mathbf{x}_2 | \mathbf{x}_1)$ can be simplified as N/V , where N is the total

number of nanoparticles uniformly dispersed in entire volume V . By expanding the expression of $\mathbf{T}(\Omega, \Omega)$ to the order of $O(\rho^3)$, where ρ is the ratio of particle radius over center-to-center distance between two inclusions, and following a similar derivation process given by Ju and Chen [17], the ensemble-averaged equivalent eigenstrain is obtained as

$$\langle \bar{\epsilon}_{\Sigma}^* \rangle = -\mathbf{Q} : \epsilon_0 \quad (37)$$

with

$$\mathbf{Q} = f[\Gamma^{\Omega} \cdot (\mathbf{S} + \mathbf{A}^{\Omega})^{-1}] + (1-f)[\Gamma^{\Omega} \cdot (\mathbf{S} + \mathbf{A}^{\Gamma})^{-1}] \quad (38)$$

and the fourth-rank interaction tensor $\Gamma^{\eta}(\eta=\Omega, \Gamma)$ has the isotropic form of

$$\Gamma_{ijkl}^{\eta} = \gamma_1^{\eta} \delta_{ij} \delta_{kl} + \gamma_2^{\eta} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (39)$$

in which

$$\begin{aligned} \gamma_1^{\eta} &= \frac{5\phi^{\Sigma}}{96\beta_{\eta}^2} \left\{ 12\nu_0(13-14\nu_0) - \frac{96\alpha_{\eta}}{3\alpha_{\eta}+2\beta_{\eta}}(1-2\nu_0)(1+\nu_0) \right\} \\ \gamma_2^{\eta} &= \frac{1}{2} + \frac{5\phi^{\Sigma}}{96\beta_{\eta}^2} \left\{ 6(25-34\nu_0+22\nu_0^2) - \frac{36\alpha_{\eta}}{3\alpha_{\eta}+2\beta_{\eta}} \right. \\ &\quad \left. \times (1-2\nu_0)(1+\nu_0) \right\} \end{aligned} \quad (40)$$

and

$$\begin{aligned} \alpha_{\eta} &= 2(5\nu_0-1) + 10(1-\nu_0) \left(\frac{\kappa_0}{\kappa_{\eta}-\kappa_0} - \frac{\mu_0}{\mu_{\eta}-\mu_0} \right) \\ \beta_{\eta} &= 2(4-5\nu_0) + 15(1-\nu_0) \frac{\mu_0}{\mu_{\eta}-\mu_0} \end{aligned} \quad (41)$$

In the above equations, κ_{η} and μ_{η} are the bulk modulus and shear modulus of phase η , respectively. ϕ^{Σ} is the total volume fraction of the inclusion domain (including the particle and its corresponding interlayer). It is noted that the expression of ensemble-averaged eigenstrain $\langle \bar{\epsilon}_{\Sigma}^* \rangle$ is a constant (position independent); thus, it will keep unchanged for an additional volume-averaging process. To simplify the symbols, we will drop the angular brackets and use the head bar to represent the ensemble-volume averaged variables.

3.2 Effective Elastic Stiffness Tensor. Based on small-deformation elastoplasticity principle, the total macroscopic (effective) strain tensor $\bar{\epsilon}$ consists of elastic part $\bar{\epsilon}^e$ and plastic part $\bar{\epsilon}^p$. The effective elastic strain $\bar{\epsilon}^e$ is related to the effective stress $\bar{\sigma}$ via $\bar{\sigma} = \bar{\mathbf{C}} : \bar{\epsilon}^e$ where the effective elastic tensor $\bar{\mathbf{C}}$ of nanocomposites can be derived with the help of general governing equations of composite materials [21]. Specifically, for nanocomposites containing randomly distributed spherical particles with interlayers, the three governing equations can be obtained as [12]

$$\begin{aligned} \bar{\sigma} &= \mathbf{C}^0 : (\bar{\epsilon} - \phi^{\Sigma} \bar{\epsilon}_{\Sigma}^*) \\ \bar{\epsilon} &= \epsilon_0 + \phi^{\Sigma} \mathbf{S} : \bar{\epsilon}_{\Sigma}^* \\ \bar{\epsilon}_{\Sigma}^* &= \mathbf{Q} : \epsilon_0 \end{aligned} \quad (42)$$

Therefore, the explicit expression of the effective elastic tensor $\bar{\mathbf{C}}$ of the nanocomposites can be shown as

$$\bar{\mathbf{C}} = \mathbf{C}^0 \cdot [\mathbf{I} + \phi^{\Sigma} \mathbf{Q} \cdot (\phi^{\Sigma} \mathbf{S} \cdot \mathbf{Q} - \mathbf{I})^{-1}] \quad (43)$$

3.3 Effective Yield Function and Hardening Rule. The overall plastic behavior of Al-based amorphous nanocomposites is attributed to the plastic deformation in the amorphous matrix since it is observed that the α -Al fcc nanoparticles contain no dislocations or other imperfections [10] and thus are assumed to

deform elastically. Accordingly, at any matrix material point $\mathbf{x}$, the disturbance strain $\epsilon'(\mathbf{x})$ due to the existence of an inclusion centered at $\mathbf{x}'$ is given by Eq. (10) and the corresponding local stress can be calculated as

$$\sigma(\mathbf{x}) = \sigma_0 + \mathbf{C}^0 : [\bar{\mathbf{G}}(\mathbf{x} - \mathbf{x}') : \bar{\epsilon}^*] \quad (44)$$

in which $\bar{\mathbf{G}}(\mathbf{x} - \mathbf{x}') = \int_{\Sigma} \Gamma(\mathbf{x} - \xi) d\xi$, ($\mathbf{x} \notin \Sigma$), and Σ is the spherical inclusion domain centered at $\mathbf{x}'$. Here, $\bar{\mathbf{G}}(\mathbf{x} - \mathbf{x}')$ is called exterior-point Eshelby's tensor and its explicit expression for spherical inclusion domain is obtained as [21]

$$\begin{aligned} \bar{G}_{ijkl}(\mathbf{x}) &= \frac{\rho^3}{30(1-\nu_0)} \times [(3\rho^2 + 10\nu_0 - 5)\delta_{ij}\delta_{kl} \\ &\quad + 15(1-\rho^2)\delta_{ij}n_k n_l + (3\rho^2 - 10\nu_0 + 5)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ &\quad + 15(1-2\nu_0-\rho^2)\delta_{kl}n_i n_j + 15(7\rho^2 - 5)n_i n_j n_k n_l \\ &\quad + 15(\nu_0 - \rho^2)(\delta_{ik}n_j n_l + \delta_{il}n_j n_k + \delta_{jk}n_i n_l + \delta_{jl}n_i n_k)] \end{aligned} \quad (45)$$

In this equation, $\rho = a/r$, in which a is the radius of the sphere domain Σ , $r = \sqrt{x_i x_i}$, and $n_i = x_i/r$. Local yielding and plastic flow are dependent on local stress field. For simplicity, the commonly used von Mises yield criterion with an isotropic hardening law is assumed for the matrix material as an illustration. Namely, the local yield function reads

$$F(\sigma, e^p) = \sqrt{\sigma : \mathbf{I}_d : \sigma} - K(e^p) \leq 0 \quad (46)$$

where e^p and $K(e^p)$ are the equivalent plastic strain and the isotropic hardening function of the matrix-only material, respectively. Moreover, $\mathbf{I}_d$ denotes the deviatoric part of the fourth-rank identity tensor $\mathbf{I}$. It is noted that, during plastic deformation, we employ the secant-moduli method [18] to update the elastic tensor of the matrix during the calculation process of local stress tensor. It is also noted that extension of the present framework to more general yield criterion and general hardening law can be derived with addition effort.

Following Ju and Sun [22], we denote by $H(\mathbf{x}|g) = \sigma(\mathbf{x}|g) : \mathbf{I}_d : \sigma(\mathbf{x}|g)$ the square of the "current stress norm" at a local point $\mathbf{x}$ for a given inclusion configuration (assembly) g . Furthermore, $\langle H \rangle_m(\mathbf{x})$ is defined as the ensemble average of $H(\mathbf{x}|g)$ over all possible realizations for a matrix point $\mathbf{x}$, which indicates

$$\langle H \rangle_m(\mathbf{x}) = H^0 + \int_g [H(\mathbf{x}|g) - H^0] P(g) dg \quad (47)$$

where $P(g)$ is the probability density function for determining an inclusion for a given configuration g , and $H^0 = \sigma^0 : \mathbf{I}_d : \sigma^0$ is the square of the far-field stress norm applied on the composite. Since nanoparticles are uniformly distributed in the amorphous matrix, $P(g)$ can be simplified as N/V , where N is the total number of nanoparticles uniformly dispersed in entire volume V . After a series of lengthy but straightforward derivations, the ensemble-averaged $\langle H \rangle_m(\mathbf{x})$ can be evaluated as

$$\langle H \rangle_m(\mathbf{x}) = \sigma_0 : \mathbf{T} : \sigma_0 \quad (48)$$

where the components of the fourth-rank isotropic tensor $\mathbf{T}$ take the form

$$T_{ijkl} = T_1 \delta_{ij} \delta_{kl} + T_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (49)$$

with

$$\begin{aligned} T_2 &= \frac{1}{2} + \frac{\phi^{\Sigma}}{36} (23 - 50\nu_0 + 35\nu_0^2) (\varphi_1)^2 \\ 3T_1 + 2T_2 &= 50\phi^{\Sigma} (1 - 2\nu_0)^2 (3\varphi_1 + 2\varphi_2)^2 \end{aligned} \quad (50)$$

and

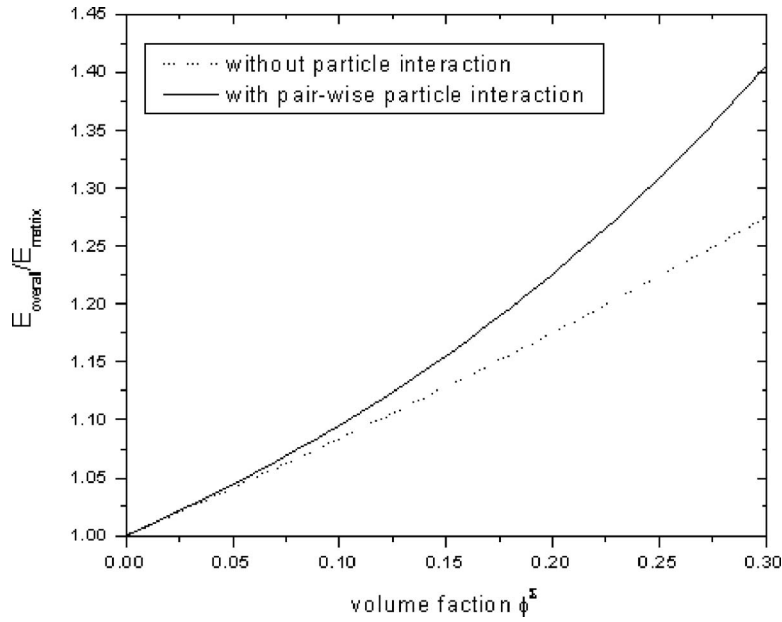


Fig. 2 Overall Young's modulus of the nanocomposites versus particle volume fraction for nanocomposites with and without considering pairwise particle interaction

$$\begin{aligned} \varphi_2 = & \sum_{\eta=\Omega, \Gamma} \frac{f_\eta}{\beta_\eta} \left\{ 6 + \frac{5\phi^\Sigma}{8(\beta_\eta)^2} \left[6(25 - 34\nu_0 + 22\nu_0^2) \right. \right. \\ & \left. \left. - \frac{36\alpha_\eta}{3\alpha_\eta + 2\beta_\eta} (1 - 2\nu_0)(1 + \nu_0) \right] \right\} \\ 3\varphi_1 + 2\varphi_2 = & \sum_{\eta=\Omega, \Gamma} \frac{f_\eta}{3\alpha_\eta + 2\beta_\eta} \left\{ 2 + \frac{25\phi}{4(\beta_\eta)^2} \left[(5 + \nu_0 - 4\nu_0^2) \right. \right. \\ & \left. \left. - \frac{6\alpha_\eta}{3\alpha_\eta + 2\beta_\eta} (1 - 2\nu_0)(1 + \nu_0) \right] \right\} \end{aligned} \quad (51)$$

Here, $f_\Omega = f$ is the volume ratio of the particle and $f_\Gamma = 1 - f$ is the volume ratio of the interlayer. Alternatively, Eq. (48) can be re-written in terms of the macroscopic (ensemble-volume averaged) stress $\bar{\sigma}$ as

$$\langle H \rangle_m(\mathbf{x}) = \bar{\sigma} : \bar{\mathbf{T}} : \bar{\sigma} \quad (52)$$

with $\bar{\mathbf{T}} = \mathbf{P} \cdot \bar{\mathbf{T}} \cdot \mathbf{P}$ and the fourth-rank tensor $\mathbf{P}$ reads

$$\mathbf{P} = [\mathbf{I} + \phi^\Sigma (\mathbf{S} - \mathbf{I}) \cdot \mathbf{Q}] \quad (53)$$

It is observed from Eq. (53) that $\langle H \rangle_m$ can be reduced to the form derived by Liu and Sun [12] for the nanocomposites if no pairwise particle interaction is considered.

The ensemble-volume averaged yield function $\bar{F}$ of the nanocomposites can be characterized directly from Eq. (46):

$$\bar{F} = (1 - \phi^\Sigma) \sqrt{\langle H \rangle_m} - K(\bar{\epsilon}^p) \leq 0 \quad (54)$$

It is noted that, for isotropic plastic hardening, $K(\bar{\epsilon}^p)$ can be simplified as

$$K(\bar{\epsilon}^p) = \sqrt{\frac{2}{3}} [\sigma_y + h(\bar{\epsilon}^p)^q] \quad (55)$$

where σ_y , h and q , and $\bar{\epsilon}^p$ denote the yield strength of the matrix, the linear and exponential isotropic hardening parameters of the matrix, and the effective equivalent plastic strain of the nanocomposites, respectively. It is noted that Eq. (54) represents the pressure-dependent nanocomposite yield function although the

matrix is assumed to be pressure-independent von Mises type yield criterion.

Based on the derived effective yield function, the effective plastic strain rate $\dot{\bar{\epsilon}}^p$ can be calculated from the associative plastic flow rule:

$$\dot{\bar{\epsilon}}^p = \dot{\lambda} \frac{\partial \bar{F}}{\partial \bar{\sigma}} \quad (56)$$

where $\dot{\lambda}$ is the plastic consistency parameter, which can be obtained from the following Kuhn-Tucker condition:

$$\dot{\lambda} \geq 0, \quad \bar{F} \leq 0, \quad \dot{\lambda} \bar{F} = 0, \quad \dot{\lambda} \dot{\bar{F}} = 0 \quad (57)$$

Therefore, the effective elastoplastic constitutive model of nanocomposites considering the pairwise particle interaction is developed, which is capable of estimating the overall elastoplastic stress-strain responses of nanocomposites under general three-dimensional loading conditions.

4 Numerical Simulation and Discussion

Specific Al-Ni-Y nanocomposites are applied as the model material in the following simulation. All phases are assumed to be isotropic. The amorphous Al matrix is assumed to have Young's modulus of 70 GPa and Poisson's ratio of 0.3. The yield strength of the matrix is taken as 1.29 GPa [9], and the plastic hardening parameters are $h = 1.0$ GPa and $q = 0.3$, respectively. The α -Al fcc nanoparticles have Young's modulus of 71 GPa and Poisson's ratio of 0.31. Since there are no mechanical data available for the RE element enriched interlayer, it is assumed that the intermetallic compound interlayer phase has Young's modulus of 500 GPa and Poisson's ratio of 0.25. Due to the much higher strength of the defect-free Al nanoparticles, plastic deformation is constrained in the amorphous matrix. Unless explicitly stated otherwise, the radius of nanoparticles is assumed to be 10 nm and the thickness of interlayers to be 3 nm.

Uniaxial loading simulation is conducted to investigate the elastoplastic behavior of the nanocomposites. Figure 2 shows that overall Young's modulus increases proportional to the inclusion volume fraction. When the inclusion volume fraction is small (less

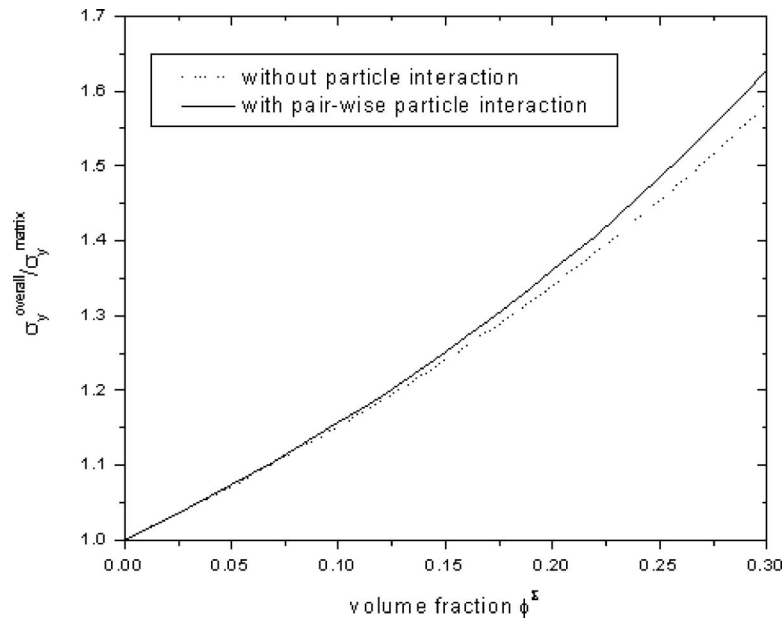


Fig. 3 Overall yield strength of the nanocomposites versus particle volume fraction for nanocomposites with and without considering pairwise particle interaction

than 10%), the ignorance of particle interaction does not result in significant error when estimating the overall elastic stiffness. With the increase of the inclusion volume fraction, the particle interaction shows more significant effect on the overall elastic stiffness. For a nanocomposite with 30% particle volume fraction, overall Young's modulus considering particle interaction is nearly 9% higher than the one without considering particle interaction, which shows that the neglect of particle interaction will introduce significant error when the inclusion volume fraction is large. The nanocomposites also demonstrate the strong strengthening effect, as shown in Fig. 3. The nanocomposite having a 30% inclusion volume fraction indicates a yield strength 1.6 times higher than that of amorphous alloys. It is shown that the effect of particle interaction on the overall yield strength is not as significant as on

overall Young's modulus and only a 3% increase of the yield strength is observed for considering the particle interaction. One-dimensional strain-stress curves for uniaxial loading tests are presented in Fig. 4. Again, considering pairwise particle interaction, higher elastic stiffness, yield strength, and hardening are observed. For comparison, the overall elastoplastic curve for nanocomposites without considering the interlayer is drawn in this figure. The existence of high strength interlayer significantly increases both elastic stiffness and yield strength of nanocomposites. Furthermore, with the introduction of the interlayer thickness as a characteristic length scale, the particle size effect can be incorporated in the current model. In Fig. 5, the overall stress-strain curves of the nanocomposites for various particle sizes are presented. For a fixed interlayer thickness, a clear particle size effect can be ob-

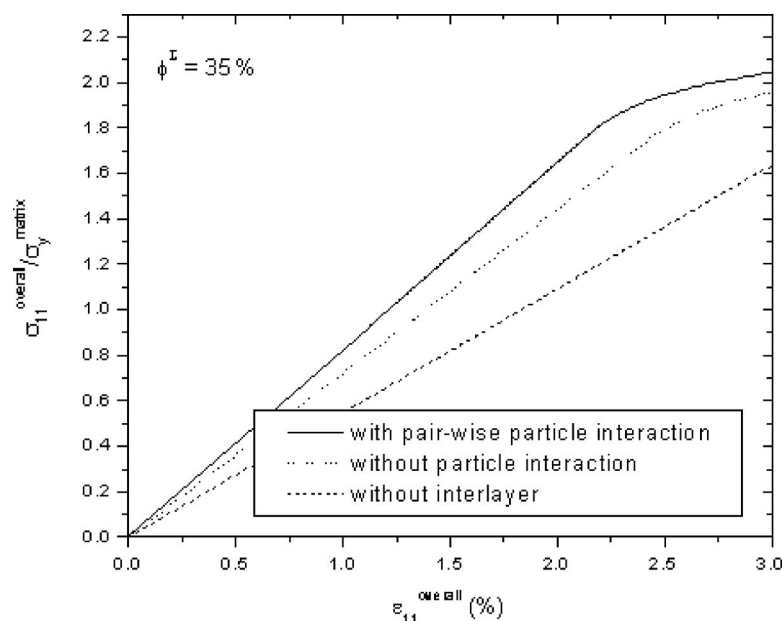


Fig. 4 Overall uniaxial nanocomposite stress-strain curves

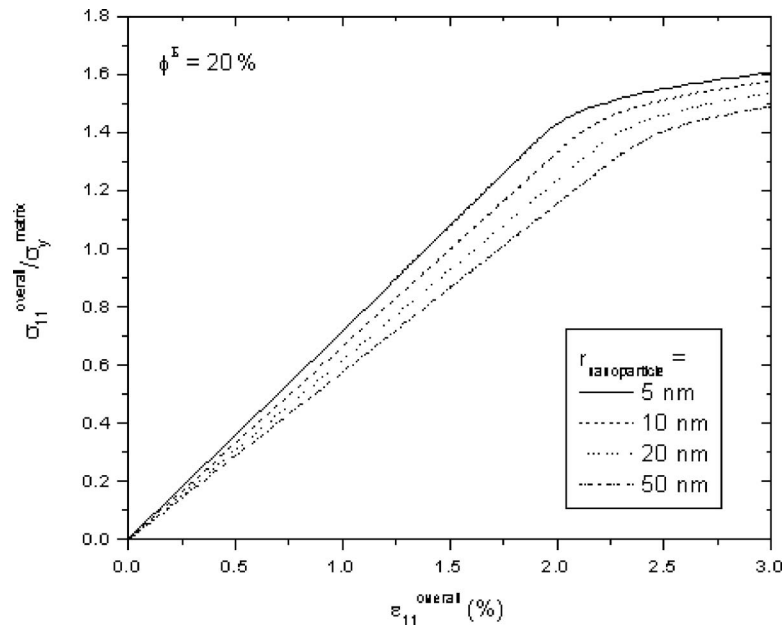


Fig. 5 Overall uniaxial nanocomposite stress-strain curves with various nanoparticle sizes

served. With small particle size, higher elastic stiffness and yield strengths are obtained, which coincide with experimental observation [23,24]. The overall yield strength of the nanocomposites is presented for a large range of particle sizes, from nanometer scale to micrometer scale, in Fig. 6. When the interlayer is nonexistent (i.e., the interlayer thickness is zero), the overall yield strength does not change for the entire particle size range, implying that for a fixed volume fraction, the particle size does not affect the overall yield strength. Alternatively, for a fixed interlayer thickness, a clear particle size effect can be observed. When the particle size corresponds to the same length scale as the interlayer thickness, a significant increase of yield strength is exhibited, proving that the existence of an interlayer significantly affects the overall mechanical properties of the nanocomposites. Since the interlayer between

the particles and the matrix is usually measured in nanometers, the nanoparticle-reinforced composites demonstrate higher yield strength than microparticle-reinforced composites.

To investigate the strengthening effect of the nanocomposites under complex loading condition, the nanocomposite yield surface is certainly of interest. Axisymmetric loading cases are specified here to study multiaxial strengthening effect. With the assumption of overall stresses as $\bar{\sigma}_{11} > 0$, $\bar{\sigma}_{22} = \bar{\sigma}_{33} > 0$, and $\bar{\sigma}_{12} = \bar{\sigma}_{13} = \bar{\sigma}_{23} = 0$, the initial yield surfaces are demarcated in the volumetric and deviatoric stress space in Fig. 7. The nanocomposite volumetric stress $\bar{\sigma}_v$ and deviatoric stress $\bar{\sigma}_d$ can be obtained from their definitions considering the axisymmetric property as $\bar{\sigma}_v = (\bar{\sigma}_{11} + 2\bar{\sigma}_{22})/3$ and $\bar{\sigma}_d = \bar{\sigma}_{11} - \bar{\sigma}_{22}$, respectively. It is shown from Fig. 7

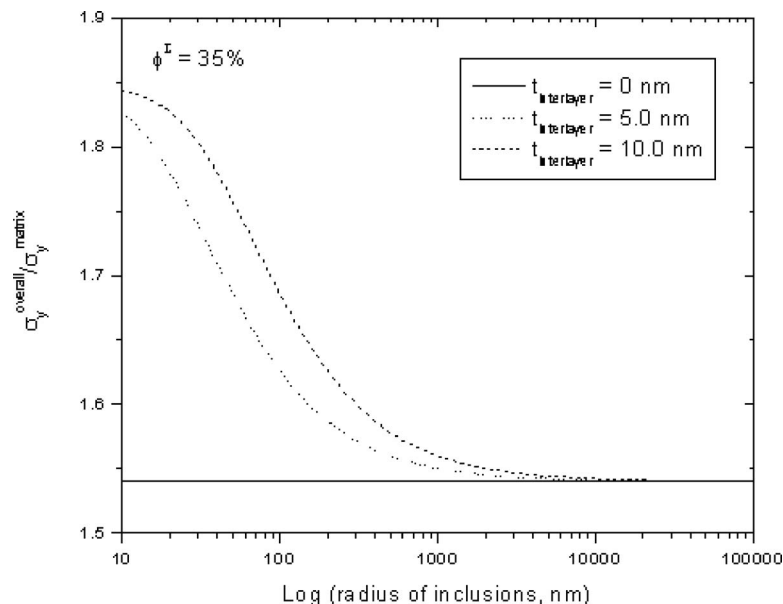


Fig. 6 Particle size effect on the overall yield strength of the nanocomposites with different interlayer thicknesses

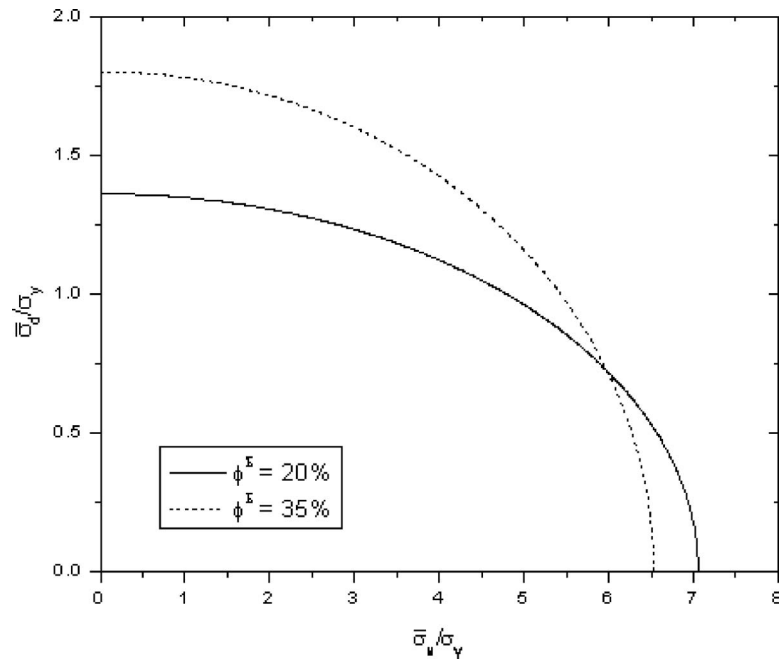


Fig. 7 Overall initial yield surfaces of the nanocomposites under axisymmetric loading

that the yielding response of the nanocomposites is not of von Mises type, even when the nanoparticles are spherical in shape and randomly distributed. A decrease in volume fraction of nanoparticles leads to an increase of the volumetric yield stress and a simultaneous decrease of the deviatoric yield stress. This trend indicates that the pressure dependence of nanocomposite yielding primarily resulted from the existence of nanoparticles. When the volume fraction of particles vanishes, the matrix-only material will be restored von Mises yielding, which is consistent with the assumption of the matrix material satisfying the von Mises J_2 -yield criterion.

5 Conclusions

A continuum micromechanics-based constitutive framework is proposed to investigate the effective elastoplastic properties of randomly dispersed nanoparticle-reinforced amorphous composites. The nanometer-scale local stress field and deformation are formulated based on the concept of eigenstrain and equivalent inclusion method considering both the particle-interlayer-matrix interaction and the particle-particle interaction. Ensemble and volume average procedures are conducted in a microscopic RVE to estimate the pairwise particle interaction and to obtain the overall elastoplastic constitutive behavior for amorphous nanocomposites with randomly distributed spherical nanoparticles. Explicit expressions of the effective elastic stiffness and yield function in terms of the constituent properties and nanostructures are obtained.

Under the uniaxial loading condition, the effects of particle interaction on the overall elastic stiffness and yield strength of nanocomposites are numerically simulated. It is shown that the consideration of pairwise particle interaction leads to a significant increase of the effective elastic stiffness when the inclusion volume fraction is large. The interlayer thickness is treated as a characteristic length scale; thus, the particle size is incorporated into the current model. One-dimensional elastoplastic strain-stress curves are presented for various particle sizes. It is demonstrated that for a fix volume fraction, composites with small particle size have much higher yield strength than that with large particle size. Thus, the particle size effect is investigated for nanocomposites within the continuum mechanics framework. The overall yield

surfaces are demarcated for nanocomposites under axisymmetric loading to demonstrate the multiaxial strengthening effect and pressure-dependent yielding response due to the existence of nanoparticles. The proposed model satisfies the continuum mechanics rules and provides a feasible means to estimate the mechanical response of nanocomposites.

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Dynamic Response of Rapidly Heated Cylindrical Rods: Longitudinal and Flexural Behavior

A very fast temperature increase, produced by a nonuniform heat generation, induces in a simply supported, isotropic, cylindrical rod both longitudinal and flexural vibrations. This paper presents an analytical method to study these vibrations and determine the stresses they provoke. The proposed procedure relies on three main steps: an exact solution for the temperature field is first obtained, by means of Fourier–Bessel expansions; quasistatic thermal stresses are then computed as a function of the calculated temperature distribution, making use of the thermoelastic displacement potential and of the solution to the equivalent isothermal two-dimensional stress problem; finally, longitudinal and flexural vibrations excited by an equivalent thermal force and thermal bending moment are determined using the mode-summation method. The influence of thermal shock duration on the maximum value of the longitudinal dynamic stress and of the ratio between the characteristic thermal time and structural response time on the dynamic bending deflection is analyzed and discussed. Finally, a comparison between the analytical model and experimental measurements is presented. The analytical model described in this paper allows the complete evaluation, within the linear elastic domain, of quasistatic and dynamic thermal stresses induced in an isotropic cylindrical rod by rapid internal heating. [DOI: 10.1115/1.2839901]

Keywords: thermally induced vibrations, dynamic thermal stresses, thermal shock, stress waves

1 Introduction

Thermal stress analyses are in most cases performed neglecting the effect of inertia force. Even if the thermal problem is time dependent, the study is conventionally solved considering the structural problem as a succession of quasistatic analyses. Thus, the dynamic effect due to the mass inertia of structures is not considered. When a heating process is very rapidly occurring, neglecting the inertia force could lead to a wrong estimation of the thermal stresses.

Studies of the dynamic response of rapidly heated structures, which take into account this effect, have been carried out in several fields of engineering and applied physics as from the 1950s, in particular, in aerospace, nuclear engineering, and high energy particle physics.

In the aerospace engineering field, the problem of thermally induced vibrations was first studied by Boley [1,2] and by Boley and Barber [3], with specific regard to flexible and slender spacecraft booms suddenly exposed to solar radiation. Boley's papers analyzed the behavior of rectangular beams and plates submitted to rapid surface heating, pointing out that a sudden, nonuniform temperature rise produces a time dependent thermal moment that deforms the structure. Boley proposed a relation between the characteristic thermal diffusion time and the structural response time and showed that, if the two parameters are in the same order of magnitude and a sudden heating process occurs, transverse vibrations of the beam take place; Boley's result, which was derived for the case of surface heating, can be adapted to the problem of

internally heated beams. Nevertheless, in this case, the structure shows a different behavior as it will be discussed later.

In the same domain, further works were published by Murozono [4] and Blandino and Thornton [5], who studied the case of slender cylindrical beams with uniform internal heating and non-uniform heat transfer coefficient on the surface. These studies confirmed that thermally induced flexural vibrations occur in a flexible structure with low fundamental frequencies.

Concerning nuclear engineering, the main contributions came from Burgreen [6,7] who studied the dynamic stresses induced in rods (analysis limited to the longitudinal behavior), thin shells, and solid spheres (structures typical of nuclear reactors and nuclear fuel) by rapid temperature pulses with uniform distribution; the effect of the duration of temperature pulses on dynamic stresses was analyzed.

In the field of high energy physics, studying the dynamic response of structures submitted to rapid internal heating is of paramount importance as in modern accelerators, short duration impacts on structures induced by highly energetic particle beams frequently occur (by accident or on purpose). Studies on these subjects were first performed by Bargmann [8] for the case of a uniformly rapidly heated rod, on the basis of Laplace transforms and by Sievers [9] for thin rods and disks, making use of Fourier and Fourier–Bessel series with some particular boundary conditions; both works only dealt with longitudinal vibrations.

Cited studies were performed from a mainly engineering point of view; many authors treated the problem from a more theoretical approach: Among the most relevant contributions are the works of Lessen [10,11], Chadwick and Sneddon [12], and Chadwick [13].

An extensive review on the subject of thermally induced waves and vibrations was made by Bargmann [14].

Virtually, all previous works on dynamic thermal stresses induced in rods and bars investigated separately the two problems of

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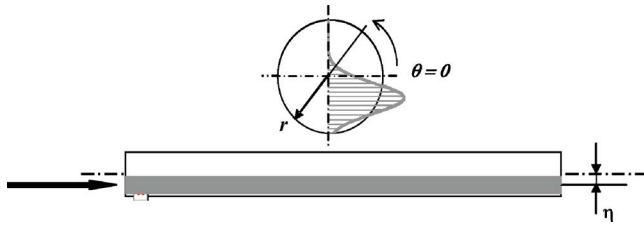


Fig. 1 Energy distribution on target rod due to proton beam impact

thermally induced flexural vibration and of longitudinal propagation of thermoelastic stress waves. Moreover, in the aerospace engineering field, the interest was mainly focused on surface heating problems, while in the nuclear engineering area, problems of rapid internal heating were studied considering only the case of uniform temperature distribution.

In the present paper, both longitudinal and flexural behaviors of rods excited by rapid nonuniform internal heating are studied; the contribution of quasistatic stresses as well as the one of the dynamic effect is considered leading to a complete evaluation of the displacements and of the thermal stress field.

The system considered is a cylindrical rod submitted to a rapid temperature rise induced by an internal nonuniform heat generation. The heat deposition is constant along the rod axis, and has an axially offset Gaussian distribution over the rod cross section (Fig. 1). Such an occurrence is common to many particle accelerator components exposed to high energy subatomic particle beams. In particular, this work was carried out to study *beam targets*: Usually, these objects are rods directly hit by a particle beam as to create a shower of secondary particles, required for physics experiments; such devices are widely employed in laboratories of particle physics. The *targets* analyzed in this paper are isotropic graphite cylinders axially hit by a proton beam: The interaction between protons and graphite nuclei rapidly generates large amounts of heat and triggers the emission, along with other particles, of a neutrino shower; the analysis presented has been applied at European Organization for Nuclear Research (CERN) for the design of targets in the frame of the CNGS experiment [15].

Though particle beams should ideally impact cylindrical targets on their axis, in practice, because of mechanical misalignments, the beam is eccentric with respect to the axis of the cylinder. This induces, on top of axial vibrations, lateral oscillations, leading, in some cases, to the collapse of the structure.

In this paper, both the longitudinal and flexural thermally induced vibrations are considered; radial waves are small enough (compared to quasistatic deformations) to be neglected, as shown by Mura [16].

An analytical method to calculate first the temperature distribution and then the quasistatic and dynamic thermal stresses is fully developed on the basis of the work carried out by Bertarelli [17], Bertarelli and Kurtyka [18], and Dallochio et al. [19].

A parametric study is also performed allowing to determine the influence of thermal shock duration on the longitudinal dynamic stresses and of the ratio of the diffusion time to the fundamental flexural period on dynamic bending stress and lateral oscillations.

In the last section, a comparison between the analytical model and experimental measurements performed with a laser vibrometer, Wilfinger [20] is presented.

2 Analytical Model: Main Parameters and Basic Hypotheses

We consider a thin cylindrical rod of radius R and length L ; the rod is supposed simply supported and is free to expand at its extremities. It is assumed that the deposited energy has a Gaussian distribution over the cross section, with standard deviation φ and eccentricity η with respect to the axis of the cylinder (Fig. 1); in

spite of this assumption, the method is valid for any energy distribution provided it is longitudinally constant and writable as a Fourier-series expansion. An isotropic material with linear elastic behavior, without damping, was assumed. Thermal and mechanical properties were considered independent of temperature.

Heating processes due to particle beam impacts are extremely fast: The thermal shock typically lasts from a few nanoseconds to some microseconds. During this time τ , the deposited energy can be considered as linearly growing. As shown by Kalbreier et al. [21], it is possible to assume that no heat diffusion occurs during the thermal shock because the characteristic thermal diffusion time t_d is much longer than the thermal shock duration τ ; hence, the rod, hit by the particle beam, sees a rapid temperature rise proportional to the deposited energy.

On the same ground, the system can be considered adiabatic during a typical time of several milliseconds (no heat exchange through the outer surface of the cylinder is taken into account; so, the total deposited energy remains constant).

The evaluation of thermal stresses is based on the linear theory of thermoelasticity; as stated above, the rapidity of the heat deposition invalidates the usual assumption that the effects of inertia may be disregarded: In fact, it will be shown that inertia plays a major role in the buildup of longitudinal and flexural stresses. However, in spite of the rapidity of the phenomenon, it can be demonstrated that the effects of radial inertia on longitudinal waves can be neglected if the frequency of the system is small compared to a certain reference value [22]: This is true for our case, as it will be proved later.

In its general expression, the linear theory of thermoelasticity states that a full coupling exists between thermal and structural effects; this means that temperature variation influences the strain field but also that the rate of dilatational strain generates heat affecting the temperature field. As shown by Boley and Weiner [23], this coupling can be neglected if the time rate of change of the axial strain is of the same order of magnitude of that of temperature: It will be seen that this is true, at least for the initial phase of the thermomechanical response, which is the one of highest interest for this analysis.

On this ground, it is possible to consider this as a weakly coupled thermoelastic problem, i.e., the elastic strains are influenced by the temperature distribution but not the inverse.

3 Thermal Analysis

3.1 Introduction. For the class of problems of interest, the temperature distribution can be supposed constant along the rod axis, thus reducing the analysis to a two-dimensional study. At the end of the thermal shock, when heat is no longer generated inside the cylinder, the following diffusion equation applies:

$$\frac{\partial^2 T(r, \theta, t)}{\partial r^2} + \frac{1}{r} \frac{\partial T(r, \theta, t)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T(r, \theta, t)}{\partial \theta^2} = \frac{1}{\kappa} \frac{\partial T(r, \theta, t)}{\partial t} \quad (1)$$

$$\nabla^2 T(r, \theta, t) = \frac{1}{\kappa} \frac{\partial T(r, \theta, t)}{\partial t}$$

3.2 Initial and Boundary Conditions. Given that the maximum energy density $U_{\max}$ is found at $r = \eta$, $\theta = 3\pi/2$, the specific energy takes the following expression:

$$U(r, \theta) = U_{\max} e^{-(r^2 + \eta^2 + 2r\eta \sin \theta)/2\varphi^2} \quad (2)$$

Having assumed that no heat diffusion occurs during the heating period, the temperature at the end of the thermal shock can be simply calculated via the following equation:

$$T_0(r, \theta) = \frac{U(r, \theta)}{c_p} \quad (3)$$

Since the analysis begins at $t = \tau$, expression (3) gives the initial

conditions for the heat conduction equation (1).

Thanks to the adiabatic hypothesis, the total energy deposited on the rod remains constant; so, it is easy to calculate the final uniform temperature T_F of the cylinder:

$$T_F = \frac{\int_r \int_\theta U(r, \theta) dr d\theta}{c_p \pi R^2} \quad (4)$$

The boundary condition, stemming from the adiabatic hypothesis, is given by

$$\left. \frac{\partial T(r, \theta, t)}{\partial r} \right|_{r=R} = 0 \quad (5)$$

Equation (5) states that the temperature gradient on the outer surface must be zero.

Initial temperature distribution given by Eq. (3) can be replaced by its Fourier-series expansion (n is the Fourier expansion index and a_n are the usual series coefficients)

$$T_0(r, \theta) = \sum_{n=0}^{\infty} a_n(r) H_n(\theta) \quad (6)$$

where $H_n(\theta) = \cos(n\theta)$ if n is even and $H_n(\theta) = \sin(n\theta)$ if n is odd.

3.3 Temperature Distribution. We introduce nondimensional variables for the radial coordinate $\bar{r}$, time coordinate $\bar{t}$ (note that $\bar{t}=0$ corresponds to $t=\tau$), and temperature $\bar{T}$, as defined in the Nomenclature. Adopting these coordinates, the diffusion equation (1) assumes the following expression:

$$\nabla^2 \bar{T}(\bar{r}, \theta, \bar{t}) = \frac{\partial \bar{T}(\bar{r}, \theta, \bar{t})}{\partial \bar{t}} \quad (7)$$

Making use of the separation of variable method, the function $\bar{T}(\bar{r}, \theta, \bar{t})$ can be reduced to the following form:

$$\bar{T}(\bar{r}, \theta, \bar{t}) = \sum_n F_n(\bar{r}) G_n(\bar{t}) H_n(\theta) \quad (8)$$

where H_n is the harmonic term defined in Eq. (6). Expression (8) must satisfy the diffusion equation (7). Solution of the previous equation can be obtained by means of standard methods for partial differential equations and can be written in the following form:

$$\bar{T}(\bar{r}, \theta, \bar{t}) = \sum_n \sum_s C_{n,s} J_n(\lambda_{n,s} \bar{r}) e^{-\lambda_{n,s}^2 \bar{t}} H_n(\theta) \quad (9)$$

where J_n is a Bessel function of the first kind of order n , $C_{n,s}$ are numerical coefficients obtained from the initial condition (3), and $\lambda_{n,s}$ are the eigenvalues of the problem obtained from the application of the adiabatic condition (5) (which can be evaluated using the approximated expressions given by Abramowitz and Stegun [24]).

The temperature distribution $\bar{T}$ as a function of time $\bar{t}$ for $\eta = 0.6$ is shown in Fig. 2. Results are scaled to the final uniform temperature T_F (4), which is proportional to the total energy deposited on the cylinder.

Several plots are presented to describe the temperature evolution at different points of the cross section; maximum temperature $\bar{T}(\eta, 3\pi/2, 0)$ takes place at the center of the Gaussian distribution, where the energy deposition has its maximum value. The heat diffusion process is virtually completed at time $\bar{t}=1$, when temperature distribution becomes practically uniform and equals to T_F ($\bar{T} \cong 1$).

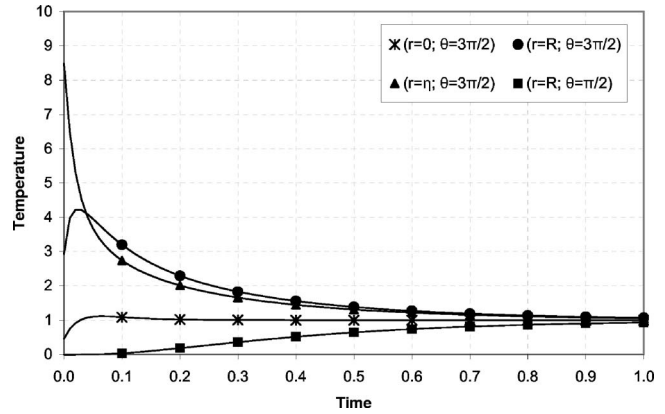


Fig. 2 Temperature $\bar{T}$ as a function of time $\bar{t}$ with eccentricity $\bar{\eta}=0.6$

4 Quasistatic Stresses

4.1 Introduction. Once the temperature distribution is known, it is possible to obtain the quasistatic stresses adapting a method developed by Goodier [25], and described in the book of Timoshenko and Goodier [26]; this method has been applied to a plane-strain case, assuming that no longitudinal expansion occurs. The exact boundary condition is subsequently restored. Stress components are calculated in two different steps: First, the stresses derived from the application of a nondimensional displacement potential $\bar{\psi}(\bar{r}, \theta, \bar{t})$ are evaluated. Goodier has shown that, in the case of pure heat conduction, the general thermoelastic equation (also known as the Duhamel–Neumann form of Hooke's law) is automatically satisfied if the nondimensional displacement potential $\bar{\psi}$ is a solution of the following equation:

$$\frac{\partial \bar{\psi}(\bar{r}, \theta, \bar{t})}{\partial \bar{t}} = \frac{1 + \nu}{1 - \nu} \bar{T} \quad (10)$$

Since $\bar{T}$ becomes uniform when $\bar{t} \rightarrow \infty$, $\bar{\psi}$ can be immediately calculated as

$$\bar{\psi}(\bar{r}, \theta, \bar{t}) = -\frac{1 + \nu}{1 - \nu} \sum_n \sum_s \frac{C_{n,s} J_n(\lambda_{n,s} \bar{r}) e^{-\lambda_{n,s}^2 \bar{t}}}{\lambda_{n,s}^2} H_n(\theta) + B \quad (11)$$

Here, B is a generic constant of integration, not affecting the displacement field.

Once $\bar{\psi}$ is known, nondimensional displacement components in radial and tangential directions $\bar{u}'$ and $\bar{v}'$ can be easily found by means of the following relations:

$$\bar{u}' = \frac{\partial \bar{\psi}}{\partial \bar{r}} \quad (12)$$

$$\bar{v}' = \frac{1}{\bar{r}} \frac{\partial \bar{\psi}}{\partial \theta}$$

Subsequently, strain and stress components, $\bar{\sigma}'_r$, $\bar{\sigma}'_\theta$, $\bar{\tau}'_{r\theta}$, and $\bar{\sigma}'_z$, can be calculated from general kinematic relations for a two-dimensional problem and Hooke's general law (Duhamel–Neumann) when the axial strain ϵ_z is equal to zero:

$$\epsilon'_r = \frac{\partial \bar{u}'}{\partial \bar{r}} \quad (13)$$

$$\epsilon'_\theta = \frac{\bar{u}'}{\bar{r}} + \frac{1}{\bar{r}} \frac{\partial \bar{v}'}{\partial \theta}$$

$$\begin{aligned}
\gamma'_{r\theta} &= \frac{1}{\bar{r}} \frac{\partial \bar{u}'}{\partial \theta} + \frac{\partial \bar{v}'}{\partial \bar{r}} - \frac{\bar{v}'}{\bar{r}} \\
\bar{\sigma}'_r - \nu(\bar{\sigma}'_\theta + \bar{\sigma}'_z) &= (\epsilon'_r - \bar{T}) \\
\bar{\sigma}'_\theta - \nu(\bar{\sigma}'_r + \bar{\sigma}'_z) &= (\epsilon'_\theta - \bar{T}) \\
\bar{\sigma}'_z - \nu(\bar{\sigma}'_r + \bar{\sigma}'_\theta) &= -\bar{T} \\
\bar{\tau}'_{r\theta} &= \frac{\gamma'_{r\theta}}{(1 + \nu)}
\end{aligned} \tag{14}$$

The stress distribution calculated from the displacement potential satisfies the thermoelastic equation, but not the boundary condition, requiring no forces on the external surface of the cylinder. Invoking the principle of superposition, a pressure field can be added in order to remove nonzero stresses on the lateral rod surface, thus restoring the correct boundary conditions. To do so, the formulation of the nondimensional Airy stress function $\bar{\Phi}(\bar{r}, \theta, \bar{t})$, as derived by Mitchell (Timoshenko and Goodier [26]), is invoked for an ordinary plane-strain problem in polar coordinates. Once $\bar{\Phi}(\bar{r}, \theta, \bar{t})$ is known, nondimensional stress components $\bar{\sigma}''_r$, $\bar{\sigma}''_\theta$, and $\bar{\tau}''_{r\theta}$ can be calculated from the following expressions:

$$\begin{aligned}
\bar{\sigma}''_r &= \frac{1}{\bar{r}} \frac{\partial^2 \bar{\Phi}}{\partial \bar{r}^2} + \frac{1}{\bar{r}^2} \frac{\partial^2 \bar{\Phi}}{\partial \theta^2} \\
\bar{\sigma}''_\theta &= \frac{\partial^2 \bar{\Phi}}{\partial \bar{r}^2} \\
\bar{\tau}''_{r\theta} &= -\frac{\partial}{\partial \bar{r}} \left(\frac{1}{\bar{r}} \frac{\partial \bar{\Phi}}{\partial \theta} \right)
\end{aligned} \tag{15}$$

4.2 Quasistatic Stresses for Zero-Axial Strain. By superposing the stress components (14) and (15), it is possible to calculate the nondimensional quasistatic stresses. Previous calculations were made in the hypothesis of zero-axial strain: The resulting axial stress $\bar{\sigma}_{z0}$ is that of a rod whose axial deformation is prevented. Hence, we obtain

$$\begin{aligned}
\bar{\sigma}_r &= \bar{\sigma}'_r + \bar{\sigma}''_r \\
\bar{\sigma}_\theta &= \bar{\sigma}'_\theta + \bar{\sigma}''_\theta \\
\bar{\tau}_{r\theta} &= \bar{\tau}'_{r\theta} + \bar{\tau}''_{r\theta} \\
\bar{\sigma}_{z0} &= \nu(\bar{\sigma}_r + \bar{\sigma}_\theta) - \bar{T}
\end{aligned} \tag{16}$$

The stresses given in (16) are nondimensional and scaled to a reference stress equal to $\sigma_{\text{ref}} = E\alpha T_F$; this value corresponds to the opposite of the compressive axial stress induced by a uniform temperature T_F in a rod with fixed ends.

The quasistatic stress distribution shown in Fig. 3 has a maximum at the end of the energy deposition $t = \tau$; these stresses tend to disappear as heat diffusion progresses and temperature distribution becomes uniform (apart from σ_{z0} which tends to σ_{ref}).

4.3 Equivalent Dynamic Loads. Axial stress in previous calculations was obtained in the hypothesis of zero-axial strain. In reality, the rod is longitudinally free at its extremities: To restore such condition, it is necessary to impose that the axial force and bending moment resulting from the axial stress at both ends of the rod are zero. To do so, we calculate the resultant axial force $\bar{F}_z(\bar{t})$ and bending moment $\bar{M}_x(\bar{t})$ with opposite sign (expressions (17)) and then apply these loads at the two extremities. By the de St. Venant's principle, at a certain distance from the ends, the actual

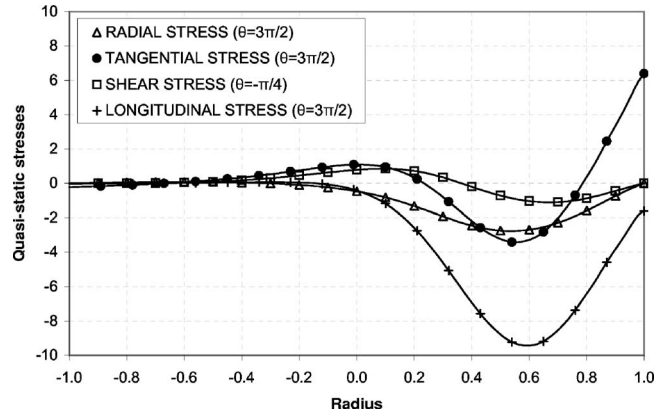


Fig. 3 Quasistatic in-plane stresses $\bar{\sigma}_r$, $\bar{\sigma}_\theta$, $\bar{\tau}_{r\theta}$ and axial stress $\bar{\sigma}_{z0}$ at zero-axial strain as a function of $\bar{r}$ ($t = \tau$, $\bar{\eta} = 0.6$)

axial stress on the rod free to expand is obtained by superimposing to $\bar{\sigma}_{z0}$ the stresses induced by $\bar{F}_z(\bar{t})$ and $\bar{M}_x(\bar{t})$.

$$\bar{F}_z(\bar{t}) = -2 \int_{-\pi/2}^{\pi/2} \int_0^1 \bar{\sigma}_{z0}(\bar{r}, \theta, \bar{t}) \bar{r} d\bar{r} d\theta \tag{17}$$

$$\bar{M}_x(\bar{t}) = -2 \int_{-\pi/2}^{\pi/2} \int_0^1 \bar{\sigma}_{z0}(\bar{r}, \theta, \bar{t}) \bar{r}^2 \sin \theta d\bar{r} d\theta$$

$$\begin{aligned}
F_z(t) &= \bar{F}_z(\bar{t})(E\alpha T_F R^2) \\
M_x(t) &= \bar{M}_x(\bar{t})(E\alpha T_F R^3)
\end{aligned} \tag{18}$$

Expressions (18) represent dimensional quantities.

5 Dynamic Stresses

5.1 Introduction. The effect of inertia is essential for the correct evaluation of a thermal shock; in fact, if the temperature change within a body is fast enough, its thermal expansion is initially prevented by the body mass, establishing a coupling between thermally originated elastic forces and inertia forces. In the case of a rod, this gives birth to dynamic stresses propagating along the rod axis as elastic stress waves: Assuming that the thermal shock time τ is very short, at $t = \tau$, the system is still in a compressive state; the stress relaxation due to elastic forces starts from the extremities of the rod; so, two stress waves appear, traveling from the extremities, superimposing at the center and reflecting at the other ends.

The structural dynamic effects provoked by the thermal shock can be studied as the response of the system to the variable loads acting at the rod ends: the force $F_z(t)$ and moment $M_x(t)$ (18). These loads, applied at the extremities of the beam and superimposed to the quasistatic stress field, are necessary to ensure the dynamic equilibrium.

Having assumed that no diffusion occurs during thermal shock, the foregoing loads grow up linearly from zero to the maximum value given by $F_z(\tau)$ and $M_x(\tau)$ at the end of the energy deposition; from this time on, $F_z(t)$ remains constant (since it is proportional to the deposited energy), while $M_x(t)$ decreases as the temperature distribution tends to become uniform because of thermal diffusion (Fig. 4). To calculate the dynamic response of the system, we have considered a piecewise linear interpolation of the curve of equivalent thermal bending moment. As shown in Fig. 4, the interpolation is quite coarse, but this did not notably affect the

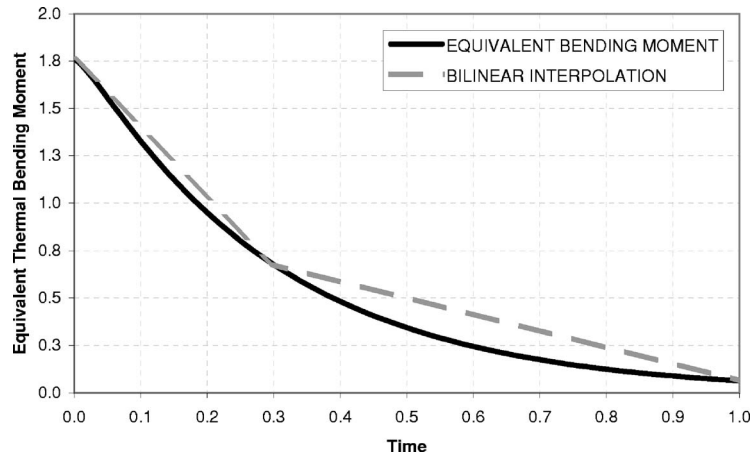


Fig. 4 Equivalent thermal bending moment $\bar{M}_x(\bar{t})$ ($\bar{t}=(t-\tau)/t_d$)

accuracy of the analysis.

Figure 5 shows the qualitative model of the dynamic loads and the corresponding unit functions $g(t)$ and $g'(t)$ (so, $F_z(t) = F_z(\tau)g(t)$ and $M_x(t) = M_x(\tau)g'(t)$).

At this stage of the analysis, we may wonder if the effects of force and moment can be treated separately. Indeed, this is the case, since $F_z(t)$, being the resultant of internal axial stresses, is always orthogonal to the rod cross section and gives no contribution to the bending moment, even when the rod is deflected.

As a consequence, no mutual influence exists between axial force and bending moment; so, the system can be considered linear, the axial and flexural behaviors can be studied separately, and the superposition principle applied.

To calculate the system time response, we have used the mode-summation method (see, e.g., Thomson) [27], which basically expands the deformation in terms of the longitudinal and flexural natural modes ($\phi_{z_i}(\bar{z})$ and $\phi_{f_i}(\bar{z})$) and of the generalized coordinates ($q_{z_i}(t)$ or $q_{f_i}(t)$) of a simply supported uniform beam, loaded at the extremities with $F_z(t)$ or $M_x(t)$, respectively. Given the slenderness of the rod, the classical Bernoulli–Euler beam theory has been assumed for the flexural behavior.

The equation of motion for each linearly independent mode is obtained by the application of Lagrange's equation, which leads to

$$\frac{d^2 q_i}{dt^2} + \omega_i^2 q_i = \frac{Q_i}{m_i} \quad (19)$$

where m_i and Q_i are the generalized masses and generalized

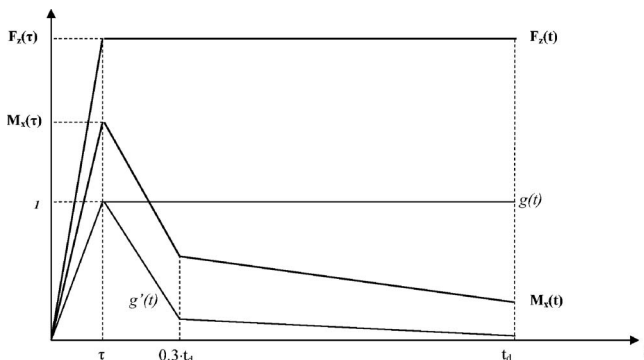


Fig. 5 Equivalent dynamic excitations (qualitative plot)

forces for the i th mode.

The generalized force Q_i is obtained from the work done by the load $F_z(t)$ or $M_x(t)$ applied at the two ends in the virtual displacement δq_i .

5.2 Flexural Modal Analysis. In the case of bending, the lateral displacement $w(z,t)$ of a simply supported beam can be expanded in terms of the natural modes and of the generalized coordinates as follows:

$$w(z,t) = \sum_i \phi_{f_i}(z) q_{f_i}(t) \quad (20)$$

The expressions of mode shapes and natural (circular) frequencies for a simply supported beam under bending are given by

$$\phi_{f_i}(z) = \sqrt{2} \sin\left(i\pi \frac{z}{L}\right) \quad (21)$$

$$\omega_{f_i} = (i\pi)^2 \sqrt{\frac{EJ}{mL^3}} \quad (22)$$

If the bending moment $M_x(t)$, as given in Fig. 5, is applied at the rod extremities, the generalized force for the i th mode takes the following expression:

$$\frac{Q_{f_i}(z)}{m_i} = \frac{M_x(\tau) \sqrt{2} i \pi}{mL} [1 - (-1)^i] g'(t) \quad (23)$$

where m is the mass of the rod and $g'(t)$ is the unit excitation function shown in Fig. 5.

Note that the modal shapes (21) cannot satisfy the natural boundary condition for the actual load case (at $z=0$, $z=L$ $\partial^2 w(0)/\partial z^2 = \partial^2 w(L)/\partial z^2 = 0$ always); however, it can be shown that w'' tends to its true value at the ends $\partial^2 w(0)/\partial z^2 = \partial^2 w(L)/\partial z^2 = M_x(t)/EJ$, provided a sufficiently large number of terms is used for expansion (20).

Generalized coordinates $q_{f_i}(t)$ are obtained by the well-known response of a single DOF system excited by the superposition of multiple ramp functions [27]. Given $t_2=0.3t_d$ and $t_3=t_d$,

$$F1_i = -\frac{M_x[0]}{m} \frac{\sqrt{2} i \pi}{L} [1 - (-1)^i]$$

$$F2_i = -\frac{M_x\left[\frac{t_2 - \tau}{t_3}\right]}{m} \frac{\sqrt{2} i \pi}{L} [1 - (-1)^i]$$

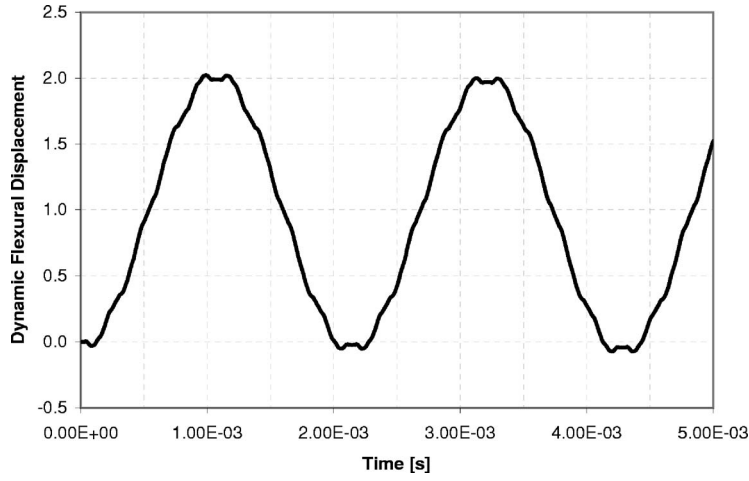


Fig. 6 Dynamic flexural response w/w_s at $z=L/2$ ($r=R$, $\varphi=3\pi/2$, $\eta=0.6$)

$$\begin{aligned}
 F3_i &= -\frac{M_x \left[\frac{t_3 - \tau}{t_3} \right]}{m} \frac{\sqrt{2}i\pi}{L} [1 - (-1)^i] \\
 F4_i &= \frac{F2_i \tau - F1_i t_2}{t_2 - \tau} \\
 F5_i &= F3_i - \frac{F1_i t_3}{\tau} - G2_i \frac{t_3 - \tau}{t_2 - \tau} \\
 q_{f_i}(t > t_2) &= \frac{F5_i}{(\omega_{f_i})^2} \left(\frac{t - t_2}{t_3 - t_2} - \frac{\sin[\omega_{f_i}(t - t_2)]}{\omega_{f_i}(t_3 - t_2)} \right) \\
 &+ \frac{F4_i}{(\omega_{f_i})^2} \left(\frac{t - \tau}{t_2 - \tau} - \frac{\sin[\omega_{f_i}(t - \tau)]}{\omega_{f_i}(t_2 - \tau)} \right) \\
 &+ \frac{F1_i}{(\omega_{f_i})^2} \left(\frac{t}{\tau} - \frac{\sin(\omega_{f_i} t)}{\omega_{f_i} \tau} \right) \\
 q_{f_i}(\tau \leq t \leq t_2) &= \frac{F4_i}{(\omega_{f_i})^2} \left(\frac{t - \tau}{t_2 - \tau} - \frac{\sin[\omega_{f_i}(t - \tau)]}{\omega_{f_i}(t_2 - \tau)} \right) \\
 &+ \frac{F1_i}{(\omega_{f_i})^2} \left(\frac{t}{\tau} - \frac{\sin(\omega_{f_i} t)}{\omega_{f_i} \tau} \right) \\
 q_{f_i}(0 < t < \tau) &= \frac{F1_i}{(\omega_{f_i})^2} \left(\frac{t}{\tau} - \frac{\sin(\omega_{f_i} t)}{\omega_{f_i} \tau} \right) \quad (24)
 \end{aligned}$$

Dynamic bending stress induced by lateral displacement $w(z, t)$ immediately follows:

$$\begin{aligned}
 \phi_{f_i}''(z) &= -\sqrt{2} \left(\frac{i\pi}{L} \right)^2 \sin \left(i\pi \frac{z}{L} \right) \\
 w''(z, t) &= \sum_i \phi_{f_i}''(z) q_{f_i}(t) \quad (25)
 \end{aligned}$$

$$\sigma_{f_d}(r, \theta, z, t) = E w''(z, t) r \sin \theta$$

where $w''(z, t)$ is the second derivative of the lateral displacement with respect to z .

In the following graphs, results are given for lateral displacement and bending stress at the middle of the rod, considering an

eccentricity $\eta=0.6R$; the results are scaled to the value of static lateral displacement w_s and static bending stress σ_{f_s} :

$$w_s \left(\frac{L}{2} \right) = \frac{M_x(\tau) L^2}{8EJ} \quad (26)$$

$$\sigma_{f_s} = \frac{M_x(\tau) R}{J} \quad (27)$$

Figure 6 shows that the dynamic response of the system is twice as large as the static one, confirming the findings of Boley [1]; the oscillations have a slight drift because the thermal moment decreases during the heat diffusion process.

Interestingly, Fig. 7 shows that the dynamic bending stress can be almost three times larger than the static one σ_{f_s} and more than six times larger if compared to the reference longitudinal stress $\sigma_{ref}=E\alpha T_F$. To the authors' knowledge, this was never evidenced in previous works.

5.3 Longitudinal Modal Analysis. The longitudinal dynamic stress is calculated by the same method used for dynamic bending. In this case, the variable of interest is the longitudinal displacement, which is given by

$$u_z(z, t) = \sum_i \phi_{z_i}(z) q_{z_i}(t) \quad (28)$$

The generalized forces are given by

$$\frac{Q_{z_i}(t)}{m_i} = \frac{F_z(\tau) \sqrt{2}}{m} [1 - (-1)^i] g(t) \quad (29)$$

Natural modes and natural frequencies are

$$\phi_{z_i}(z) = \sqrt{2} \cos \left(i\pi \frac{z}{L} \right) \quad (30)$$

$$\omega_{z_i} = \frac{i\pi}{L} \sqrt{\frac{E}{\rho}} \quad (31)$$

Generalized coordinates $q_{z_i}(t)$ are given by the response of a single DOF system excited by a ramp function

$$F_{z_i} = -\frac{F_z(\tau)}{m} \sqrt{2} [1 - (-1)^i]$$

$$q_{z_i}(t \geq \tau) = \frac{F_{z_i}}{(\omega_{z_i})^2} \left(1 - \frac{\sin(\omega_{z_i} t)}{\omega_{z_i} \tau} + \frac{\sin[\omega_{z_i}(t - \tau)]}{\omega_{z_i} \tau} \right) \quad (32)$$

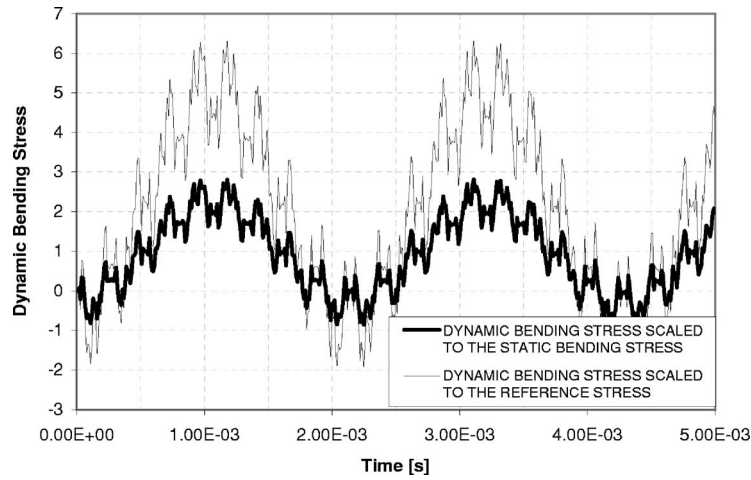


Fig. 7 Scaled dynamic bending stress (σ_{td}/σ_{ts} and σ_{td}/σ_{ref}) at $z=L/2$ ($r=R$, $\theta=3\pi/2$, $\eta=0.6R$)

$$q_{z_i}(t < \tau) = \frac{F_{z_i}}{(\omega_{z_i})^2} \left(\frac{t}{\tau} - \frac{\sin(\omega_{z_i}t)}{\omega_{z_i}\tau} \right)$$

Finally, the dynamic longitudinal stress component is calculated as follows:

$$\begin{aligned} \phi'_{z_i}(z) &= -\sqrt{2} \frac{i\pi}{L} \sin\left(i\pi \frac{z}{L}\right) \\ u'_z(z, t) &= \sum_i \phi'_{z_i}(z) q_{z_i}(t) \\ \sigma_{z_d}(z, t) &= E u'_z(z, t) \end{aligned} \quad (33)$$

where $u'_z(z, t)$ denotes the first derivative of u_z , with respect to z , i.e., the longitudinal strain.

Figure 8 shows the evolution over time of the longitudinal dynamic stress. The values are scaled with respect to σ_{ref} ; the maximum longitudinal dynamic stress is exactly two times larger than the reference stress value.

Figure 9 shows the progression of stress relaxation and the first reflection of the longitudinal stress wave; t_0 is the first fundamental period of longitudinal vibrations, $t_0=2L/c_0$; ($c_0=(E/\rho)^{1/2}$ is the velocity of longitudinal wave propagation). At time $t=\tau$, the

dynamic tensile stress, relaxing the quasistatic compressive stress, only affects the end parts of the rod (elsewhere $\sigma_{z_d}=0$); at time $t=\tau+t_0/4$, the two dynamic stress waves traveling in opposite directions have already begun to superimpose, attaining $\sigma_{z_d}/\sigma_{ref}=2$. At $t=t_0/2$, each of the two waves has reached the opposite end, and reflection starts to play its role.

5.4 Influence of Thermal Shock Duration on Maximum Longitudinal Dynamic Stress. In order to extend the applicability of this method to a broader class of problems, we want to analyze the influence of thermal shock duration τ on maximum dynamic longitudinal stress. The dynamic longitudinal stress is considered in the middle of the rod, where the maximum value is reached; it can be easily verified that the maximum stress at $\bar{z}=1/2$ always occurs at time $t=\tau+t_0/4$.

As shown by Eq. (34), the thermal shock response spectrum H , defined as the ratio between the dynamic and the reference stress, is now function only of the thermal shock parameter τ .

$$H(\tau) = \frac{\sigma_{z_d}\left(\frac{1}{2}, \tau + \frac{t_0}{4}\right)}{\sigma_{ref}} \quad (34)$$

By letting $\tau_0=\tau/t_0$, after some manipulations, we obtain

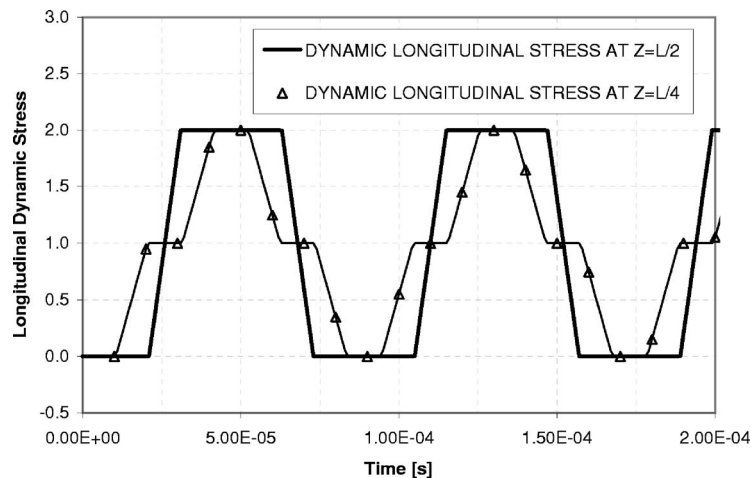


Fig. 8 Dynamic axial stress scaled to the reference stress $\sigma_{z_d}/\sigma_{ref}$ as a function of time at different location along the cylinder

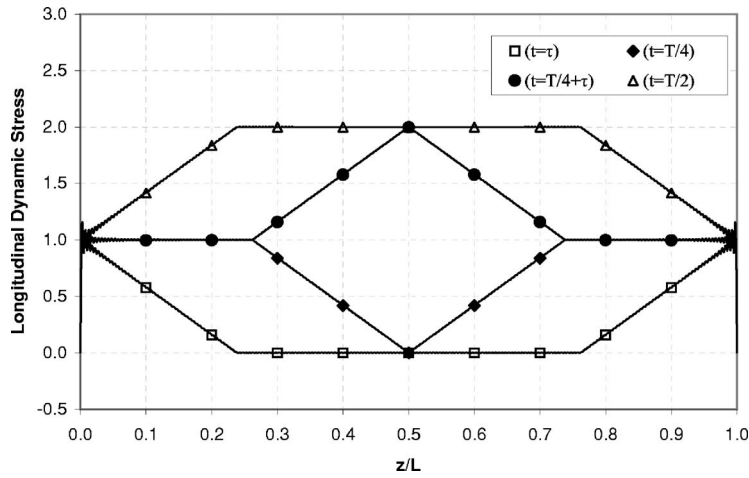


Fig. 9 Scaled dynamic axial stress σ_{zd}/σ_{ref} along rod length at different instants

$$H(\tau_0) = \frac{4}{\pi} \left[\sum_i \left[\frac{1 - (-1)^i}{2} \left[\frac{1}{i} + \frac{(-1)^{(i-1)/2} (1 - \cos(i2\pi\tau_0))}{i^2 2\pi\tau_0} \right] \right] \times (-1)^{(i-1)/2} \right] \quad (35)$$

Figure 10 shows that the maximum longitudinal dynamic stress is initially twice as large as the reference value. This relation holds until the parameter τ is smaller than half of the first fundamental period $t_0/2 = L/c_0$; if the thermal shock is longer than this, the heating process will be slower than the stress relaxation phenomenon occurring through wave propagation and the maximum longitudinal dynamic stress will never be reached; in other words, the temperature rise time τ is longer than the time required for the stress wave to propagate along the rod at the speed of sound c_0 .

This relation between maximum longitudinal dynamic stress, first fundamental period of the system and thermal shock duration τ , is very important in the design of systems submitted to thermal shocks; in fact, given a heating duration τ , the natural frequency of the system could be chosen, by modifying its geometry, so as to limit the maximum dynamic longitudinal stress.

5.5 Influence of the Ratio Between Thermal and Structural Characteristic Response Time on the Dynamic Maximum Deflection. Some considerations should also be done on the effect of the thermal shock parameter τ on the dynamic flexural response. In this case, τ is typically some orders of magnitude smaller than the first natural period of flexural oscillation; thus, varying τ has a negligible influence on the maximum dynamic flexural deflection. More interestingly, for the flexural behavior, the characteristic heat diffusion time t_d should be compared to the first fundamental period of bending oscillation $t_f = 2\pi/\omega_{f1}$, as illustrated by Boley [1,2] and Boley and Barber [3], who identified this ratio as the key parameter for the description of thermally induced oscillations; following his notation, we define $B = \sqrt{t_d/t_f}$.

In his papers, Boley analyzed the ratio of dynamic maximum deflection w_{dyn} to static maximum deflection w_{st} as a function of the parameter B (also known as Boley number) showing that, for beams submitted to rapid surface heating, thermally induced vibrations occur only when B is close to or smaller than 1.

In this paper, we have found that the dynamic behavior dependence upon B for internally heated beams is quite different.

We have first identified a range of values for B , from zero to 30,

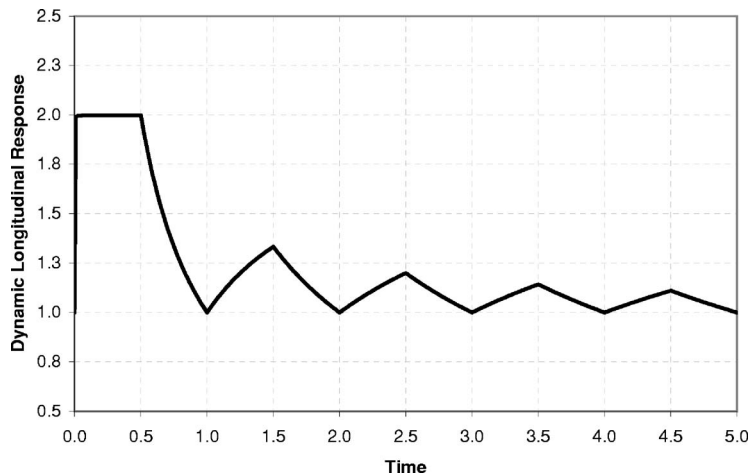


Fig. 10 Thermal shock longitudinal dynamic response $H = \sigma_{zd}(0.5, \tau + t_0/4)/\sigma_{ref}$ as a function of $\tau_0 = \tau/t_0$

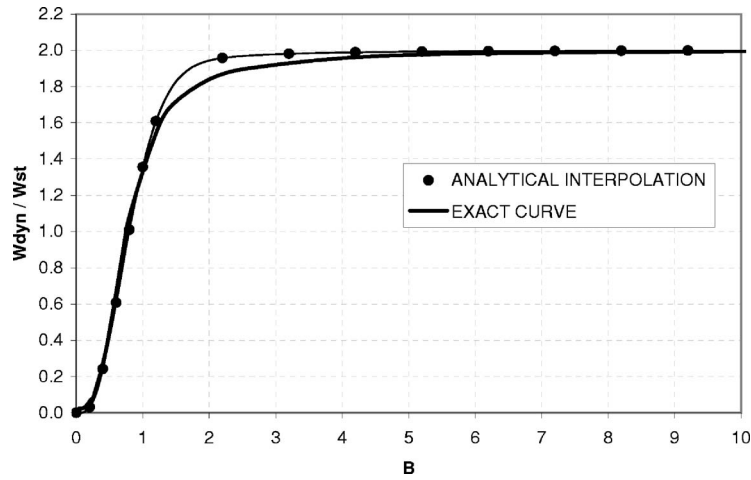


Fig. 11 Variation of the ratio of dynamic maximum deflection to static maximum deflection with parameter B at $z=L/2$

which covers several engineering applications. Values of B smaller than 1 correspond to flexible structures with low fundamental frequencies made up of materials with high thermal conductivity; higher values of B correspond to stiffer structures or to structures made up of materials with low thermal conductivity.

The variation of the ratio of dynamic maximum deflection to static maximum deflection as a function of B is plotted in Fig. 11 for the case of rapid internal heating. We can observe that for values of B larger than 2, the dynamic maximum deflection is roughly twice the correspondent static deformation; practically, this means that thermally induced flexural vibrations play a relevant role for values of B larger than 1.

For engineering design purposes, we present an analytical function that interpolates the exact curve (see Fig. 11):

$$\frac{w_{\text{dyn}}}{w_{\text{st}}} = \left(\frac{2B^2}{0.1 + B^2} \right) (1 - e^{-\pi B^2/2.3}) \quad (36)$$

It is also possible to evaluate the variation of the ratio of maximum dynamic bending stress to static bending stress with parameter B (see Fig. 12). As already shown in Sec. 5.2, this ratio could reach a value up to 3 (see Fig. 7).

Results showed above are quite different from the one presented by Boley in his papers [1–3]. Nevertheless, we can confirm that Boley's number is the key parameter that governs the dynamic phenomenon of thermally induced vibration; but, the range

of values for B and the relation between B and the dynamic flexural response of the structure are different depending on thermal boundary conditions.

5.6 Global Axial Stress. The global axial stress, as already mentioned, can be evaluated superimposing the zero-strain quasi-static component with the dynamic bending stress and the dynamic longitudinal stress:

$$\bar{\sigma}_{z_{\text{tot}}} = \frac{\sigma_{z0} + \sigma_{f_d} + \sigma_{z_d}}{\sigma_{\text{ref}}} \quad (37)$$

Figure 13 gives the evolution over time of the axial global stress scaled to the reference stress. It is important to observe that the peak value for the global stress at $r=R$, $\theta=3\pi/2$, and $\eta=0.6R$ is roughly six times larger than the reference stress and occurs at a time 100 times longer than the shock duration ($t_f \cong 1$ ms against $\tau \cong 10$ μ s).

This gives a valuable indication for the design of accelerator components submitted to thermal shocks.

5.7 Discussion on Radial Inertia Effect and Weak Thermoelastic Coupling. At this point of the analysis, we want to clarify with a brief discussion the initial assumptions of neglecting the influence of radial inertia on the longitudinal dynamic behavior.

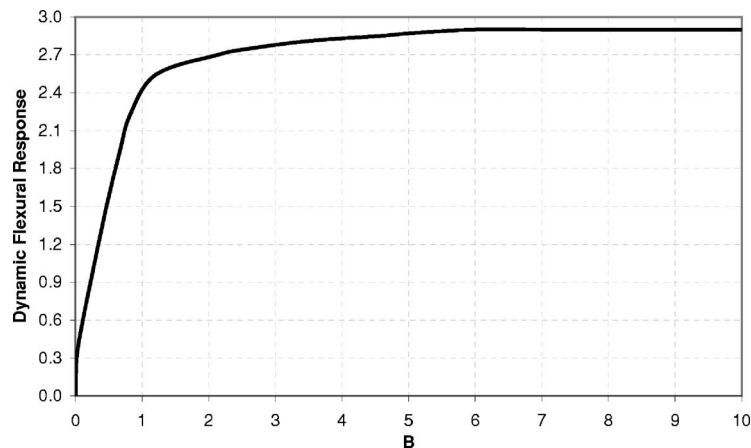


Fig. 12 Variation of the ratio of maximum dynamic bending stress to static bending stress with parameter B at $z=L/2$

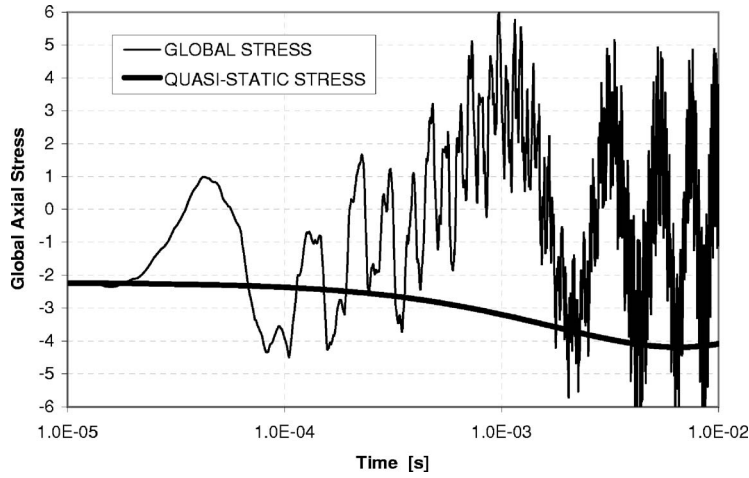


Fig. 13 Scaled global axial stress $\sigma_{z_{\text{tot}}}/\sigma_{\text{ref}}$ and quasistatic axial stress $\sigma_{z0}/\sigma_{\text{ref}}$ as a function of time at $r=R$, $\theta=3\pi/2$, $\eta=0.6R$ (logarithmic scale)

ior and of weak thermoelastic coupling.

For radial inertia, we follow the considerations made by Graff [22]. He found that, beyond a limiting working frequency, radial inertia must be considered and the simplified theory of longitudinal wave propagation in cylindrical rods cannot be applied.

Following Graff's notation, we define several nondimensional quantities: velocity of longitudinal wave propagation $\bar{c}$, wave number $\bar{\gamma}$, and pulse frequency $\bar{\omega}$:

$$\begin{aligned}\bar{c} &= \frac{c}{c_0} \\ c_0 &= \sqrt{\frac{E}{\rho}} \\ \bar{\gamma} &= \beta \nu \gamma \\ \bar{\omega} &= \bar{\gamma} \bar{c} \\ \beta &= \frac{R}{\sqrt{2}}\end{aligned}\quad (38)$$

Here, β is the radius of gyration of the cylinder, ν is Poisson's ratio, c is the actual velocity of longitudinal wave propagation, and c_0 is the reference value from the classical wave equation. After some simple manipulations, we obtain

$$\bar{\gamma} = \frac{\nu \omega R}{\sqrt{2} c} \quad (39)$$

If $\bar{\gamma} \ll 1$, the simplified classical theory of wave propagation can be applied and the effects of radial inertia can be neglected.

We can make some remarks on expression (39): If $\bar{\gamma} \ll 1$, we can assume $c=c_0$ (as confirmed also by Suhubi [28], for the case of infinitely long cylinders) and $\omega=\pi c_0/L$. Replacing in (39), we obtain

$$\frac{\nu \pi R}{\sqrt{2} L} \ll 1 \quad (40)$$

Since ν can assume values between 0 and 0.5, it is possible to write that

$$\frac{R}{L} \ll 1 \quad (41)$$

If (41) is verified, then the restriction on radial inertia is valid. For the application studied in this paper, $L=100$ mm and $R=2.5$ mm;

so, expression (41) is valid.

Concerning the weak thermoelastic coupling hypothesis, we follow the study of Boley and Weiner [23] showing that this assumption is valid if the strain rate is of the same order of magnitude of the temperature rate.

The coupled thermoelastic heat conduction equation in indicial notation is given by

$$K \nabla^2 T = \rho c_p \dot{T} + (3\lambda + 2\mu) \alpha T \dot{\epsilon}_{kk} \quad (42)$$

In (42), λ and μ are Lamé's constants and $\dot{\epsilon}_{kk}$ is the trace of the strain rate tensor. From expression (42), neglecting radial and circumferential terms, we immediately obtain the thermoelastic coupling term Γ ; if $\Gamma \ll 1$, then the weak coupling assumption can be applied:

$$\begin{aligned}K \nabla^2 T &= \rho c_p \dot{T} (1 + \Gamma) \\ \Gamma(r, \theta, z, t) &= \left[\frac{(3\lambda + 2\mu) \alpha T}{\rho c_p} \right] \left(\frac{\dot{\epsilon}_z}{\dot{T}} \right)\end{aligned}\quad (43)$$

In expression (41), $\dot{\epsilon}_z = \partial/\partial t (\epsilon_{z_d}(z, t) + \epsilon_{f_d}(z, r, \theta, t))$ is the time derivative of the dynamic axial strain due to flexural and longitudinal oscillations.

Figure 14 shows the evolution over time of the thermoelastic coupling term Γ evaluated at $z=L/2$ in the coldest zone of the rod cross section ($R, \pi/2$), where the temperature change rate is small and the strain rate is high; it is possible to observe that the parameter is always much smaller than unit; so, the weak coupling assumption is valid, at least in the first milliseconds of the analysis.

6 Comparison With Experimental Results

We present here a comparison between the analytical model described in this paper and the experimental measurements performed at CERN on a CNGS target rod hit by a proton beam. The experimental test exploits the procedure elaborated by Wilfinger [20]: A laser Doppler vibrometer suitably positioned picks up the lateral velocity of the rod surface at $z=L/2$; the flexural displacement is obtained by integrating the recorded signal.

Figure 15 shows the comparison between experimental data and analytical calculation: We can observe that the frequency and the shape of the flexural oscillations of the rod are in very good agreement; the amplitude is scaled with respect to the intensity and position of the proton beam impacting the rod (η, θ, φ , and U_{max} are directly dependent on the proton beam parameters). The dis-

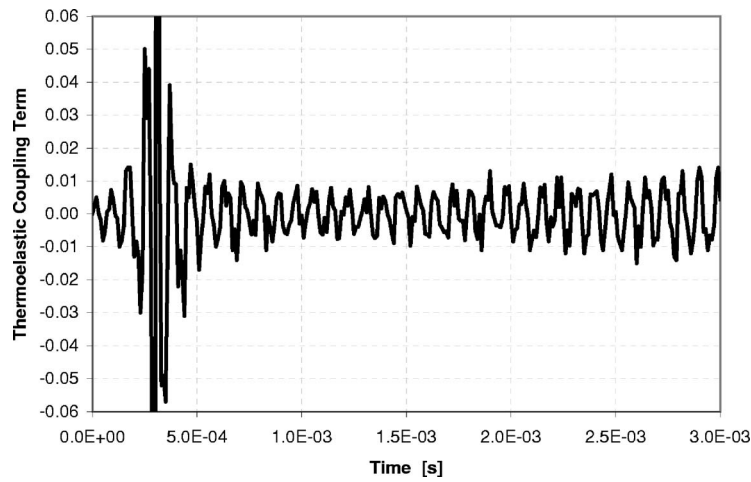


Fig. 14 Thermoelastic coupling term as a function of time at $r=R$, $\theta=\pi/2$

crepancy between the curves shown in Fig. 15 is mainly due to damping, not considered in the analytical model.

Experimental measurements showed that CNGS target rods behaved as if they were completely free at the extremities rather than simply supported because of mechanical play on the bearings. Flexural dynamic displacement shown in Fig. 15 has been obtained using expressions for modal shapes and natural circular frequencies derived by Blevins [29], instead of those given in (21) and (22).

For a free-free rod, the expressions of the natural circular frequencies and of the modal shapes are as follows:

$$\omega_{fi} = \chi_i^2 \sqrt{\frac{EJ}{mL^3}} \quad (44)$$

$$\phi_{fi}(z) = \cosh\left(\omega_{fi} \frac{z}{L}\right) + \cos\left(\omega_{fi} \frac{z}{L}\right) - \xi_i \left[\sinh\left(\omega_{fi} \frac{z}{L}\right) + \sin\left(\omega_{fi} \frac{z}{L}\right) \right] \quad (45)$$

In expression (44), coefficients χ_i can be evaluated numerically if $i < 5$ while $\chi_i = (2i+1)\pi/2$ if $i > 5$.

Similarly, in expression (45), coefficients ξ_i can be evaluated numerically if $i < 5$ and $\xi_i = 1$ if $i > 5$.

By using formulas (44) and (45), it is possible to obtain the displacement $w(z,t)$ by means of expression (20). Figure 15 shows that the period of flexural oscillation is roughly 1 ms; if we consider the simply supported rod, we can observe a period of flexural oscillation of more than 2 ms, as shown in Fig. 6. The difference is due to the distinct support conditions of the rod.

7 Summary and Conclusions

The analytical model presented in this paper allows the complete evaluation, within the linear elastic domain, of quasistatic and dynamic thermal stresses induced in an isotropic cylindrical rod by rapid internal heating.

We have considered a heat deposition constant over the rod length with a Gaussian distribution over the cross section, which is typically provoked by subnuclear particle beams directly impacting the rod. However, the method can be extended to any other problem of thermal shock on cylindrical rods with an arbitrary energy distribution on the cross-section, longitudinally constant.

As compared to previous studies, the approach used in this

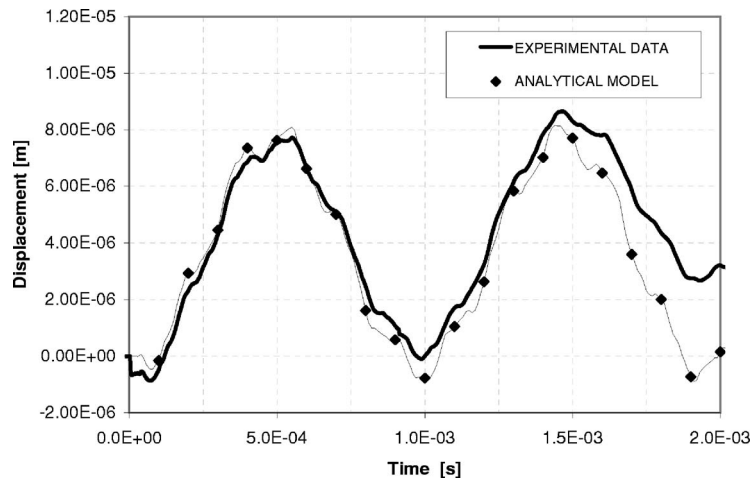


Fig. 15 Flexural displacement at the rod center $z=L/2$; comparison between analytical model and experimental data

paper, to the authors' knowledge, is original and allows a fast evaluation of the complex stress field induced by thermal shocks acting on isotropic cylindrical rods.

The solution of the problem is divided into three main steps: the evaluation of the temperature distribution as a function of space and time, the evaluation of the quasistatic thermal stress components, and the study of the dynamic problem both for the longitudinal and flexural behaviors.

On the basis of the aforementioned assumptions, the heat diffusion equation is solved using a Fourier–Bessel expansion. Once the temperature distribution is known, quasistatic stresses have been calculated by means of the thermoelastic displacement potential in the plane-strain hypothesis; Airy stress function is also applied to restore the free boundary condition on the lateral surface of the cylinder.

Due to the rapidity of the heat deposition, the inertia effect of the structure cannot be neglected: Two equivalent variable loads $F_z(t)$ and $M_x(t)$ applied at the extremities of the rod are introduced to model the effect of the fast nonuniform temperature rise and restore the correct boundary conditions; the response of the system is evaluated by way of the mode-summation method.

The global axial stress can be assessed via the superposition of the quasistatic axial stress at zero-axial strain with the longitudinal and flexural dynamic stresses induced by $F_z(t)$ and $M_x(t)$. An analytical expression has been developed to describe the influence of the thermal shock duration τ on the maximum dynamic longitudinal stress.

The influence of the ratio between thermal and structural characteristic response times (Boley number B) on the dynamic maximum deflection has been studied and extended to the case of rapid internal heating: It has been confirmed that B is the key parameter for evaluating the dynamic behavior of a rapidly heated slender structure; however, this behavior depends on the way the heating occurs and on the thermal boundary conditions.

In the last section, a comparison between the analytical model and an experimental measurement is presented; curves show that the analytical calculations give very good prediction of the real behavior of the structure.

The model developed permits to quickly obtain good estimates of the thermoelastic behavior of isotropic rods submitted to thermal shocks. Among other findings, it is worth noting that, for the case of interest, dynamic bending stress can be up to three times larger than the corresponding quasistatic stress.

The results of this study could be usefully applied for the design of slender structures submitted to thermal shocks typically used in particle accelerator technology, but easily extendable to aerospace and nuclear applications.

Nomenclature

$A = \pi R^2$	= cross-section area
E	= Young's modulus
$J = \pi R^4/4$	= cross-section moment of inertia
K	= thermal conductivity
L	= length of the cylinder
R	= radius of the cylinder
T	= temperature
T_F	= final uniform temperature
$\bar{T} = T/T_F$	= nondimensional temperature
$U_{\max}$	= maximum deposited specific energy
$c_0 = \sqrt{E/\rho}$	= velocity of longitudinal waves (classical wave equation)
c_p	= specific heat
$m = \rho AL$	= mass of the rod
r	= radial coordinate
$\bar{r} = r/R$	= nondimensional radial coordinate
q_f	= flexural generalized coordinate for modal analysis

q_z	= longitudinal generalized coordinate for modal analysis
t	= time coordinate
$\bar{t} = (t - \tau)/t_d$	= nondimensional time coordinate
$t_0 = 2L/c_0$	= fundamental period of longitudinal vibrations
$t_d = R^2/\kappa$	= thermal diffusion time
u_z	= longitudinal displacement
w	= lateral displacement (rod deflection)
z	= longitudinal coordinate
$\bar{z} = z/L$	= nondimensional longitudinal coordinate
Γ	= thermoelastic coupling term
$\bar{\Phi} = \Phi/(E\alpha T_F)$	= nondimensional Airy stress function
α	= thermal expansion coefficient
η	= eccentricity of the energy distribution with respect to the rod axis
$\bar{\eta} = \eta/R$	= nondimensional eccentricity
θ	= angular coordinate
$\kappa = K/\rho c_p$	= thermal diffusivity
$\lambda = \nu E/[(1 + \nu)(1 - 2\nu)]$	= first Lamé's constant
$\mu = G = E/2(1 + \nu)$	= second Lamé's constant
ν	= Poisson's ratio
ρ	= density
$\sigma_{\text{ref}} = E\alpha T_F$	= reference axial stress
τ	= thermal shock duration
φ	= standard deviation of the energy (Gaussian) distribution
$\bar{\varphi} = \varphi/R$	= dimensionless standard deviation of the energy distribution
ϕ_z	= longitudinal modal shape
ϕ_f	= flexural modal shape
ψ	= displacement potential
$\bar{\psi} = \psi/(\alpha T_F \kappa)$	= nondimensional displacement potential

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Homogenization Based 3D Continuum Damage Mechanics Model for Composites Undergoing Microstructural Debonding

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This paper develops a microscopic homogenization based continuum damage mechanics (HCDM) model framework for fiber reinforced composites undergoing interfacial debonding. It is an advancement over the 2D HCDM model developed by Raghavan and Ghosh (2005, "A Continuum Damage Mechanics Model for Unidirectional Composites Undergoing Interfacial Debonding," Mech. Mater., 37(9), pp. 955–979), which does not yield accurate results for nonproportional loading histories. The present paper overcomes this shortcoming through the introduction of a principal damage coordinate system (PDCS) in the HCDM representation, which evolves with loading history. The material behavior is represented as a continuum constitutive law involving a fourth order orthotropic tensor with stiffness characterized as a macroscopic internal variable. The current work also extends the model of Raghavan and Ghosh to incorporate damage in 3D composites through functional forms of the fourth order damage tensor in terms of macroscopic strain components. The model is calibrated by homogenizing the micromechanical response of the representative volume element (RVE) for a few strain histories. This parametric representation can significantly enhance the computational efficiency of the model by avoiding the cumbersome strain space interpolations. The proposed model is validated by comparing the CDM results with homogenized micromechanical response of single and multiple fiber RVEs subjected to arbitrary loading history.

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Keywords: continuum damage mechanics, homogenization, interfacial debonding, cohesive zone element, principal damage coordinate system

1 Introduction

Structural failure of composite materials is inherently a multiple scale phenomenon coupling different scales of damage initiation and progression. Microstructural damage mechanisms and structural failure properties are sensitive to the local variations in morphology, such as clustering and variations in reinforcement shape or size. An analysis of composite materials with microstructural heterogeneities is conventionally done with phenomenological macroscopic properties, obtained either from macroscopic experiments or by homogenizing response functions at smaller length scales. Such homogenization schemes are important to develop continuum constitutive or damage models to be used in a macroscopic computational analysis. Continuum damage mechanics (CDM) models, developed in Refs. [1–5], provide a framework for macroscopic anisotropic constitutive equations reflecting damage induced stiffness reduction or softening. Effective damage parameters and variables are introduced in CDM to represent average material deterioration with diffused damage evolution at the microscale. CDM models are of two types, viz., phenomenological and micromechanical models. The phenomenological CDM models [5–9] employ scalar, second order, and fourth order damage tensors using mathematically and thermodynamically consistent

formulations of damage mechanics. In general, they do not explicitly account for the microstructural morphology and the evolving microscopic damage mechanisms. In most cases, damage parameters are identified in a heuristic manner through a set of macroscopic experiments for different materials. Micromechanics based approaches [10–15] on the other hand, conduct a micromechanical analysis of a representative volume element (RVE) with subsequent homogenization to predict material behavior with evolving damage. With the exception of a few, e.g., Refs. [2,12,16–18] most damage models account for neither the evolution of damage nor the effect of loading history in the damage parameters. A significant error can consequently accrue in the solution of problems, especially those that involve nonproportional loading. Some homogenization studies, e.g., Refs. [12,17], have overcome this shortcoming through the introduction of simultaneous RVE-based microscopic and macroscopic analyses in each load step. However, such approaches can be computationally very expensive since detailed micromechanical analyses need to be conducted in each load step at every integration point in elements of the macroscopic structure.

Raghavan and Ghosh [16] and Ghosh [19] have recently developed a computationally efficient, 2D anisotropic homogenization based CDM (HCDM) model for unidirectional composites with microstructural damage in the form of fiber-matrix interface decohesion. The model is constructed by homogenizing evolving damage variables in micromechanical analyses of a representative microstructural volume element. Micromechanical analyses of fiber-matrix debonding in nonuniform composite microstructures are conducted by the Voronoi cell finite element model (FEM)

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[20,21]. The HCDM model has been successfully used for a macroscopic analysis in multiscale modeling of composites undergoing fiber-matrix interfacial debonding in Ref. [19]. In a multiscale modeling framework, the use of a CDM model in regions of relatively low and nonlocalized damage in the multiscale analysis makes the overall computing very efficient. However, a top-down localization process for a pure microscopic analysis in critical regions of crack propagation or localized instability should necessarily accompany the macroscopic analysis to accurately predict catastrophic failure [19,22,23]. Physically based error measures are devised to initiate level change to regions of detailed microstructural analysis only at locations where the assumptions of RVE-based homogenization break down.

The anisotropic 2D HCDM model in Ref. [16] does not incorporate the effects of the path dependent load history on the damage variables; hence, its predictions are not accurate for, e.g., non-proportional loading cases. The present paper is aimed at overcoming the limitations of this model through the introduction of a principal damage coordinate system (PDCS), which evolves with the loading history. The PDCS has been used for a phenomenological damage model in Ref. [24] to predict formability of viscoplastic materials. The current work also extends the model in Ref. [16] to incorporate damage in 3D composites through functional forms of the fourth order damage tensor in terms of macroscopic strain components. Development and calibration of the parametric functions require micromechanical RVE analysis along different strain loading paths. This parametric representation can significantly enhance the computational efficiency of the model by avoiding the cumbersome strain space interpolations in Ref. [16]. The paper starts with a review of the 2D HCDM model of Ref. [16] in Sec. 2. Section 3 discusses the 3D HCDM model with a brief description of the micromechanics model. The accuracy of the model is demonstrated through numerical examples in Sec. 4.

2 Review of Micromechanical Homogenization Based 2D CDM Model

The 2D continuum damage model developed in Ref. [16] requires micromechanical analyses of composite microstructures undergoing explicit forms of damage, e.g., interfacial debonding or matrix cracking. The powerful Voronoi cell FEM (VCFEM) that has been developed in Refs. [20,21] is used for this analysis. A typical microstructural RVE model of Voronoi cell elements is shown in Fig. 1. In contrast to conventional FEM formulations, VCFEM makes the following assumptions. In each Voronoi cell element Ω_e , the stress fields σ_{ij}^m in the matrix phase Ω_m and σ_{ij}^c in the inclusion phase Ω_c are independent and equilibrated, and are expressed as

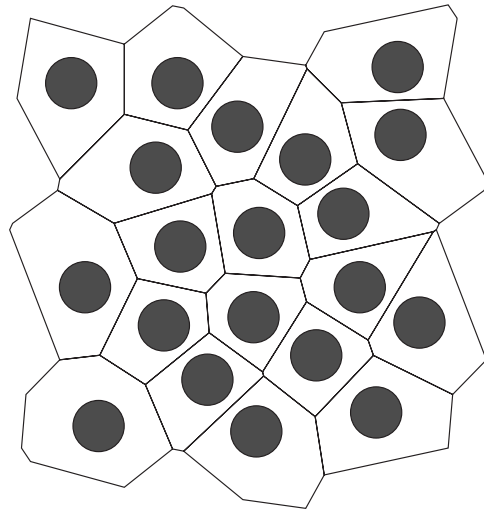


Fig. 1 A 2D microstructural RVE model of Voronoi cell elements with 20 circular fibers

$$\{\sigma^m\} = [\mathbf{P}^m]\{\beta^m\} \quad \text{in } \Omega_m \quad \text{and} \quad \{\sigma^c\} = [\mathbf{P}^c]\{\beta^c\} \quad \text{in } \Omega_c \quad (1)$$

where the matrices $[\mathbf{P}^{m/c}]$ are obtained from assumed stress functions and $\{\beta^{m/c}\}$ are unknown coefficients to be solved. Compatible displacement fields u_i^e are assumed on each Voronoi cell element boundary $\partial\Omega_e$ and interpolated in terms of displacements of the boundary nodes q_i^e as

$$\{\mathbf{u}^e\} = [\mathbf{L}^e]\{\mathbf{q}^e\} \quad (2)$$

Compatible displacement fields u_i^m and u_i^c are assumed on the matrix and inclusion sides of the matrix-inclusion interface $\partial\Omega_c$, respectively, and are interpolated from displacements q_i^m and q_i^c of the interface nodes as

$$\{\mathbf{u}^m\} = [\mathbf{L}^c]\{\mathbf{q}^m\} \quad \text{on } \partial\Omega_c^m \quad \text{and} \quad \{\mathbf{u}^c\} = [\mathbf{L}^c]\{\mathbf{q}^c\} \quad \text{on } \partial\Omega_c^c \quad (3)$$

where interpolation matrices $[\mathbf{L}^{e/c}]$ are constructed using standard linear or hierarchical shape functions.

In an incremental formulation for evolving damage, the potential energy functional for each element is expressed in terms of the incremented stresses and displacements as

$$\begin{aligned} \Pi_e(\sigma_{ij}^m, \Delta\sigma_{ij}^m, \sigma_{ij}^c, \Delta\sigma_{ij}^c, u_i^e, \Delta u_i^e, u_i^m, \Delta u_i^m, u_i^c, \Delta u_i^c) = & - \int_{\Omega_m} \frac{1}{2} S_{ijkl}^m \Delta\sigma_{ij}^m \Delta\sigma_{kl}^m d\Omega - \int_{\Omega_m} S_{ijkl}^m \sigma_{kl}^m \Delta\sigma_{ij}^m d\Omega - \int_{\Omega_c} \frac{1}{2} S_{ijkl}^c \Delta\sigma_{ij}^c \Delta\sigma_{kl}^c d\Omega \\ & - \int_{\Omega_c} S_{ijkl}^c \sigma_{kl}^c \Delta\sigma_{ij}^c d\Omega + \int_{\partial\Omega_e} (\sigma_{ij}^m + \Delta\sigma_{ij}^m) n_j^e (u_i^e + \Delta u_i^e) d\partial\Omega - \int_{\partial\Omega_c^m} (\sigma_{ij}^m + \Delta\sigma_{ij}^m) n_j^c (u_i^m \\ & + \Delta u_i^m) d\partial\Omega + \int_{\partial\Omega_c^c} (\sigma_{ij}^c + \Delta\sigma_{ij}^c) n_j^c (u_i^c + \Delta u_i^c) d\partial\Omega - \int_{\partial\Omega_c^m \cup \partial\Omega_c^c} \int_{u_n^m - u_n^c}^{u_n^m + \Delta u_n^m - u_n^c - \Delta u_n^c} T_n^m d(u_n^m \\ & - u_n^c) d\partial\Omega - \int_{\partial\Omega_c^m \cup \partial\Omega_c^c} \int_{u_t^m - u_t^c}^{u_t^m + \Delta u_t^m - u_t^c - \Delta u_t^c} T_t^m d(u_t^m - u_t^c) d\partial\Omega - \int_{\Gamma_{im}} (t_i + \Delta t_i) (u_i^e + \Delta u_i^e) d\Gamma \end{aligned} \quad (4)$$

Here, $(1/2)S_{ijkl}\sigma_{ij}\sigma_{kl}$ is the complementary energy density, ϵ_{ij}^m and ϵ_{ij}^c are strain fields in the matrix and inclusion phases, respectively, of each Voronoi element, and $\mathbf{t}$ is the prescribed traction on the boundary Γ_{lm} . The prefix Δ corresponds to increments, and subscripts n and t correspond to the normal and tangential directions at the matrix-inclusion interface. $\mathbf{n}^e$ and $\mathbf{n}^c$ are the outward normal on $\partial\Omega_e$ and $\partial\Omega_c$, respectively. The two terms on the matrix-inclusion interface $\partial\Omega_c^m \cup \partial\Omega_c^c$ provide the work done by the interfacial tractions $\mathbf{T}^m = T_n^m \mathbf{n}^m + T_t^m \mathbf{t}^m$ due to interfacial separation $(\mathbf{u}^m - \mathbf{u}^c)$. T_n^m and T_t^m are the normal and tangential components that are described by bilinear cohesive laws developed in Ref. [25]. In this model, the relation between the traction $T(\delta) = \sqrt{T_n^2 + T_t^2}$ and the effective opening displacement $\delta(\delta) = \sqrt{\delta_n^2 + \beta^2 \delta_t^2}$ in the cohesive zone model is given through a free energy potential as

$$t = \frac{\partial \phi(\delta)}{\partial \delta} \quad (5)$$

where δ_n and $\delta_t(\delta) = \sqrt{\delta_n^2 + \delta_t^2}$ are the displacement jumps in normal and tangential directions, respectively, and the factor β provides weight to the contribution of the tangential displacement jump. The t - δ relation in the bilinear model is expressed as

$$T = \begin{cases} \frac{\sigma_{\max}}{\delta_c} \delta & \text{if } \delta \leq \delta_c \text{ (hardening region)} \\ \frac{\sigma_{\max}}{\delta_c - \delta_e} (\delta - \delta_e) & \text{if } \delta_c < \delta \leq \delta_e \text{ (softening region)} \\ 0 & \text{if } \delta > \delta_e \text{ (complete debonding)} \end{cases} \quad (6)$$

Consequently, the normal and tangential tractions are derived

from the relations $T_n = \partial \phi / \partial \delta_n$ and $T_t = \partial \phi / \partial \delta_t$, where $i = 1, 2$. For a positive normal displacement jump δ_n , the traction at the interface increases linearly to a maximum value of $\sigma_{\max}$ corresponding to δ_c , then starts to decrease, and finally reaches zero at a value of δ_e . The unloading behavior in the hardening region follows the same slope as that of the loading path. In the softening region, unloading is assumed to follow a different linear path back from the current position to the origin with a reduced stiffness. This is expressed as

$$t = \frac{\sigma_{\max}}{\delta_{\max}} \frac{\delta_{\max} - \delta_e}{\delta_c - \delta_e} \delta, \quad \delta_c < \delta_{\max} < \delta_e \quad \text{and} \quad \delta < \delta_{\max} \quad (7)$$

The reloading follows the hardening slope and then continues along the softening slope. The normal and the tangential tractions vanish when interface debonds completely, i.e., $\delta \geq \delta_e$. Also, the tangential traction-displacement behavior is the same for both positive and negative tangential separations. If the normal displacement is negative, i.e., during compression, stiff penalty springs with high stiffness are introduced between the node pairs at the interface to prevent penetration.

Progressive debonding in composite microstructures is solved using an incremental approach. In each increment, a set of element and global equations are solved for stresses and displacements. Equations in each element are obtained by substituting increments of stress interpolations (Eq. (1)) and increments of displacement interpolations (Eqs. (2) and (3)) in the element energy functional of Eq. (4) and by setting variations with respect to the stress coefficients $\Delta \boldsymbol{\beta}^m$ and $\Delta \boldsymbol{\beta}^c$ to zero. This results in the weak forms of the element kinematic relations,

$$\begin{aligned} & \begin{bmatrix} \int_{\Omega_m} [\mathbf{P}^m]^T [\mathbf{S}^m] [\mathbf{P}^m] d\Omega & [\mathbf{0}] \\ [\mathbf{0}] & \int_{\Omega_c} [\mathbf{P}^c]^T [\mathbf{S}^c] [\mathbf{P}^c] d\Omega \end{bmatrix} \begin{Bmatrix} \boldsymbol{\beta}^m + \Delta \boldsymbol{\beta}^m \\ \boldsymbol{\beta}^c + \Delta \boldsymbol{\beta}^c \end{Bmatrix} \\ &= \begin{bmatrix} \int_{\partial\Omega_e} [\mathbf{P}^m]^T [\mathbf{n}^e] [\mathbf{L}^e] d\delta\Omega & - \int_{\partial\Omega_c} [\mathbf{P}^m]^T [\mathbf{n}^c] [\mathbf{L}^c] d\delta\Omega & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & \int_{\partial\Omega_c} [\mathbf{P}^c]^T [\mathbf{n}^c] [\mathbf{L}^c] d\delta\Omega \end{bmatrix} \begin{Bmatrix} \mathbf{q} + \Delta \mathbf{q} \\ \mathbf{q}^m + \Delta \mathbf{q}^m \\ \mathbf{q}^c + \Delta \mathbf{q}^c \end{Bmatrix} - \begin{Bmatrix} \int_{\Omega_m} [\mathbf{P}^m]^T \{\boldsymbol{\epsilon}^m\} d\Omega \\ \int_{\Omega_c} [\mathbf{P}^c]^T \{\boldsymbol{\epsilon}^c\} d\Omega \end{Bmatrix} \quad (8) \end{aligned}$$

or in a condensed form

$$[\mathbf{H}^e] \{\boldsymbol{\beta} + \Delta \boldsymbol{\beta}\} = [\mathbf{G}^e] \{\mathbf{q} + \Delta \mathbf{q}\} - \{\mathbf{R}^e\} \quad (9)$$

Here, $[\mathbf{n}^e]$ and $[\mathbf{n}^c]$ are matrices defined in terms of direction cosines of unit outward normal vectors to the element boundary and matrix-inclusion interface, respectively. Equation (9) is linear and is solved to express the stress coefficients in terms of the nodal displacements.

The total potential energy functional of the RVE containing N_{vc} Voronoi cell elements, as shown in Fig. 1, is $\Pi = \sum_{e=1}^{N_{vc}} \Pi_e$. The weak forms of the global traction continuity conditions are solved by setting the variation of the total energy functional Π with respect to $\Delta \mathbf{q}$, $\Delta \mathbf{q}^m$, and $\Delta \mathbf{q}^c$, to zero. This results in the weak form of the traction reciprocity conditions as

$$\sum_{e=1}^N \begin{bmatrix} \int_{\partial\Omega_e} [\mathbf{L}^e]^T [\mathbf{n}^e]^T [\mathbf{P}^m] d\partial\Omega & [\mathbf{0}] \\ - \int_{\partial\Omega_m^c} [\mathbf{L}^c]^T [\mathbf{n}^c]^T [\mathbf{P}^m] d\partial\Omega & [\mathbf{0}] \\ [\mathbf{0}] & \int_{\partial\Omega_c^c} [\mathbf{L}^c]^T [\mathbf{n}^c]^T [\mathbf{P}^c] d\Omega \end{bmatrix} \begin{Bmatrix} \boldsymbol{\beta}^m + \Delta\boldsymbol{\beta}^m \\ \boldsymbol{\beta}^c + \Delta\boldsymbol{\beta}^c \end{Bmatrix} = \sum_{e=1}^N \begin{Bmatrix} \int_{\Gamma_{im}} [\mathbf{L}^e]^T \{\bar{\mathbf{t}} + \Delta\bar{\mathbf{t}}\} d\Omega \\ - \int_{\partial\Omega_m^c} [\mathbf{L}^c]^T (\{\mathbf{n}^c\} T_n^m(u_n + \Delta u_n, u_t + \Delta u_t) + \{\mathbf{t}^c\} T_t^m(u_n + \Delta u_n, u_t + \Delta u_t)) d\partial\Omega \\ - \int_{\partial\Omega_c^c} [\mathbf{L}^c]^T (\{\mathbf{n}^c\} T_n^m(u_n + \Delta u_n, u_t + \Delta u_t) + \{\mathbf{t}^c\} T_t^m(u_n + \Delta u_n, u_t + \Delta u_t)) d\partial\Omega \end{Bmatrix} \quad (10)$$

or in the condensed form

$$\sum_{e=1}^N [\mathbf{G}^e]^T \{\boldsymbol{\beta} + \Delta\boldsymbol{\beta}\} = \sum_{e=1}^N \{\mathbf{R}_2^e\} \quad (11)$$

Substituting Eq. (9) into Eq. (11) yields

$$\sum_{e=1}^N [\mathbf{G}^e]^T ([\mathbf{H}^e]^{-1} [\mathbf{G}^e] \{\mathbf{q} + \Delta\mathbf{q}\} - \{\mathbf{R}_1^e\}) = \sum_{e=1}^N \{\mathbf{R}_2^e\} \quad (12)$$

The normal and tangential components of the interfacial separation are expressed as

$$u_n + \Delta u_n = \{\mathbf{n}^c\}^T [\mathbf{L}^c] \{\mathbf{q}^m + \Delta\mathbf{q}^m - \mathbf{q}^c - \Delta\mathbf{q}^c\} \\ u_t + \Delta u_t = \{\mathbf{t}^c\}^T [\mathbf{L}^c] \{\mathbf{q}^m + \Delta\mathbf{q}^m - \mathbf{q}^c - \Delta\mathbf{q}^c\} \quad (13)$$

Following the evaluation of nodal displacements, stress coefficients are calculated in each element using Eq. (9). The stresses at any location within the element may then be assessed from Eq. (1). As discussed in Ref. [20], these relations are solved for displacements and stresses in the microstructural RVE by imposing displacement periodicity conditions on the RVE boundary. Equation (12) is nonlinear due to the relation between interfacial tractions and interfacial displacements in the cohesive laws. A Newton–Raphson iteration method is consequently invoked to solve for the increments of nodal displacement on the element boundaries and matrix-inclusion interfaces.

2.1 Anisotropic Damage Model With Fourth Order Damage Tensor. The general form of CDM models [1] introduces a fictitious stress $\tilde{\Sigma}_{ij}$ acting on an effective resisting area ($\tilde{A}$), which is caused by the reduction of the original resisting area A due to material degradation from the presence of microcracks and stress concentration in the vicinity of cracks. In Refs. [7,16], the effective stress $\tilde{\Sigma}_{ij}$ is related to the actual Cauchy stress Σ_{ij} through a fourth order damage effect tensor M_{ijkl} as

$$\tilde{\Sigma}_{ij} = M_{ijkl}(\mathbf{D}) \Sigma_{kl} \quad (14)$$

where M_{ijkl} is a function of a damage tensor $\mathbf{D} (= D_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)$. $\mathbf{D}$ can be a zeroth, second, or fourth order tensor, depending on the model employed. The hypothesis of equivalent elastic energy is used to evaluate M_{ijkl} and establish a relation between the damaged and undamaged stiffnesses [8,26]. The hypothesis, detailed in Refs. [2,18], specifically assumes that the elastic comple-

mentary energy W_C in a damaged material with the actual stress is equal to that in a hypothetical undamaged material with the fictitious effective stress, i.e.,

$$W_C(\Sigma, \mathbf{D}) = \frac{1}{2} (E_{ijkl}(\mathbf{D}))^{-1} \Sigma_{ij} \Sigma_{kl} = W_C(\tilde{\Sigma}, \mathbf{0}) = \frac{1}{2} (E_{ijkl}^o)^{-1} \tilde{\Sigma}_{ij} \tilde{\Sigma}_{kl} \quad (15)$$

where $\Sigma = \Sigma_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$, E_{ijkl}^o is the elastic stiffness tensor in the undamaged state and $E_{ijkl}(\mathbf{D})$ is the stiffness in a damaged state. From Eqs. (14) and (15), the relation between the damaged and undamaged stiffnesses is established as

$$E_{ijkl} = (M_{pqij})^{-1} E_{pqrs}^o (M_{rskl})^{-T} \quad (16)$$

where $-T$ corresponds to the transpose of the inverse of the fourth order $\mathbf{M}$ tensor. With the choice of an appropriate order of the damage tensor and the assumption of a function for M_{ijkl} , Eq. (16) can be used to formulate a damage evolution model using micromechanics and homogenization. The anisotropic CDM model involving fourth order damage tensor proposed in Ref. [16] introduces a damage evolution surface to delineate the interface between damaged and undamaged domains in the strain (e_{ij}) space as

$$F = \frac{1}{2} P_{ijkl} e_{ij} e_{kl} - \kappa(\alpha W_d) = 0 \quad (17)$$

where W_d is the dissipation of the strain energy density due to stiffness degradation expressed as (see Ref. [27]):

$$W_d = \int \frac{1}{2} e_{ij} e_{kl} dE_{ijkl} \quad (18)$$

Assuming associativity rule in the stiffness space, the evolution of the fourth order secant stiffness is obtained as

$$\dot{E}_{ijkl} = \dot{\lambda} \frac{\partial F}{\partial \left(\frac{1}{2} e_{ij} e_{kl} \right)} = \dot{\lambda} P_{ijkl} \quad (19)$$

P_{ijkl} is a fourth order symmetric negative definite tensor that corresponds to the direction of the rate of stiffness degradation tensor $\dot{E}_{ijkl}$. P_{ijkl} is expressed as a function of strain e_{ij} , α is a scaling parameter, and κ is a function of W_d . Calibration of the model requires evaluation of $\kappa(W_d)$, α , and P_{ijkl} in Eq. (17).

The function $\kappa(W_d)$ is evaluated in Ref. [16] for a reference

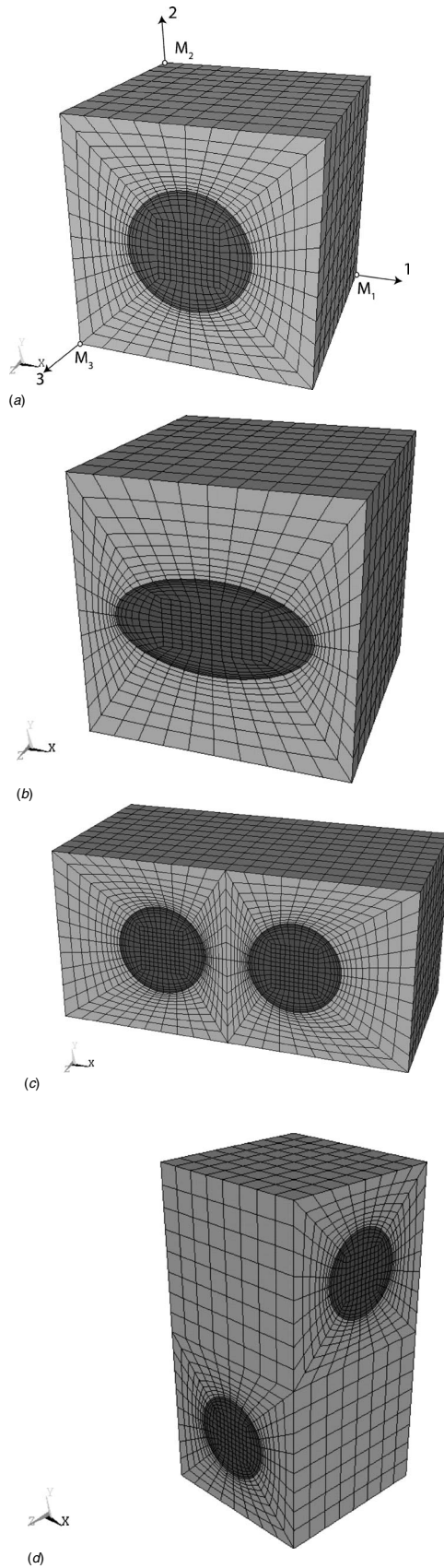


Fig. 2 Finite element mesh for (a) RVE with a cylindrical fiber, (b) RVE with an elliptical fiber, (c) RVE with two parallel fibers, and (d) RVE with two perpendicular fibers

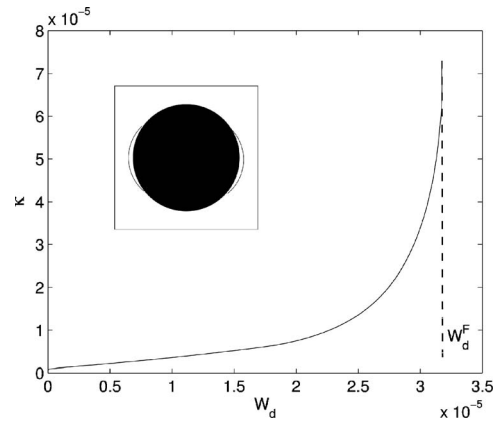


Fig. 3 κ - W_d plot for a RVE with circular fiber subjected to uniaxial tension

loading path, and results for all other strain paths are scaled with respect to this reference value. For the reference loading path ($e_{11} \neq 0$, all other $e_{ij} = 0$); setting $P_{1111} = 1$, κ is determined from damage surface of Eq. (17) as

$$\kappa = \frac{1}{2} e_{11}^2 \quad (20)$$

Micromechanical simulation of RVE with a circular fiber (see Fig. 2(a) or inset in Fig. 3) is performed for this load path, and the secant stiffness E_{ijkl} is evaluated at the end of each increment by applying unit macroscopic strains, described in Ref. [16]. The function $\kappa(W_d)$ is then determined by evaluating W_d at each strain increment. The κ - W_d plot in Fig. 3 shows that W_d increases with evolving macroscopic strain and attains a maximum value of W_d^F corresponding to saturation of damage in the microstructure. The scaling parameter α , accounting for the variability of W_d^F with the loading path, is obtained by a simple scaling relation as

$$\alpha(e_{11}, e_{22}, e_{12}) = \frac{W_d^F(e_{11}, e_{22}, e_{12})}{W_d^F(e_{11} \neq 0, e_{22} = 0, e_{12} = 0)} \quad (21)$$

For composites with interfacial debonding, the direction of stiffness degradation rate varies significantly with increasing damage; hence, P_{ijkl} is a function of the total macroscopic strain e_{ij} . To evaluate this dependence, the macroscopic strain space is discretized into a uniform grid that is created by periodic intercepts on each proportionally loaded strain path. Values of P_{ijkl} at each grid point are evaluated by homogenizing the RVE-based micromechanical analysis results. The evaluation of P_{ijkl} involves integration of Eq. (19) with satisfaction of consistency condition in Eq. (17) at each strain increment. Once the values of P_{ijkl} are evaluated at the grid points, their values for any arbitrary macroscopic strain can be determined by interpolating in the (e_{11}, e_{22}, e_{12}) strain space using shape functions of linear hexahedral elements as

$$P_{ijkl}(e_{11}, e_{22}, e_{12}) = \sum_{\alpha=1}^8 (P_{ijkl})_{\alpha} N_{\alpha}(e_{11}, e_{22}, e_{12}) \quad (22)$$

Details of the parameter evaluation process for the anisotropic damage model and the subsequent stress update algorithm using the HCDM model are given in Ref. [16].

2.2 Limitations of the Model in Reference [16]. The HCDM model in Ref. [16] provides a general framework for describing evolving damage at the microscale using continuum constitutive laws at the macroscale. A comparison of CDM results with the

homogenized microscopic response shows excellent agreement [16] for proportional loading conditions. However, the model suffers from limitations as discussed below.

1. While components of P_{ijkl} in Eq. (19) depend on the current location in the (e_{11}, e_{22}, e_{12}) strain space, their dependence on the strain path or history is not explicitly accounted for. To understand the error incurred due to this assumption, a VCFEM based micromechanical analysis is conducted for nonproportional loading. A RVE consisting of a circular fiber of 20% volume fraction in a square matrix, as shown in Fig. 3, is modeled. The elastic matrix has $E_m=4.6$ GPa and $\nu_m=0.4$, and the elastic fiber has $E_f=210$ GPa and $\nu_f=0.3$. The interface properties in Eq. (6) are $\delta_c=5.0 \times 10^{-5}$ m, $\delta_e=20 \times 10^{-4}$ m, and $\sigma_m=0.02$ GPa. The macroscopic FEM that uses the HCDM model consists of a single four-noded quadrilateral (QUAD4) element. The material is subjected to a nonproportional strain path, viz., initial uniaxial tension $e_{11} \neq 0, e_{22}=e_{12}=0$, followed by a strain path corresponding to $e_{12}/e_{11} \neq 0, e_{22}=0$. Results by the two models, viz., HCDM and averaging micromechanics, are compared in Fig. 4. The first part of the loading, corresponding to proportional loading, shows excellent agreement. However, significant error is incurred in predicting the response for the subsequent nonproportional portion of the loading. This necessitates the incorporation of loading history effect in the HCDM formulation.
2. The 2D CDM model in Ref. [16] describes material damage behavior for interfacial debonding in unidirectional composites subjected to in-plane loading. A 3D CDM model is desirable to describe the damage in fiber reinforced composites with randomly oriented fibers and subjected to multiaxial loading histories. The corresponding components of P_{ijkl} in Sec. 2.1 will be functions of the six independent components of strain tensor e_{ij} . According to the algorithm described in Ref. [16], this will necessitate evaluation and storage of each component of P_{ijkl} at discrete points of the macroscopic strain evolution paths in a six dimensional strain space. A large number of micromechanical RVE analyses should be carried out followed by homogenization. Subsequently, in macroscopic analysis using the HCDM model, all independent components of P_{ijkl} will be interpolated in the six dimensional strain space. The discrete strain space representation and interpolation makes the model construction and use computationally prohibitive and cumbersome. Functional forms of P_{ijkl} are more desirable and will be derived in the present work.

3 Homogenization Based 3D Continuum Damage Mechanics Model

3.1 Micromechanical Model of the RVE Including Debonding. A 3D micromechanical model for analyses of composite microstructures undergoing fiber-matrix interfacial debonding has been developed by the authors in Ref. [28]. In this model, the interface is comprised of a set of nonlinear cohesive springs with bilinear traction-displacement relations discussed in Sec. 2. The bilinear cohesive zone models have yielded satisfactory agreement with experiments for composites described in Refs. [21,29]. In Ref. [28], 3D interface elements using the cohesive laws are developed in the user defined element (UEL) subroutine of the commercial FE code ABAQUS. As detailed in Ref. [28], the interface elements have 16 nodes with a quadratic displacement interpolation, corresponding to a total of 48 degrees of freedom with nine integration points. They are compatible with the 20 noded quadratic brick elements that are used to model the fiber and matrix phases in ABAQUS, as shown in Fig. 2(a). In the initial unloaded state, the interface nodes on the matrix and fiber surfaces share the same coordinates. With the application of external

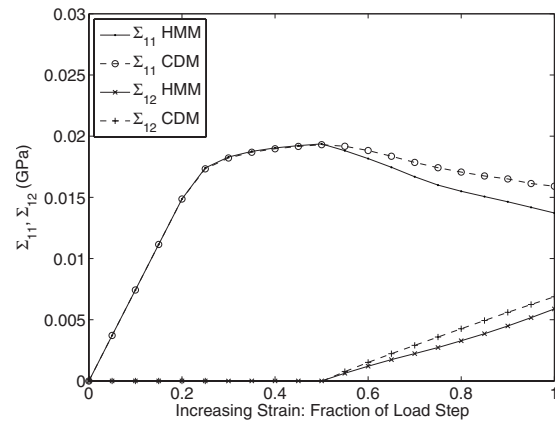


Fig. 4 Comparison of macroscopic stress-strain curve obtained using the 2D homogenized CDM model and HMM under nonproportional loading

load and deformation, the surfaces move and separate from one another. The relative normal and tangential tractions for the interface elements are calculated at the element integration points according to the traction separation law in Sec. 2.

Volume averaged variables such as stresses Σ_{ij} and strains e_{ij} , required to develop and calibrate the HCDM model, are evaluated by homogenization of micromechanical solutions using the expressions

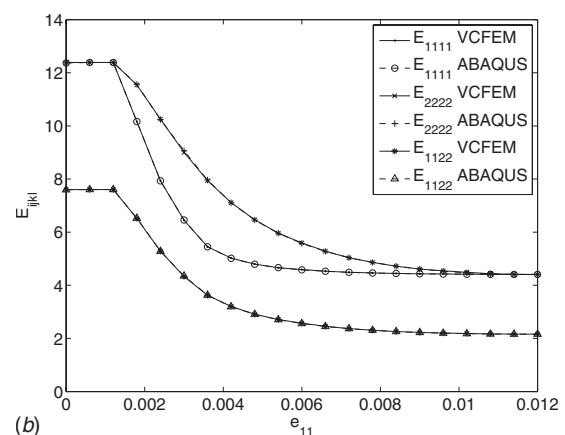
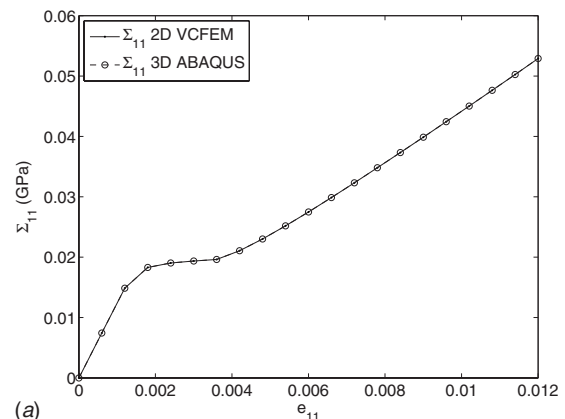


Fig. 5 Comparison of results of 2D plane strain VCFEM and 3D ABAQUS models: (a) macroscopic stress-strain curve and (b) degradation of secant stiffness

$$\Sigma_{ij} = \frac{1}{Y} \int_Y \sigma_{ij}(Y) dY$$

and

$$e_{ij} = \frac{1}{Y} \int_Y \epsilon_{ij}(Y) dY + \frac{1}{2Y} \int_{\partial Y_{\text{int}}} ([u_i]n_j + [u_j]n_i) dS \quad (23)$$

where σ_{ij} and ϵ_{ij} are microscopic stress and strain, respectively and Y is the RVE domain. Y_{int} corresponds to the fiber-matrix interface domain and $[u_i]$ denotes the jump in the displacement components across the interface with outward normal n_i . The components of the homogenized elastic stiffness tensor E_{ijkl} are calculated by solving six independent boundary value problems. In each case, only a single unit strain component is applied on the RVE, while the other strain components are kept to zero; i.e., Case 1, $e_{11}=1.00$, all other components 0.0; Case 2, $e_{22}=1.00$, all other components=0.0; and so on. Periodicity displacement conditions are enforced by constraining nodes on opposite faces of the RVE boundary to deform in a periodic manner. A given macroscopic strain is applied on the RVE by decomposing the displacement on the boundary into a macroscopic averaged and a periodic part as discussed in Refs. [30,31], i.e.,

$$u_i = e_{ij}x_j + \tilde{u}_i \quad (24)$$

Since the periodic part $\tilde{u}_i$ is equal on corresponding nodes of opposite faces of the RVE (say, n_1^p and n_2^p), the total displacement at these nodes are related as

$$(u_i)_{n_2^p} - (u_i)_{n_1^p} = e_{ij}\Delta x_j \quad (25)$$

where Δx_i are the relative nodal coordinates. The macroscopic strains are then applied, in conjunction with the periodicity constraints, by fixing a corner node and specifying the displacement on master nodes M_1 , M_2 , and M_3 that belong to orthogonal faces, as shown in Fig. 2(a).

The RVE with a circular fiber in Sec. 2.2 is now analyzed in 3D, subject to uniaxial tension. Figures 5(a) and 5(b) show a comparison of the homogenized stress-strain curve and the in-plane secant stiffness components obtained by the 2D VCFEM model and the 3D ABAQUS model with debonding. Results by the two methods show excellent agreement. The stiffness components rapidly decay during the initial stages of debonding and eventually stabilize at values corresponding to that of a RVE with a void, when the interface separates completely. The out-of-plane stiff-

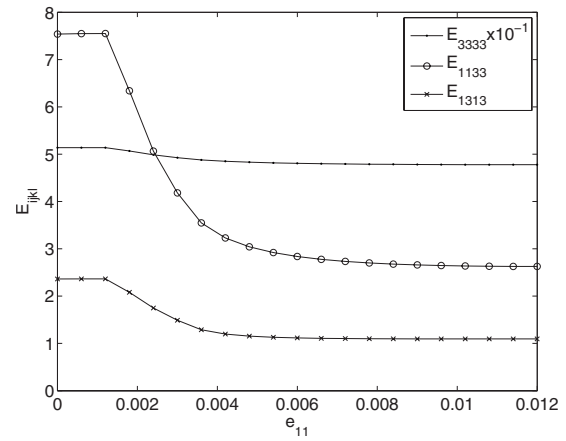


Fig. 6 Degradation of components of secant stiffness in third direction

ness components obtained from the 3D model are plotted in Fig. 6. As expected, the stiffness component E_{3333} along the length of the fiber is significantly higher than other components. This component undergoes very little degradation as fibers continue to support the load even after the interface has debonded.

3.2 HCDM Model in the Principal Damage Coordinate System (PDCS). For a second order damage tensor D_{ij} , the damage effect tensor M_{ijkl} in Eq. (14) has been defined in Ref. [32] as

$$M_{ijkl} = (\delta_{ik} - D_{ik})^{-1} \delta_{jl} \quad (26)$$

It has been shown in Ref. [32] that D_{ij} is symmetric, and it can describe the damage states that have at least an orthotropic symmetry. For any arbitrary D_{ij} , the corresponding effective stress tensor, obtained by substituting Eq. (26) into Eq. (14), may be unsymmetric. An implicit method of rendering the stress tensor symmetric has been suggested in Ref. [33], which corresponds to a representation of the stress tensor in a fixed global coordinate system as

$$\Sigma_{ij} = \frac{\tilde{\Sigma}_{ik}(\delta_{kj} - D_{kj})^{-1} + (\delta_{il} - D_{il})^{-1}\tilde{\Sigma}_{lj}}{2} \quad (27)$$

The corresponding inverse of the damage effect tensor $[M(D_{ij})]^{-1}$ is represented in a matrix form as

$$[M(D_{ij})]^{-1} = \begin{bmatrix} 1-D_{11} & 0 & 0 & 0 & -D_{13} & -D_{12} \\ 0 & 1-D_{22} & 0 & -D_{23} & 0 & -D_{12} \\ 0 & 0 & 1-D_{33} & -D_{23} & -D_{13} & 0 \\ 0 & -\frac{1}{2}D_{23} & -\frac{1}{2}D_{23} & 1-\frac{1}{2}(D_{22}+D_{33}) & -D_{12} & -D_{13} \\ -\frac{1}{2}D_{13} & 0 & -\frac{1}{2}D_{13} & -\frac{1}{2}D_{12} & 1-\frac{1}{2}(D_{11}+D_{33}) & -D_{23} \\ -\frac{1}{2}D_{12} & -\frac{1}{2}D_{12} & 0 & -\frac{1}{2}D_{13} & -\frac{1}{2}D_{23} & 1-\frac{1}{2}(D_{11}+D_{22}) \end{bmatrix} \quad (28)$$

This can be substituted in Eq. (16) to update the damaged stiffness E_{ijkl} from the initial undamaged stiffness E_{ijkl}^0 .

Numerical examples in Ref. [16] have shown that material symmetry is considerably affected by damage evolution in composite

microstructures. Different load paths will yield different damage profiles in the microstructure, and this will alter the initial material symmetry in E_{ijkl}^0 in different ways. In a fixed coordinate system, a RVE exhibiting, e.g., orthotropy in E_{ijkl}^0 can exhibit general an-

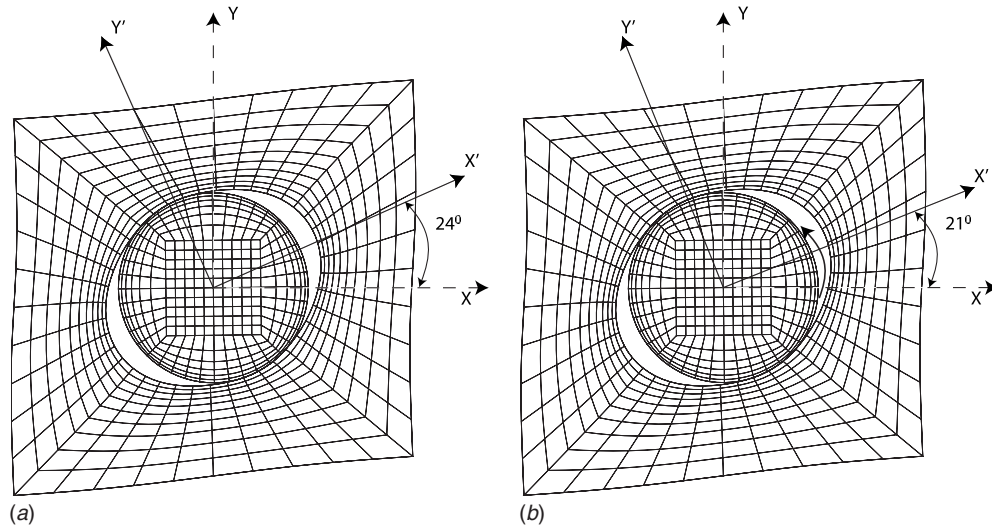


Fig. 7 Rotation of the PDCS for (a) proportional and (b) nonproportional loading paths

isotropy with evolving damage under multiaxial loading. In the fixed coordinate system, the anisotropic E_{ijkl} will couple normal and shear strain components in the elastic energy expression. However, when the strains are represented in a coordinate system that corresponds to the principal damage axes, the coupling terms in the stiffness E_{ijkl} reduce to near vanishing values and the initial symmetry properties are retained.

The present work assumes orthotropy of the homogenized stiffness matrix in the PDCS. The damage effect tensor M_{ijkl} corresponding to Eq. (28) has a diagonal representation in this coordinate system; consequently, the initial material symmetry is retained throughout the loading process. The determination of the continuously evolving PDCS requires the determination of the second order damage tensor D_{ij} and the subsequent evaluation of its eigenvectors at each step of the incremental loading process. For known values of E_{ijkl}^0 and E_{ijkl} , Eq. (16) results in a system of nonlinear algebraic equations in D_{ij} . Since there are nine independent components of the orthotropic stiffness tensor E_{ijkl} and six independent components of the symmetric D_{ij} , a nonlinear least squares minimization solver is used to solve for D_{ij} . Subsequently, the eigenvectors of D_{ij} , viz., $\mathbf{e}_{D1}$, $\mathbf{e}_{D2}$, and $\mathbf{e}_{D3}$ are evaluated and the transformation matrix $[\mathbf{Q}]^D = [\mathbf{e}_{D1} \ \mathbf{e}_{D2} \ \mathbf{e}_{D3}]$ is formed. $[\mathbf{Q}]^D$ leads to the rotation of the global coordinate system to the PDCS.

3.2.1 Numerical Example on the Evolution of PDCS. To understand the evolution of PDCS on load history, the problem considered in Sec. 2.2 is solved, subject to the following two load histories:

1. Case a: Proportional loading with strain path $e_{12}/e_{11} \neq 0$, $e_{22}=0$ throughout.
2. Case b: Nonproportional loading with $e_{11} \neq 0$, $e_{22}=e_{12}=0$ in the first half of the loading, followed by a strain path of $e_{12}/e_{11} \neq 0$, $e_{22}=0$ till the end.

The final state of macroscopic strain e_{ij} for both cases is identical. Along each of these load paths, the homogenized secant stiffness of the damaging material E_{ijkl} is calculated, followed by determination of the PDCS. Figures 7(a) and 7(b) show the orientation of the PDCS in the final deformed configuration for the two cases. For the proportional loading Case a, the orientation of the damage axes jumps to and remains fixed at 24 deg with respect to the global axes throughout the damaging process. For Case b, the PDCS coincides with the global coordinate system in simple tension during the first half of loading. In the last half of loading, the PDCS continuously rotates to a final position of 21 deg orienta-

tion. Certainly, in this case, the PDCS rotation should be incorporated in the HCDM model to account for the damage and load history.

3.3 Orthotropic Damage Model With Fourth Order Damage Tensor

3.3.1 Damage Evolution Laws in the PDCS. The damage evolution surface of Eq. (17) is rewritten in the PDCS as

$$F' = \frac{1}{2} e'_{ij} P'_{ijkl} e'_{kl} - \kappa'(W_d) = 0 \quad (29)$$

where the prime in the superscript denotes quantities expressed in the PDCS using the transformation laws

$$E'_{ijkl} = Q_{ip} Q_{jq} Q_{kr} Q_{ls} E_{pqrs} \quad \text{and} \quad e'_{ij} = Q_{ik} Q_{jl} e_{kl} \quad (30)$$

and Q_{ij} is the transformation matrix. The corresponding rate of stiffness degradation in PDCS is

$$\dot{E}'_{ijkl} = \dot{\lambda} \frac{\partial F'}{\partial \left(\frac{1}{2} e'_{ij} e'_{kl} \right)} = \dot{\lambda} P'_{ijkl} \quad (31)$$

3.3.2 Parametric Forms in the HCDM Model and Their Calibration

3.3.2.1 Damage state variable κ' . As described in Sec. 2.1, the function $\kappa'(\alpha W_d)$ is evaluated for the reference loading path ($e_{11} \neq 0$, all other $e_{ij}=0$) using Eq. (29) as

$$\kappa' = \frac{1}{2} (e'_{11})^2 \quad (32)$$

The κ' - W_d plots for four different strain paths are shown with circular marks in Fig. 8. The loading cases are

1. simple shear: $e_{12}=0.012$
2. tension-torsion: $e_{11}=0.006$ and $e_{12}=0.010$
3. multiaxial tension-torsion: $e_{11}=0.009$, $e_{22}=0.002$, and $e_{12}=0.006$
4. uniaxial tension: $e_{11}=0.012$

In these plots, W_d remains zero until κ' exceeds a threshold value corresponding to the initiation of debonding induced damage. Subsequently, W_d increases rapidly, signaling substantial material deterioration during the initial stages of damage. Eventually, W_d saturates at a value W_d^F corresponding to configuration with ar-

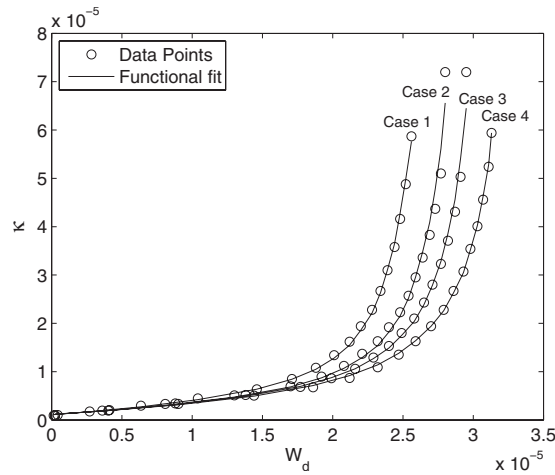


Fig. 8 Functional representation of the κ - W_d relation

rested debond or fully debonded interface. Insignificant or no degradation occurs after this. It is observed that all of the RVEs have the same nature, and only the value of W_d^F varies for different strain histories. The variability of the saturation damage energy W_d^F with loading paths in the 3D strain space can be taken into account using the scaling factor $\alpha(e_{ij})$ defined in Eq. (21). However, such an approach requires evaluation and storage of $\alpha(e_{ij})$ for a large number of individual loading paths for interpolation, as discussed in Sec. 2.2.

To avoid this in the 3D analysis, novel functional forms of κ' are introduced to explicitly describe its dependence on the macroscopic strain components e_{ij} , as well as on W_d . Three invariant forms of the strain components, consistent with the anisotropic material property model, are used in these functions. These are

$$I_1 = \frac{1}{3}(Ae_{11} + Be_{22} + Ce_{33})$$

$$J_2 = F(e_{22}^d - e_{33}^d)^2 + G(e_{33}^d - e_{11}^d)^2 + H(e_{11}^d - e_{22}^d)^2 + L(e_{12}^d)^2 + M(e_{13}^d)^2 + N(e_{23}^d)^2$$

$$J_3 = O(e_{11}^d e_{22}^d e_{33}^d) + P(e_{11}^d (e_{23}^d)^2) + Q((e_{12}^d)^2 e_{33}^d) + R(e_{13}^d e_{12}^d e_{23}^d) + S((e_{13}^d)^2 e_{22}^d) \quad (33)$$

where $e_{ij}^d = e_{ij} - \frac{1}{3}\delta_{ij}e_{kk}$ is the deviatoric strain tensor. The invariants I_1 , J_2 , and J_3 are, respectively, linear, quadratic, and cubic

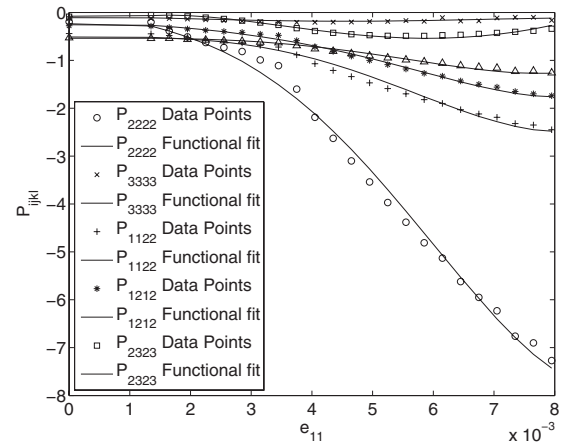


Fig. 9 Functional representation of variation of P'_{ijkl} with macroscopic strain for uniaxial tension

functions of e_{ij} . The constants $A, B, C, F, G, \dots, S$ are introduced to characterize the state of anisotropy in the damaged material. The function J_2 is similar to that used in the anisotropic yield criterion for elastoplasticity proposed by Hill [34]. The functional form of κ' is developed to conform with the plots of Fig. 8, and is expressed as

$$\kappa'(I_1, J_2, J_3, W_d) = b_0 + f(I_1, J_2, J_3)[1 + b_1 \tan(b_2 W_d)] \quad (34)$$

The form separates its dependence on the dissipation energy and strains. The latter dependence is represented by a polynomial function of the invariants, i.e.,

$$f(I_1, J_2, J_3) = a_0 + a_1 I_1 + a_2 J_2 + a_3 J_3 + a_4 I_1^2 + a_5 I_1 J_2 + \dots \quad (35)$$

The constants $A, B, C, \dots$ in Eq. (33), b_0, b_1, b_2 in Eq. (34), and $a_0, a_1, \dots$ in Eq. (35) are determined by a nonlinear least squares minimization of the difference between results of micromechanical analysis and those from the functional form in Eq. (34), i.e.,

$$\text{minimize} \sum_{i=1}^{N_{\text{ref}}} [\kappa'_{\text{ref}} - \kappa'(I_1, J_2, J_3, W_d)]_i^2 \quad (36)$$

A micromechanical analysis of the RVE is conducted for N_{ref} different strain histories to explicitly compute the values of κ'_{ref} . A fifth order polynomial function in Eq. (35) yields good convergence properties for the least squares residual. Figure 8 satisfactorily compares the κ' - W_d plots by the function in Eq. (34) with

Table 1 Constants in the parametric representation of I_1 , J_2 , and J_3 in Eq. (33) for various RVEs

Constants in Eq. (33)	RVE in Fig. 2(a)	RVE in Fig. 2(b)	RVE in Fig. 2(c)	RVE in Fig. 2(d)	RVE in Fig. 1
A	0.32	0.30	0.07	0.22	1.39
B	0.32	0.59	0.66	0.21	1.32
C	0.02	0.89	0.69	0.22	—
F	0.02	0.35	0.02	0.65	0.01
G	0.02	0.35	1.85	0.71	0.01
H	0.42	1.64	1.85	0.65	0.02
L	0.18	0.41	0.58	1.76	2.20
M	0.38	2.28	2.60	0.23	—
N	0.38	2.35	0.06	1.76	—
O	0.00	1.87	0.03	0.45	1.50
P	0.16	0.98	2.27	0.43	—
Q	0.34	0.05	2.03	0.43	0.30
R	0.13	0.41	0.08	0.07	—
S	0.16	0.13	1.67	0.12	—

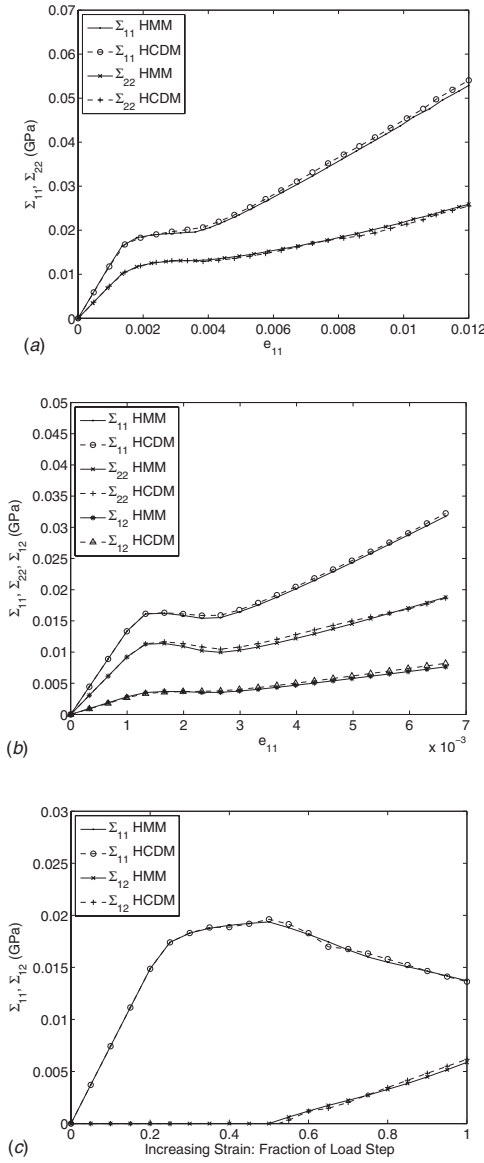


Fig. 10 Comparison of macroscopic stress-strain curve obtained using HCDM and HMM for a 3D RVE with a cylindrical fiber (Fig. 2(a)) for load cases (a) L1, (b) L2, and (c) L3

those from the micromechanical analysis for the different strain paths.

3.3.2.2 Damage evolution surface parameter P'_{ijkl} . In the incremental finite element formulation for evolving damage, the backward Euler method is used to integrate the rate of stiffness degradation in Eq. (31). For a strain increment from step n to step $n+1$, the parameter P'_{ijkl} may be expressed as

$$(P'_{ijkl})_{n+1} = \frac{(E'_{ijkl})_{n+1} - (E'_{ijkl})_n}{\lambda_{n+1} - \lambda_n} \quad (37)$$

where $(E'_{ijkl})_{n+1}$ is the secant stiffness at the end of the increment. As explained in Sec. 3.1, this is calculated by unloading to the origin from the current state of stress. Substituting this into the damage evolution, Eq. (29) at the end of the increment yields the incremented form

$$\frac{1}{2}(e'_{ij})_{n+1} \left(\frac{(E'_{ijkl})_{n+1} - (E'_{ijkl})_n}{\lambda_{n+1} - \lambda_n} \right) (e'_{kl})_{n+1} - \kappa'_{n+1} = 0 \quad (38)$$

where $\kappa'_{n+1}(I_1, J_2, J_3, W_d)$ represents the size of the parametric damage surface. The dissipation energy $(W_d)_{n+1}$ at the end of the interval is evaluated by using the backward Euler integration method. The parameter λ_{n+1} is evaluated as

$$\lambda_{n+1} = \lambda_n + \frac{\frac{1}{2}(e'_{ij})_{n+1}((E'_{ijkl})_{n+1} - (E'_{ijkl})_n)(e'_{kl})_{n+1}}{\kappa'_{n+1}} \quad (39)$$

$(P'_{ijkl})_{n+1}$ is then determined from Eq. (37). The direction of the rate of stiffness degradation varies continuously with damage evolution due to macroscopic strain. From Eq. (31), this implies that P'_{ijkl} also varies accordingly. The method of strain space interpolation in Ref. [16] would require P'_{ijkl} evaluation and storage at a large number of points for a wide range of strain combinations. This enterprise can become prohibitively exhaustive for 3D problems. To avoid this, a polynomial function form is derived for the components P'_{ijkl} in terms of the anisotropic invariants of strain defined in Eq. (33) as

$$P'_{ijkl}(I_1, J_2, J_3) = c_0^{ijkl} + c_1^{ijkl} I_1 + c_2^{ijkl} J_2 + c_3^{ijkl} J_3 + c_4^{ijkl} I_1^2 + c_5^{ijkl} I_1 J_2 + \dots \quad (40)$$

Again, the coefficients c_p^{ijkl} in Eq. (40) are determined by the nonlinear least squares solver. In this method, the square of the difference in P'_{ijkl} obtained from micromechanical analysis and the functional form for a few representative strain paths is minimized, i.e.,

$$\text{minimize} \sum_{i=1}^{N_{\text{ref}}} [(P'_{ijkl})_{\text{ref}} - P'_{ijkl}(I_1, J_2, J_3)]_i^2 \quad (41)$$

The subscript “ref” corresponds to data points obtained by micromechanical analysis. Figure 9 shows a comparison of the micromechanical results and the calibrated function in Eq. (40) for a RVE under uniaxial tension. With a fifth order polynomial function (Eq. (40)), the root mean square error is observed to be less than 3%. The coefficients can be used subsequently for computing P'_{ijkl} for any given strain e_{ij} during the macroscopic analysis.

3.4 Implementation of the HCDM Model in a Macroscopic Analysis Module. The HCDM model is implemented in the macroscopic FEM in ABAQUS using the user material interface (UMAT). In an incremental solution process, subscripts n and $n+1$ correspond to values at the beginning and end of the n th increment, respectively. At each element integration point, the stresses $(\Sigma_{ij})_{n+1}$ are obtained from known values of the strain $(e_{ij})_{n+1}$ and state variables at n using the HCDM constitutive model. The essential steps in the UMAT update algorithm in the n th increment are described below.

1. For given $(e_{ij})_{n+1}$, evaluate $(I_1)_{n+1}$, $(J_2)_{n+1}$, and $(J_3)_{n+1}$ using Eq. (33) and, subsequently, $(P'_{ijkl})_{n+1}$ using Eq. (40).
2. Initialize variables at the start of an iteration algorithm for solving the damage evolution problem.
 - Assume that the starting value of the PDCS rotation tensor $(Q_{ij})_{n+1}^0 = (Q_{ij})_n$.
 - Evaluate the starting value of the damage function $(F'_{n+1})^0 = \frac{1}{2}(e'_{ij})_{n+1}(P'_{ijkl})_{n+1}(e'_{kl})_{n+1} - \kappa'_n$.
 - If $(F'_{n+1})^0 \leq 0$, there is no additional damage. In this case, proceed to step 7 with unchanged secant stiffness tensor $(E_{ijkl})_{n+1} = (E_{ijkl})_n$.
3. For the l th iteration, damage evolution takes place if $\frac{1}{2}(e'_{ij})_{n+1}(P'_{ijkl})_{n+1}(e'_{kl})_{n+1} - \kappa'_n > 0$. In this case, determine

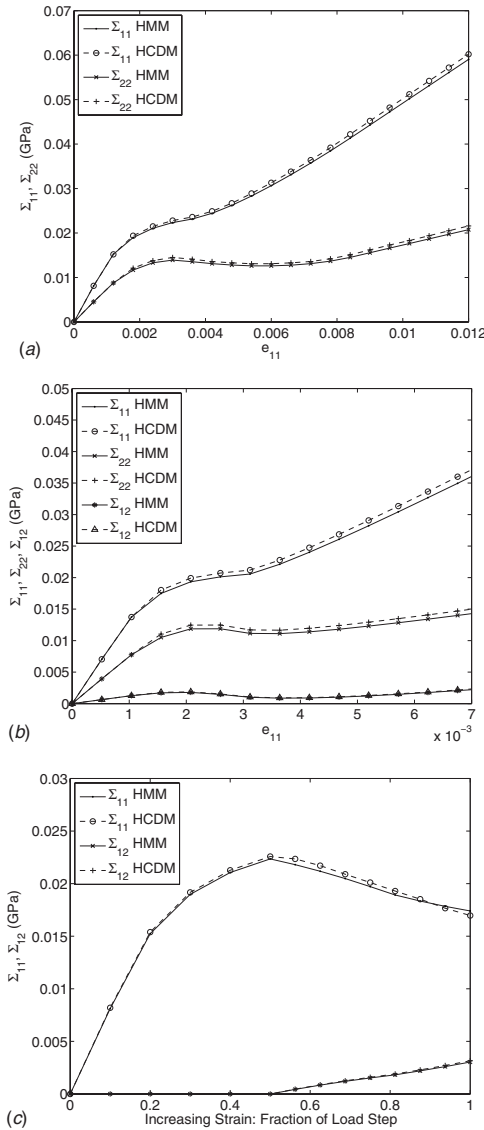


Fig. 11 Comparison of macroscopic stress-strain curve obtained using HCDM and HMM for RVE with an elliptical fiber (Fig. 2(b)) for load cases (a) L1, (b) L2, and (c) L3

- $(\kappa'_{n+1})^I = \frac{1}{2}(e'_{ij})_{n+1}((P'_{ijkl})_{n+1})^I(e'_{kl})_{n+1}$
- W_d by inverting the $\kappa' - W_d$ relation in Eq. (34) as

$$((W_d)_{n+1})^I = \frac{1}{b_2} \tan^{-1} \left(\frac{1}{b_1} \left(\frac{(\kappa'_{n+1})^I - b_0}{(f_{n+1})^I(I_1, J_2, J_3)} - 1 \right) \right) \quad (42)$$

- Using the backward Euler method to integrate $\dot{W}_d$, determine

$$(\lambda_{n+1})^I = \lambda_n + \frac{2(((W_d)_{n+1})^I - (W_d)_n)}{(e'_{ij} P'_{ijkl} e'_{kl})^I_{n+1}} \quad (43)$$

- Update the secant stiffness using the relation

$$(E'_{ijkl})^I_{n+1} = (E'_{ijkl})^I_n + (\lambda_{n+1}^I - \lambda_n)(P'_{ijkl})^I_{n+1} \quad (44)$$

- Determine the PDCS rotation matrix $(Q_{ij})^I_{n+1}$ from the eigenvectors of $(D_{ij})^I_{n+1}$, corresponding to the updated secant stiffness $(E_{ijkl})^I_{n+1}$ using the procedure in Sec. 3.2. If convergence in the rotation matrix is achieved, i.e.,

$$\max |((Q_{ij})_{n+1})^I - ((Q_{ij})_{n+1})^{I-1}| \leq TOL, \quad \forall i, j = 1, 2, 3$$

then proceed to step 7. Otherwise, return to step 3 and continue the iteration.

- Update macroscopic stresses with the converged value of the secant stiffness matrix as

$$(\Sigma_{ij})_{n+1} = (E_{ijkl})^I_{n+1} (e_{kl})_{n+1} \quad (45)$$

4 Numerical Examples for Validating the HCDM Model

A few numerical simulations are conducted in this section for validating the orthotropic 3D HCDM model. The HCDM results are compared with homogenized micromechanics (HMM) solutions of the RVE. The HMM model is obtained by homogenizing the micromechanical response of the RVE using the asymptotic homogenization methods discussed in Ref. [16,22]. The macroscopic FEM implementing the HCDM model for its constitutive relations consists of a single eight-noded quadrilateral element. Five different 3D and 2D RVEs are considered for development of the HCDM model using micromechanics.

- Unidirectional 3D uniform composite microstructure, where RVE is a unit cell containing a single cylindrical fiber of volume fraction 20%. This is shown in Fig. 2(a).
- Unidirectional 3D uniform composite microstructure, where RVE is a unit cell containing a single fiber of an elliptical cross section, as shown in Fig. 2(b). The volume fraction is 20% and the aspect ratio is $a/b=2$.
- Unidirectional 3D composite microstructure, where RVE contains two nonuniformly placed cylindrical fibers, as shown in Fig. 2(c). The fiber volume fraction is 20%.
- Cross-ply 3D composite microstructure with its RVE containing two cylindrical fibers of volume fraction 20% oriented at 90 deg with respect to each other. This is shown in Fig. 2(d).
- Random 2D composite microstructure, as shown in Fig. 1. The RVE contains 20 circular fibers of volume fraction 21.78%. The periodic domain or the RVE is generated by repeating the core fiber configuration periodically and tessellating the overall domain as developed in Ref. [35]. This leads to an aggregate of Voronoi cells, constituting the RVE.

The material properties of the elastic matrix are $E_m=4.6$ GPa and $\nu_m=0.4$, and the elastic fiber have $E_c=210$ GPa and $\nu_c=0.3$. The cohesive zone properties for the interface are $\delta_c=5.0 \times 10^{-5}$ m, $\delta_e=20 \times 10^{-4}$ m, and $\sigma_m=0.02$ GPa.

Micromechanical analyses of the RVEs are conducted by enforcing periodic displacement boundary conditions and imposing the macroscopic strain fields in the entire RVE. Both proportional and nonproportional macroscopic strain loading conditions are applied as follows:

- L1, proportional uniaxial tension loading: $e_{11} \neq 0$, all other $e_{ij}=0$ for the entire loading process. This is taken as the reference loading path.
- L2, proportional combined tension/shear loading: $e_{11} \neq 0$, $e_{22} \neq 0$, $e_{12} \neq 0$, all other $e_{ij}=0$ for the entire loading process.
- L3, nonproportional loading: $e_{11} \neq 0$, all other $e_{ij}=0$ (uniaxial tension in the first half of the loading); $e_{11} \neq 0$, $e_{12} \neq 0$, all other $e_{ij}=0$ (combined tension/shear in the second half of the loading)

Contour plots of the microscopic stress in the different 3D RVEs, subjected to uniaxial tension in the x_1 direction, are shown in Figs. 15(a)–15(d).

The various parameters in the HCDM model are calibrated for the RVEs (a)–(e) following the procedure outlined in Sec. 3.3. The constants $A, B, C, F, G, \dots, S$ in Eq. (33) for the strain invariants

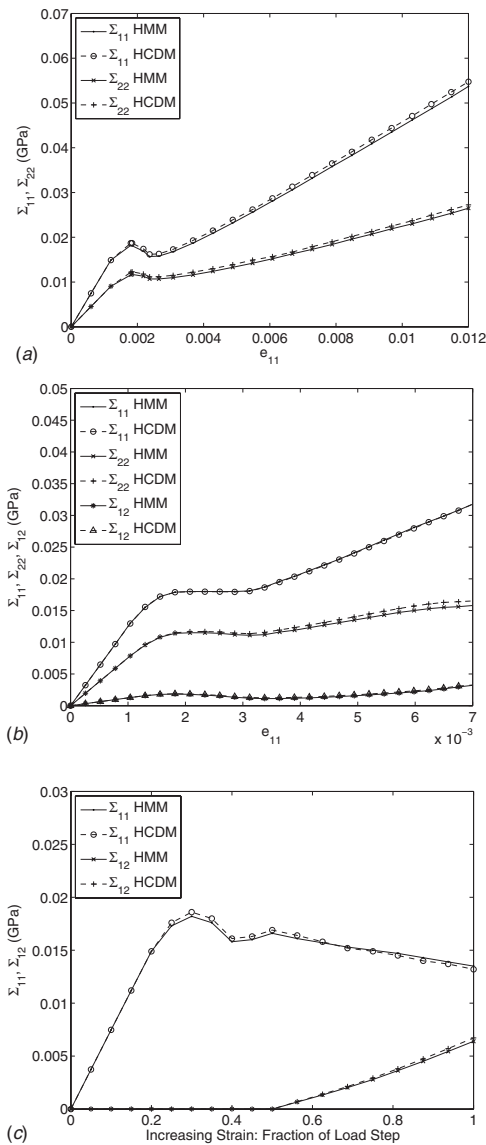


Fig. 12 Comparison of macroscopic stress-strain curve obtained using HCDM and HMM for RVE with two parallel fibers (Fig. 2(c)) for load cases (a) L1, (b) L2, and (c) L3

are evaluated for all five RVEs and are reported in Table 1. The constants exhibit symmetry with respect to x_1 and x_2 directions for the RVE with a cylindrical fiber, and with respect to x_1 and x_3 directions for the RVE with two perpendicular fibers. No particular symmetry is observed for the other RVEs. Since the calibrated constants in the expressions for κ' and P'_{ijkl} are numerous, only a few representative values are given for the RVE with a single cylindrical fiber (a). These are $b_0=0.1E-6$, $b_1=12.44$, $b_2=0.44E5$, $a_0=0.42$, $a_1=3.08$, $a_2=2.44$, and $a_3=74.2$. For P'_{1111} , some of the constants are $c_0^{1111}=-1.16$, $c_1^{1111}=-0.144$, $c_2^{1111}=-0.615$, and $c_3^{1111}=-0.144$.

Figures 10–14 compare the macroscopic stress-strain plots obtained using HCDM with those by homogenizing the results of micromechanical RVE analyses for the three load cases considered. The excellent match in most cases corroborates the satisfactory performance of the HCDM model. An important observation from these results is the sensitivity of the HCDM behavior to the microstructural architecture in response to different loads. The RVE (a) subjected to loading (L1) shows rapid material degradation with increasing strain as the interface undergoes debonding in Fig. 10(a). The stiffness stabilizes at a strain of 0.0012 when the

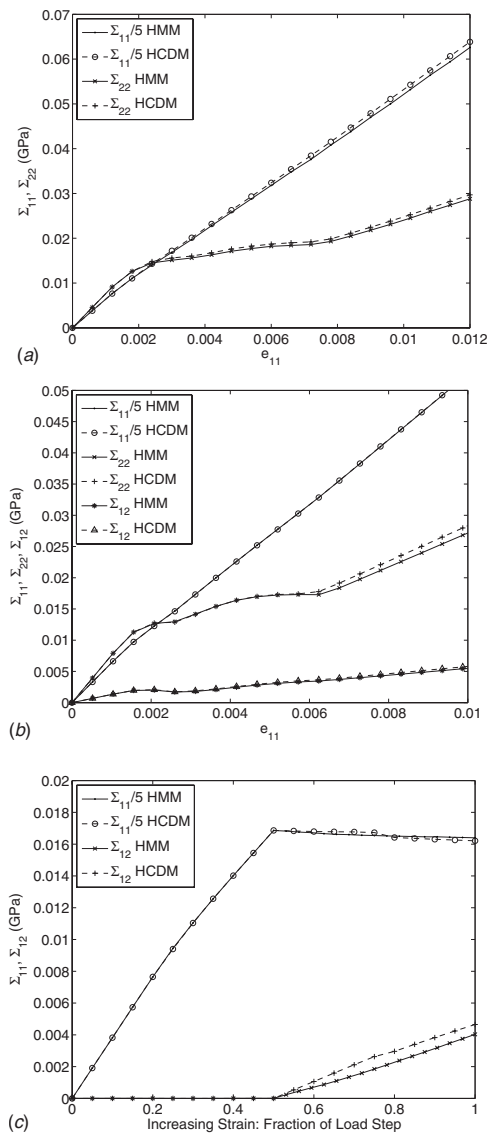


Fig. 13 Comparison of macroscopic stress-strain curve obtained using HCDM and HMM for RVE with two perpendicular fibers (Fig. 2(d)) for load cases (a) L1, (b) L2, and (c) L3

interface debonds completely. Similar trends are seen for the combined loading (L2) in Fig. 10(b), with material degradation followed by constant stiffness corresponding to a locked state. For the nonproportional loading (L3) in Fig. 10(c), the PDCS representation results in a remarkable improvement of accuracy when compared to results in Fig. 4. The stress Σ_{11} in this case keeps reducing in the second half of the loading. For the RVE (b) with the elliptical fiber, stress concentration at the major axis, as shown in Fig. 15(b), causes uneven debonding of the interface. This results in different behaviors in different directions even for uniaxial tension, as shown in Figs. 11(a)–11(c).

The debonded configuration of the RVE (c) with two parallel fibers in Fig. 15(c) shows that nonuniform spacing of fibers causes one side of each fiber to debond more than the other. The stress-strain plots in Figs. 12(a)–12(c) for various loading paths show more rapid degradation than that for RVE (a). Figures 13(a)–13(c) show macroscopic stress-strain plots for the cross-ply composite RVE (d), shown in Fig. 2(d). The Σ_{11} component of stress shows insignificant amount of softening for all three loading cases. This is because for a tensile load in the x_1 direction, the major share of the load is supported by the fiber in the x_1 direction. This fiber

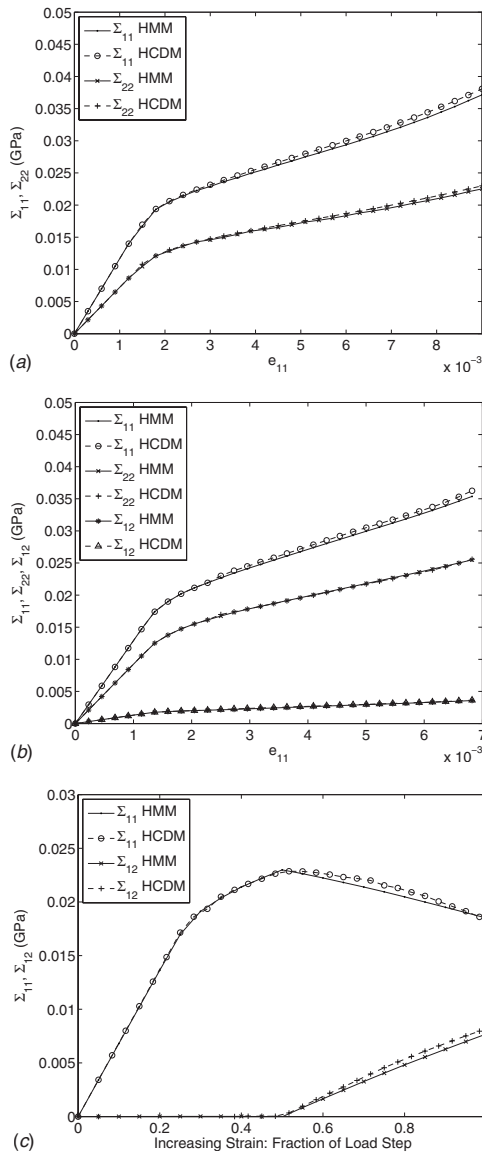


Fig. 14 Comparison of macroscopic stress-strain curve obtained using HCDM and HMM for RVE with 20 circular fibers (Fig. 1) for load cases (a) L1, (b) L2, and (c) L3

continues to support the tensile load even after debonding occurs in this fiber. Degradation of the elastic stiffness in the x_1 direction occurs mainly due to debonding at the interface of the fiber in the x_3 direction, which is not significant. On the other hand, the Σ_{22} plot for the uniaxial tension (L1) case in Fig. 13(a) shows the effect of considerable damage. The initial elastic response is followed by degradation before stabilizing at a lower value of stiffness. There are two phases of rapid degradation corresponding to debonding of fibers in the x_3 and x_1 directions, respectively. The first is due to rapid separation of the fiber in the x_3 direction, whose axis is perpendicular to the loading direction (see Fig. 15(d)). Due to symmetry, degradation in the x_2 direction is equally affected by debonding of fibers in the x_1 and x_3 directions. When the RVE is subjected to tensile strain in the x_1 direction, imposed periodicity conditions in the three orthogonal directions causes the material to experience tension in these directions. This is a consequence of Poisson's effect and the constraint to produce zero macroscopic strains in the x_2 and x_3 directions, which leads to interfacial separation of the fiber in x_3 direction. This second rapid degradation is due to this effect. Finally, when the two interfaces

have debonded completely, the stress response in the x_2 direction corresponds to that of a RVE containing two voids. A similar behavior is also observed for the combined loading (L2) shown in Fig. 13(b). The shear stress is seen to be affected more prominently by the debonding of fiber in the x_1 direction. Figures 14(a)–14(c) show good agreement between the response of HCDM model and the HMM for the 2D multiple fiber RVE with random fiber distribution. The material degradation in this case is not as rapid as with the RVE (a).

The error in stress predicted by HCDM is attributed to the error in the functional representation of $\kappa'(W_d)$ and P'_{ijkl} , and the assumption of orthotropy. It is evident from the examples discussed that the material degradation and, consequently, the variation of damage parameters depend on the shape, distribution, and orientation of the fibers in the RVE. This emphasizes the need for comprehensive 3D micromechanics based continuum damage model for use in macroscopic analysis modules.

5 Conclusions

An accurate and computationally efficient 3D HCDM is presented in this paper for fiber reinforced composites undergoing interfacial debonding. An orthotropic damage model in the PDCS using fourth order damage tensor, which characterizes the stiffness as an internal variable, is found to be appropriate for predicting the damage behavior for a wide range of proportional and non-proportional loading. An investigation of the effect of loading history on the orientation of principal damage axes reveals that the PDCS experiences rotation along a nonproportional macroscopic strain evolution path. This effect is accounted for by expressing the evolution laws in PDCS. Functional forms of various damage parameters (κ' , P'_{ijkl}) in terms of the strain invariants I_1 , J_2 , and J_3 , and damage dissipation energy W_d of the strain tensor, are developed to express variation of damage variables with the evolving damage, and are calibrated by performing micromechanical RVE analyses for a few imposed strain loads. The functional forms of the parameters overcome the serious limitations of constant damage parameters that are conventionally assumed in CDM models. The representation of the CDM in the evolving PDCS also adds significantly to its versatility to a wide variety of loading conditions. The orthotropy assumption in the PDCS is found to yield reasonable accuracy. The model's robustness is evident from the good agreement between HCDM and HMM response of different RVEs for a variety of loading paths.

The HCDM model developed in this paper is a very helpful tool for making macroscopic damage predictions once the model has been calibrated for a given RVE. It can be used as a design tool to optimize the mechanical properties of structures/components by considering the variety of microstructural configurations. However, a different set of calibrated parameters is needed once the RVE architecture changes. Transferability from one RVE to another would require scaling parameters formed out of geometric parameters in the microstructure. This is a nontrivial task and is currently being pursued. Another issue with this model is the assumption of the existence of a RVE, wherein macroscopic variables are nearly uniform and microscopic periodicity can be assumed. Identification of RVEs can be done by methods described by the second author in Ref. [36]. However, as shown in Ref. [37], the RVE keeps evolving in size with the evolution of microscopic damage. Continuous adjustments are needed in the model for their applicability to problems with significant damage. One possible fix is to invoke a scale transfer for a complete microscopic analysis in regions of high localized damage as conducted in Ref. [19,22,23], thereby avoiding the use of CDM in this region. Alternatively, higher order CDMs that account for the growth of the RVE can be developed. These are topics of ongoing research.

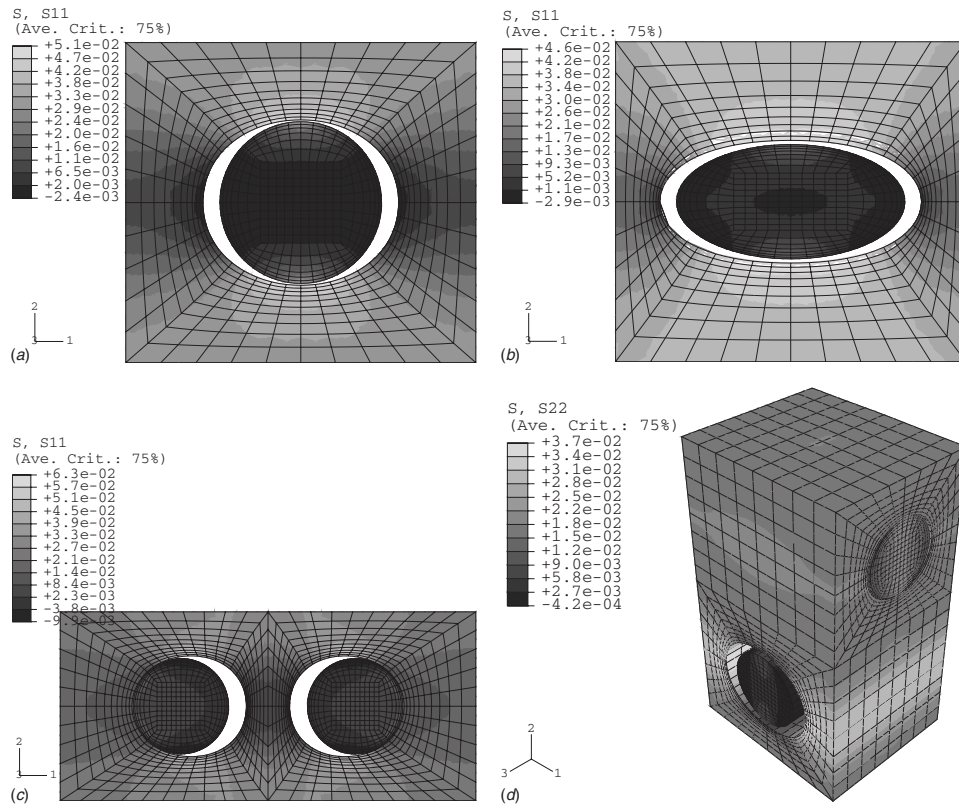


Fig. 15 Microscopic stress contour plot of RVE with (a) a cylindrical fiber, (b) an elliptical fiber, (c) two parallel fibers, and (d) two perpendicular fibers, subjected to uniaxial tension of $\epsilon_{11}=0.004$

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The Effects of Vibrations on Particle Motion in a Viscous Fluid Cell

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The effects of small vibrations on particle motion in a viscous fluid cell have been investigated experimentally and theoretically. A steel particle was suspended by a thin wire at the center of a fluid cell, and the cell was vibrated horizontally using an electromagnetic actuator and an air bearing stage. The vibration-induced particle amplitude measurements were performed for different fluid viscosities (58.0 cP and 945 cP), and cell vibration amplitudes and frequencies. A viscous fluid model was also developed to predict the vibration-induced particle motion. This model shows the effect of fluid viscosity compared to the inviscid model, which was presented earlier by Hassan et al. (2004, "The Effects of Vibrations on Particle Motion in an Infinite Fluid Cell," ASME J. Appl. Mech., 73(1), pp. 72–78) and validated using data obtained for water. The viscous model with modified drag coefficients is shown to predict well the particle amplitude data for the fluid viscosities of 58.5 cP and 945 cP. While there is a resonance frequency corresponding to the particle peak amplitude for oil (58.0 cP), this phenomenon disappeared for glycerol (945 cP). This disappearance of resonance phenomenon is explained by referring to the theory of mechanical vibrations of a mass-spring-damper system. For the sinusoidal particle motion in a viscous fluid, the effective drag force has been obtained, which includes the virtual mass force, drag force proportional to the velocity, and the Basset or history force terms. [DOI: 10.1115/1.2839658]

Keywords: hydrodynamics, viscosity, particle, vibration, frequency, amplitude

1 Introduction

Advanced materials and new drugs can be produced under microgravity where sedimentation and buoyancy-induced motion are suppressed even in systems that contain fluids and solid particles with different densities. However, many semiconductor and crystal growth experiments conducted in the past aboard the Space Shuttle and Mir Space Station have yielded unexpected results possibly due to small amplitude vibrations.

These vibrations called g-jitter exist on the space platforms and may totally alter the fluid behavior under microgravity, which may lead to different crystal properties. Recent protein crystal growth experiments conducted by Gamache et al. [1] have shown that small vibrations can induce movements of protein crystals, which in turn can cause significant fluid motions around the growing crystal. Thus, to improve material processing in space, the effects of small vibrations on fluid and fluid-particle systems need to be better understood. To this end, the motion of a solid particle in a fluid cell filled with a viscous liquid and subjected to small horizontal vibrations has been investigated experimentally and theoretically over a wide range of vibration conditions.

In the past, many articles have been published on flow-induced vibrations of solid structures. However, the reverse situation of vibration-induced particle motion in a viscous fluid has not been fully explored and understood for different particle Reynolds number ranges. Some theoretical and experimental studies exist on fluid flow for different vibration frequencies; so, those results would be utilized in this work.

Stokes [2], Basset [3], and Boussinesq [4] derived expressions for the hydrodynamic force exerted on a sphere, which is sub-

jected to harmonic and arbitrary motions, respectively. They omitted the inertia terms (nonlinear terms) in order to simplify the Navier–Stokes equations. Oseen [5] linearized the nonlinear terms up to the first order and showed that the ratio of the inertial to the viscous terms cannot be negligible at distances ($1/Re$) as assumed by Stokes for a steady creeping flow no matter how small the particle diameter is. Later, Proudman and Pearson [6] showed the inaccuracy of Oseen's equations near the edge of the particle, and introduced solutions near and far from the body, and matched them asymptotically.

Odar [7] dealt with the forces acting on a sphere accelerating in an otherwise quiet and viscous fluid. He stated that the general situations in which both fluid and body move are difficult to study, since the motion of the fluid could be curved and unsteady. Other important studies performed to determine the terminal velocities of a spherical particle in an oscillating liquid include those by Baird et al. [8], Ikeda [9], Jameson and Davidson [10], and Tunstall et al. [11]. Molinier et al. [12] dealt with the motion of a sphere in a column where there was a circulation of a viscous oil. An important reference that summarizes the works of above authors and others on the motion of drops, bubbles, and particles up to 1978 is by Clift et al. [13]. It covers flows at low and high particle Reynolds numbers, the drag force induced, and the wall effects on the particle motion.

Feinman [14] studied experimentally the effect of viscosity on the terminal velocity of a particle in a sinusoidal velocity field. He neglected the existence of a Basset force (or history force) given by Eq. (1),

$$F_B = 6R_0^3(\pi\mu\rho_F)^{1/2} \int_0^t \frac{\gamma(t')}{(t-t')^{1/2}} dt' \quad (1)$$

where the Basset force on an oscillating particle results from the past motion of the particle. One can see from Eq. (1) that the Basset force is an unsteady force, which depends on the particle's past acceleration relative to the fluid, $\gamma(t)$, fluid viscosity and

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density, and particle radius. For highly viscous fluids and particles of large diameters, the Basset force becomes large at high acceleration rates.

Feinman's experiments [14] were based on Houghton's analysis [15] of the particle trajectories given by the Mathieu equation. He derived the minimum frequency required for particle levitation and stability. The range of viscosities tested in Feinman's experiments was, however, quite limited (1–7 cP), which made his predictions for the virtual mass coefficient applicable only to his experiment. Houghton [15] analyzed the nonlinear drag (Newton's law) on free particles in a sinusoidal velocity field leading to the Mathieu equation. He found that stable particle trajectories may occur in certain ranges of sinusoidal velocity amplitude and frequency.

Mei et al. [16] used a finite-difference method to numerically calculate the unsteady drag on a stationary sphere with small fluctuations in the freestream velocity at finite Reynolds numbers. Their results indicated that the drag force would increase linearly with frequency ω when the frequency is very low and asymptotically as the square root of frequency $\sqrt{\omega}$ at high frequencies. The latter behavior of the drag force is similar to that of the Basset force. This drag force dependence on frequency suggests that the Basset force is not constant at all frequencies. Their approach in analyzing the total drag force will be used later in this paper to justify the use of an equivalent drag coefficient.

Maxey and Riley [17] developed an equation of motion for a small rigid sphere in a nonuniform flow. Their analysis was based on the errors in Tchen's equation [18]. Even though no assumption was made, their equation cannot be simplified in order to be applied to predict experimental data. Lovalenti and Brady [19] derived the hydrodynamic force acting on a rigid particle in very low Reynolds number flows and found that the expression for the hydrodynamic force is not simply an additive combination of the results obtained from unsteady Stokes' and steady Oseen's equations. Recently, Abbad and Souhar [20] investigated the history force acting on oscillating fluid spheres in a quiescent viscous liquid at low Reynolds numbers. Unfortunately, no analytical expression was developed for the particle amplitude.

Coimbra and Rangel [21] derived an analytical model for the periodic motion of a small particle in a viscous fluid. In their paper, relative scaling of the virtual mass, Stokes drag, and history forces was presented. They showed that when the scaling parameter S is near unity, the magnitude of the history force would be approximately three times larger than those of the virtual mass and the Stokes drag forces. Their results will be utilized later in this paper to interpret the effect of the viscosity on the importance of the Basset force in highly viscous fluids.

Our recent papers related to vibration-induced particle motion in an inviscid fluid have experimentally and theoretically examined an infinitely large cell (Hassan et al. [22]) and a semi-infinite cell where wall effects were present (Hassan et al. [23]). In both papers, theoretical models were derived for the particle amplitude A_p , assuming an inviscid fluid and irrotational flow, and shown to predict well the experimental data obtained using fluid cells filled with water. In both experiments, a steel particle attached to a thin wire was immersed in a water-filled cell, which was subjected to horizontal vibrations of different amplitudes and frequencies. Using an image analysis technique, the particle amplitude A_p , a half of the peak-to-peak amplitude, was accurately determined. The fluid cell was sufficiently large in size compared to the particle so that it could be regarded as an infinite cell (Hassan et al. [22]). The semi-infinite cell was achieved by inserting one or more flat plates into the cell and narrowing the distance between the particle and the nearest vertical cell wall (Hassan et al. [23]).

For both infinite and semi-infinite cells, a resonance phenomenon was detected experimentally and predicted theoretically by the inviscid models. At the resonance frequency, the particle amplitude would increase to an infinite value, and a change would occur in the phase relationship between the particle and cell mo-

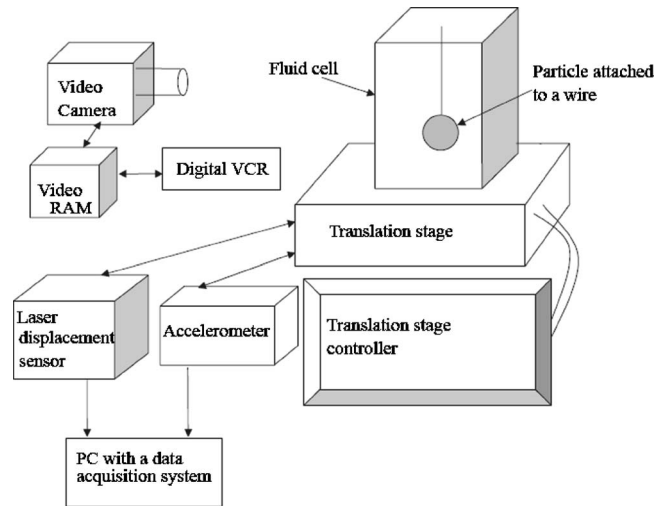


Fig. 1 Experimental setup

tions. At sufficiently high vibration frequencies, the particle amplitude asymptotically reached a constant value independent of the cell vibration frequency, and the asymptotic particle amplitudes were well predicted by the inviscid models. A further inviscid analysis was performed by Hassan et al. [24] to show the existence of a vibration-induced hydrodynamic force, which causes the particle to drift toward the nearest wall. The theoretical predictions of the particle drift compared well with the experimental data obtained in a fluid cell filled with water.

Although the viscosity effect on the particle motion was determined to be small for water since the inviscid model could predict the experimental data rather well, the particle motion is expected to be strongly damped in a highly viscous fluid. To quantitatively determine the effect of fluid viscosity on the particle amplitude and resonance phenomenon observed in water-filled cell experiments, new experiments have been carried out using mineral oil (58 cP) and glycerol (945 cP).

A viscous model has also been developed based on an analogy with a mechanical vibration system, and the viscous model predictions will be compared with the new experimental data. The mechanical representation consists of a mass attached to a damper-spring system having certain values of stiffness and damping coefficients. It is also known from the theory of vibrations [25] that as the damping coefficient increases, the resonance phenomenon disappears completely. In addition to investigating the particle amplitude in a viscous fluid cell, which is subjected to external vibrations, the questions to be addressed in this paper included the suppression of a resonance phenomenon in the particle motion similar to a highly damped mass-spring system.

2 Experimental Apparatus and Procedure

The experimental apparatus used consisted of a test section, a personal computer (PC)-controlled linear stage, and a video camera/recording system as shown in Fig. 1 and described in detail elsewhere [22–24]. Each of the major components is briefly described below.

A computer-controlled translation stage was used to vibrate the fluid cell with submicron resolution and repeatability. It was controlled to move horizontally with a specified amplitude and frequency in a near sinusoidal manner. The test section was a liquid-filled transparent container 110 mm in height and 70 mm in width and length, and made of smooth, polished acrylic plates. A thin platinum wire of 125 μm diameter and 70 mm wire length was used to suspend a spherical particle at the center of the fluid cell. The effects of the wire diameter as well as the wire length had been studied previously by Hassan et al. [22].

Only a spherical steel particle ($\rho_s = 7.83 \text{ kg/cm}^3$) of diameter $d = 12.7 \text{ mm}$ has been used in the experiments. The liquid used to fill the fluid cell was viscous oils of different viscosities, 58.0 cP and 945 cP. A digital color video camera (Hitachi D.S.P VK C-370) with an interchangeable lens was used to capture the particle motion with sufficient magnification and proper illumination from the rear using a light source placed far from the fluid cell to avoid any heating effect. The edge of the particle was captured at a rate of 30 frames per second for at least 2 min, and the shutter speed was set at $1/1000 \text{ s}^{-1}$ to obtain sharp images. The video data were recorded on a mini-DV videocassette by means of a digital video cassette recorder (VCR) (JVC No. AG-7355). The particle edges recorded in a video tape were analyzed using an image analysis program. In total, 255 consecutive frames of particle edge images captured over 8.64 s were analyzed to obtain the particle amplitude and frequency data.

In conducting the experiments, the vibration frequency, f , was first set at 0.25 Hz and the amplitude was changed from 0.5 mm to 2.0 mm over small increments. Then, the frequency was increased by 0.25 Hz up to 7.0 Hz, and the experiment was run again with the same amplitudes as before for 0.25 Hz. At frequencies above 4.0 Hz, the cell amplitude was changed from 0.5 mm to only 1.0 mm, due to the limitation of the linear stage at high frequencies. The experiments were also conducted such that at a fixed cell vibration amplitude of 0.5 mm, the cell vibration frequency was increased from 0.25 Hz to 7.0 Hz. After reaching 7.0 Hz, the previous procedure was also repeated with different imposed cell amplitudes.

The phase of the particle motion with respect to the cell motion was identified by fixing a rod with a sharp marker on the vibration isolation table, and vibrating the fluid cell while the particle edge and the marker position were recorded simultaneously. This way, the direction and speed of the cell motion could be detected in the video image with respect to the fixed marker rod. A pixel to micron conversion factor was obtained by lowering a platinum wire of known diameter to the bottom of the fluid cell and recording its image. Then, by using the same image analysis program, the wire diameter in pixels was calculated and the conversion factor was computed.

3 Theoretical Analysis of the Particle Motion in a Viscous Fluid in an Infinite Cell

In this section, an approximate theoretical model will be developed to predict the particle amplitude in viscous fluids. The system modeled consists of a particle suspended by a thin wire in a fluid cell much larger than the particle and horizontally vibrated in a sinusoidal manner. The dimensionless amplitude $\tilde{A}_{pi}$ of a particle oscillating in a large fluid cell filled with an inviscid fluid and vibrated sinusoidally with a dimensionless amplitude of $\tilde{a}$ in one direction was obtained by Hassan et al. [22] as

$$\tilde{A}_{pi} = \frac{\tilde{\lambda}\tilde{a}}{\tilde{\eta} - 1} \quad (2)$$

where the subscript i stands for the inviscid fluid, and $\tilde{\eta}$ and $\tilde{\lambda}$ are given by

$$\tilde{\lambda} = \frac{2(\tilde{\rho} - 1)}{2\tilde{\rho} + 1} \quad (3)$$

$$\tilde{\eta} = \frac{\tilde{\lambda}}{\tilde{\omega}^2} \quad (4)$$

Here, the dimensionless parameters for the fluid density, wire length, cell vibration frequency, particle and cell amplitudes, particle displacement from the equilibrium position, and time are defined as follows:

$$\tilde{\rho} = \frac{\rho_s}{\rho_L}, \quad \tilde{L} = \frac{L}{R_0}, \quad \tilde{\omega} = \frac{\omega}{\sqrt{g/L}}, \quad \tilde{A}_{pv} = \frac{A_{pv}}{R_0},$$

$$\tilde{a} = \frac{a}{R_0}, \quad \tilde{X}_p = \frac{X_p}{R_0}, \quad \tilde{t} = \omega t$$

For a particle moving in a cell filled with a viscous fluid, an additional term has to be added to the inviscid model to take into account the viscous drag force. The dimensionless equation of motion for a particle in a viscous fluid is then given by

$$\ddot{\tilde{X}}_p + \tilde{\alpha}\dot{\tilde{X}}_p + \tilde{\eta}\tilde{X}_p = \tilde{\lambda}\sin\tilde{t} \quad (5)$$

where $\ddot{\tilde{X}}_p$ and $\dot{\tilde{X}}_p$ are the acceleration and velocity of the vibrating particle in the cell frame of reference. The second term on the left hand side of Eq. (5) represents the damping of particle motion by viscous drag, and the parameter $\tilde{\alpha}$ is given by

$$\tilde{\alpha} = \frac{3C_d}{8(\tilde{\rho} + 0.5)} \quad (6)$$

where C_d is the drag coefficient. The third term in Eq. (5) represents the restoring force due to gravity.

For the present experiments involving an oscillating particle in a viscous fluid, the particle Reynolds number ranged from 0.001 to 0.5. Hence, it is possible to linearize Eq. (5) for slow particle motions or low particle Reynolds numbers using

$$C_d = \frac{k}{\text{Re}} = \frac{k\mu}{\rho_L|\dot{X}_p|d} \quad (7)$$

where k is a constant in the drag coefficient usually taken to be 24 for a spherical particle in steady flow when Re is small (< 1). Equation (5) includes the added mass force but not the Basset force. In this work, the effect of the Basset force will be approximately accounted for in the drag force term by modifying the drag coefficient C_d as described below.

From the numerical results of Mei et al. [16], it was found that the fluctuating part of the unsteady drag force can be decomposed into three major components: a quasisteady component, which is a nonlinear function of Re, an unsteady component due to the added mass and the fluid acceleration, and finally the modified Basset force, which increases with frequency and decreases as Re increases. For the case of a particle in an oscillatory motion with an instantaneous speed, $\dot{X}_p$, and the direction of motion changing over each half-period, the drag coefficient should be greater than the steady motion case of Eq. (7) with $k=24$. Hence, the value of k is assumed to increase above 24 for more viscous fluids, larger particles, and at higher vibration frequencies.

Substituting Eqs. (7) and (6) into (5), the equation of particle motion can be given in dimensionless form by

$$\ddot{\tilde{X}}_p + \tilde{\beta}\dot{\tilde{X}}_p + \tilde{\eta}\tilde{X}_p = \tilde{\lambda}\sin\tilde{t} \quad (8)$$

where the viscous damping coefficient $\tilde{\beta}$ is given by

$$\tilde{\beta} = \frac{3k\mu}{2d^2(2\rho_s + \rho_L)\omega} \quad (9)$$

A particular solution has the same form as the right hand side of Eq. (5) (i.e., a sinusoidal motion) but with a phase angle due to the damping term. Since the periodic motion is obtained experimentally by running the system long enough until the transient motion has disappeared, this solution can be written in complex form as

$$\tilde{\tilde{X}}_p = \tilde{A}_{pv}e^{i(\tilde{t}-\varphi)} \quad (10)$$

where the dimensionless particle amplitude $\tilde{A}_{pv}$ and the phase angle are given, respectively, by

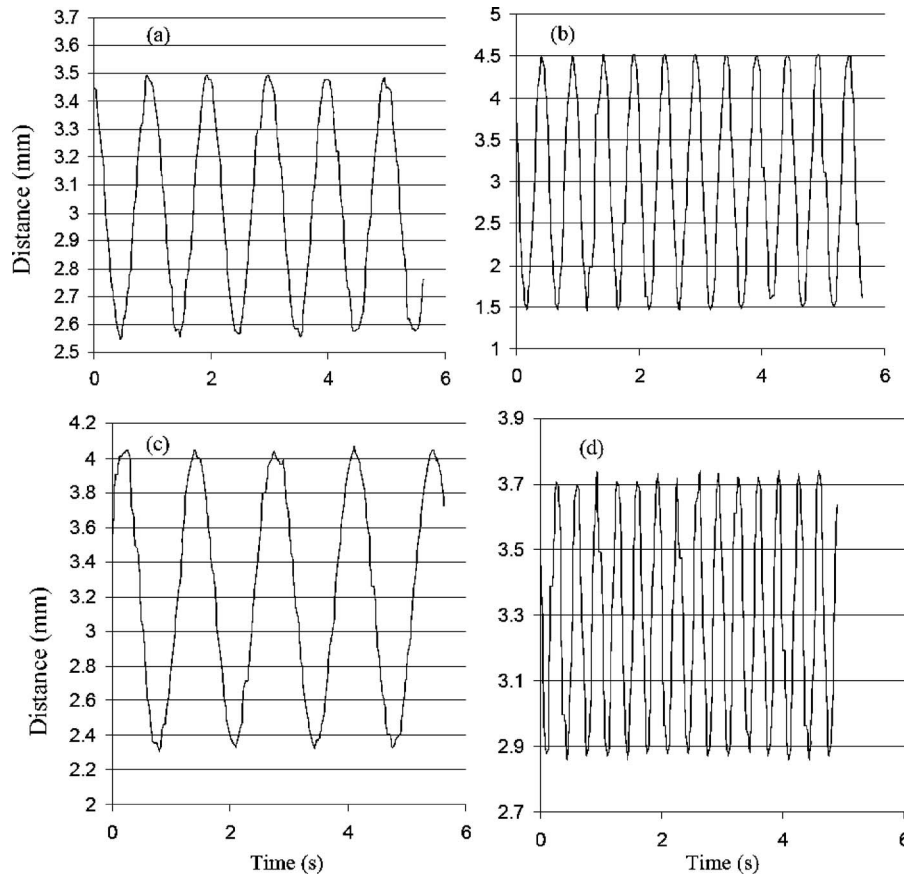


Fig. 2 Instantaneous displacement of a particle for different viscosities and vibration frequencies: (a) oil (58.0 cP, $f=1.0$ Hz, $a=1.0$ mm); (b) oil (58.0 cP, $f=2.0$ Hz, $a=1.0$ mm); (c) glycerol (945 cP, $f=0.75$ Hz, $a=8.0$ mm), and (d) glycerol (945 cP, $f=3.0$ Hz, $a=1.0$ mm)

$$\tilde{A}_{pv} = \frac{\tilde{\lambda}\tilde{a}}{\sqrt{(\tilde{\eta}-1)^2 + \tilde{\beta}^2}} \quad (11)$$

$$\tan \varphi = \frac{\tilde{\beta}}{\tilde{\eta}-1} \quad (12)$$

The subscript v indicates the viscous fluids. For an inviscid fluid ($\mu=0$), the particle amplitude in Eq. (11) reduces to Eq. (2).

From Eq. (11), in microgravity ($g \rightarrow 0$), the particle amplitude reaches an asymptotic value $\tilde{A}_{pv} \rightarrow \tilde{A}_{pv,0g}$, which can be expressed as follows by using Eq. (2) for $\tilde{\eta} \rightarrow 0$ as $g \rightarrow 0$ and $\tilde{\omega} \rightarrow \infty$:

$$\tilde{A}_{pv,0g} = \frac{-\tilde{\lambda}\tilde{a}}{\sqrt{1 + \tilde{\beta}^2}} \quad (13)$$

A comparison of Eqs. (11) and (2) shows that the viscous damping would reduce the particle amplitude, but its effect would vary with the viscosity, particle diameter, and drag coefficient parameter k . Unlike the inviscid model, the viscous model does not predict the existence of a resonance phenomenon since the denominator of Eq. (11) can never become zero. However, as shown later, a maximum particle amplitude can be reached at a certain frequency. The viscous model prediction of the particle amplitude given by Eq. (11) will be quantitatively compared against the experimental data in the next section.

The phase angle given by Eq. (12) varies with the viscosity, particle diameter, drag coefficient parameter k , and vibration frequency. This is in contrast with the inviscid model prediction,

which only indicated a phase difference of either zero or $\pi/2$ rad depending on whether the vibration frequency is below or above the resonance frequency. The phase angle is predicted to also vary with the wire length, particle, and fluid densities. The tangent of the phase angle increases with the fluid viscosity, but decreases with the square of the particle diameter. The particle amplitude is still proportional to the cell amplitude, but in contrast to the inviscid model, the amplitude depends on the particle diameter. The validity of the phase angle relation for a vibrating system, as given by Eq. (12), was qualitatively confirmed experimentally for fluids of different viscosities.

4 Results and Discussion

Typical particle responses to the external vibration are shown in Fig. 2 for imposed cell amplitudes of 8.0 mm, 4.0 mm, 2.0 mm, and 1.0 mm and vibration frequencies ranging from 1.0 Hz to 3.0 Hz. The wire length was 70.0 mm. The instantaneous particle position data obtained from consecutive particle images for different fluid viscosities are plotted as a function of time. The data clearly indicate nearly sinusoidal motions of the particle having the same frequency as that of the cell vibration. From these data, the particle amplitude, A_{pv} , was obtained by dividing in half the difference between the averages of the maximum and minimum positions.

Figures 3 and 4 compare the measured and predicted variations of the dimensionless particle amplitude with the cell vibration frequency for a cell amplitude of 1.0 mm and wire length of 70.0 mm ($=L-R_0$). The viscous oil used was a mineral oil of 58 cP viscosity. Although no resonance phenomenon was pre-

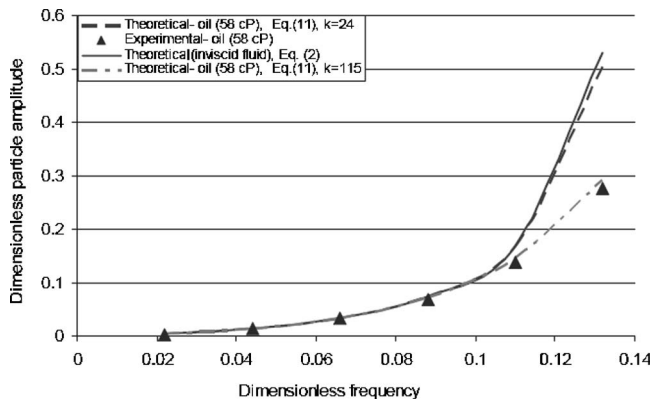


Fig. 3 Experimental and theoretical variations of particle amplitude with cell vibration frequency; cell amplitude: 1.0 mm, steel particle diameter: 12.7 mm

dicted to occur for a particle oscillating in viscous fluids, the particle amplitude reached a maximum value at a dimensionless frequency of 0.13, as shown in Figs. 3 and 4. If the fluid were inviscid, the dimensionless resonance frequency predicted by the inviscid model would be 0.15.

Figures 3 and 4 show that with a constant value of k equal to 24 in the drag coefficient equation, the viscous model predictions are nearly identical with those of the inviscid model, and both show good agreement with the experimental data for dimensionless vibration frequencies below 0.084 and above 0.264. However, between these dimensionless frequencies, both the present viscous model with $k=24$ and inviscid model significantly overpredicted the measured particle amplitudes. This is due to the neglect of hydrodynamic forces such as the Basset force, which are expected to become significant as the particle amplitude and acceleration relative to the surrounding fluid increase. This inadequacy of the steady drag coefficient in the viscous model is consistent with the effect of the Basset force discussed by Mei et al. [16] and by the values of Coimbra and Rangel's [21] scaling parameter S between 0.2 and 1 indicating a large effect of the Basset force in harmonic motions.

If larger values of k are used in the drag coefficient equation to approximately account for the effect of the Basset force, the present viscous model can predict the measured particle amplitudes for all the cell vibration frequencies covered in the experiments, as shown in Figs. 3 and 4. The values of k that could predict the data at dimensionless frequencies of 0.11–0.13 and 0.17–0.35 were 115 and 190, respectively. A higher value of k was needed at higher vibration frequencies, because the particle accel-

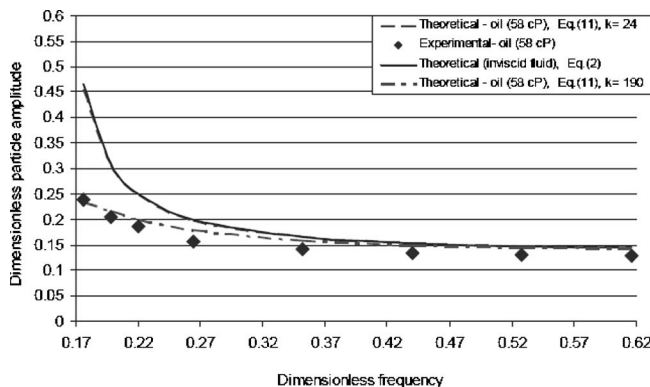


Fig. 4 Experimental and theoretical variations of particle amplitude with dimensionless vibration frequency; cell amplitude: 1.0 mm; steel particle diameter: 12.7 mm

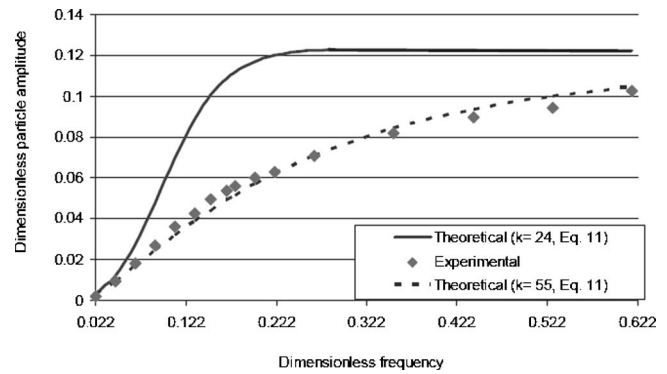


Fig. 5 Experimental and theoretical variations of particle amplitude with dimensionless cell frequency for glycerol oil (945 cP) and for different values of k ; cell amplitude: 1.0 mm

eration relative to the surrounding fluid and the Basset force increased with the vibration frequency at a constant cell amplitude.

As the liquid viscosity was further increased to 945 cP (glycerol), there was a clear deviation of the theoretical predictions made with $k=24$ from the experimental values as shown in Fig. 5 at almost all the dimensionless vibration frequencies tested. Only for the lowest values of $\tilde{\omega} < 0.044$, the value of $k=24$ could be used to well predict the experimental data. The differences between the experimental results and theoretical predictions for $\tilde{\omega} > 0.044$ diminished with an increase in the value of k from the steady drag value of $k=24$ to $k=55$, which yielded good agreement between the viscous model predictions and glycerol (945 cP) data. However, this value of $k=55$ is much smaller than the optimum values (115 and 190) found for the mineral oil (58 cP), because of the smaller effect of the Basset force experienced by the particle due to generally smaller particle amplitudes and accelerations in glycerol.

For sufficiently high dimensionless vibration frequencies, the dimensionless particle amplitude is theoretically predicted from Eq. (11) to reach a constant value equal to

$$\tilde{A}_{pv\infty} = \tilde{\lambda}\tilde{a} \quad (14)$$

which is independent of the fluid viscosity and the vibration frequency. The particle amplitude data agreed well with the predictions of Eq. (14) as shown in Fig. 4 for the mineral oil (58.0 cP); however, for glycerol (945 cP), the particle amplitude did not quite reach the asymptotic value even at the highest dimensionless vibration frequency of $\tilde{\omega}=0.62$ tested in the present work. The dimensionless particle amplitude of slightly greater than 0.1 in glycerol was 23% lower than the dimensionless amplitude of 0.123 predicted by Eq. (14) and experimentally obtained for the same particle in water by Hassan et al. [22].

The dimensionless particle amplitude predicted by Eq. (11) can be shown to reach a maximum value at a certain dimensionless vibration frequency $\tilde{\omega}_m$,

$$\tilde{\omega}_m = \sqrt{\frac{\tilde{\zeta}^4}{\tilde{\zeta}^2 - 2\tilde{\eta}^2}} \quad (15)$$

where $\tilde{\zeta}$ is the dimensionless resonance frequency for a particle in an inviscid fluid given by

$$\tilde{\zeta} = \sqrt{\tilde{\lambda}} = \sqrt{\frac{2(\tilde{\rho} - 1)}{2\tilde{\rho} + 1}} \quad (16)$$

and

$$\tilde{\eta}^2 = \left(\frac{3k\mu}{8d^2(\rho_s + 0.5\rho_L)} \right)^2 \left(\frac{L}{g} \right) \quad (17)$$

The dimensionless resonance frequency $\tilde{\zeta}$ for an inviscid fluid derived by Hassan et al. [22] is recovered when the viscosity is set to zero in Eq. (17) in the viscous model. Furthermore, Eq. (15) indicates that the dimensionless vibration frequency $\tilde{\omega}_m$ is always greater than the dimensionless resonance frequency $\tilde{\zeta}$ for an inviscid fluid provided that

$$\tilde{\psi} < \frac{\tilde{\zeta}}{\sqrt{2}} \quad (18)$$

A clear increase in $\tilde{\omega}_m$ to above the dimensionless resonance frequency $\tilde{\zeta}$ can be seen in Fig. 5.

Substituting the values of $\tilde{\psi}$ and $\tilde{\zeta}$ from Eq. (17) into Eq. (18) leads to the following inequality relating the fluid viscosity with the fluid density, wire length, gravitational acceleration, particle diameter, and density for a given value of constant k in the drag coefficient:

$$\mu < \frac{4d^2}{3k} \sqrt{2 \left[(\rho_s + 0.5\rho_L)(\rho_s - \rho_L) \frac{g}{L} \right]} \quad (19)$$

If Eq. (19) is not satisfied, there would be no resonance phenomenon in which the particle amplitude becomes infinite. For the case of a steel particle of 12.7 mm diameter immersed in a fluid cell filled with glycerol (945 cP), the inequality given by Eq. (19) is satisfied for $k=24$, and Eqs. (15)–(17) with $k=24$ predict the maximum dimensionless particle amplitude to occur at a dimensionless vibration frequency of 0.352. But, this prediction does not agree with the experimental data, as shown in Fig. 5. In fact, the particle amplitude data showed no peak at any frequency. Since the value of $k=24$ accounts for only the steady viscous drag but not the Basset force which becomes more dominant for higher viscosity and particle accelerations as shown in Figs. 3–5, the value of k needed to be increased to 55 in order to give better predictions of the particle amplitude data as discussed earlier. With this value of $k=55$, Eq. (19) is not satisfied and the viscous theory predicts the absence of a resonance phenomenon as shown by the experimental data.

5 Conclusion

A study of vibration-induced particle and fluid motion has been conducted theoretically and experimentally. A theoretical model of a particle suspended in a viscous fluid cell by a thin wire was developed to predict the vibration-induced particle motion. A series of experiments were conducted using a steel particle in a fluid-filled rectangular container under different vibration conditions. Compared to the inviscid theory developed previously by Hassan et al. [22], the viscous theory predicted that the amplitude of a particle oscillating in a viscous fluid would additionally depend on the fluid viscosity and particle diameter.

The viscous theory was able to predict the experimentally measured amplitudes of the steel particle for a moderately viscous mineral oil (58.0 cP) and highly viscous glycerol (945 cP); however, the drag coefficient had to be increased for large particle amplitudes and highly viscous fluids to account for additional hydrodynamic forces such as the Basset force. Further experiments using other viscous fluids and particles are needed to develop a correlation for the modified drag coefficient by understanding its dependence on fluid and particle properties as well as the cell vibration frequency.

The resonance phenomenon was found to exist for a moderately viscous mineral oil (58.0 cP) but disappear for highly viscous glycerol (945 cP), similar to a mass-spring-damper system with different damping coefficients.

Acknowledgment

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Nomenclature

a	= cell vibration amplitude, m
A_p	= particle amplitude in the cell frame of reference, m
C_d	= drag coefficient
d	= particle diameter, m
F_B	= Basset force, N
f	= vibration frequency, Hz
g	= gravitational acceleration, m/s ²
k	= constant in a drag coefficient
L	= distance from the wire attachment point to the center of mass of the particle, m
Re	= Reynolds number
R_0	= particle radius, m
S	= scaling number ($=R_0^2\omega/9\nu$)
t	= time, s
X_p	= horizontal displacement of the particle with respect to the cell, m
$\gamma(t)$	= relative acceleration of the particle with respect to the fluid, m/s ²
μ	= dynamic liquid viscosity, kg/m s
φ	= phase angle, rad
ν	= kinematic liquid viscosity, m ² /s
ρ_L	= liquid density, kg/m ³
ρ_s	= particle density, kg/m ³
ω	= cell vibration frequency ($=2\pi f$), rad/s

Subscripts

F	= fluid
L	= liquid
S	= solid

Superscript

$\sim$	= dimensionless parameter
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Dynamic Biaxial Plastic Buckling of Circular Shells

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Of particular interest is the experimental study of the complex dynamic plastic buckling of circular metallic shells and their energy absorption capacity. Initially proposed by Baleh and Abdul-Latif (2006), "Quasi-Static Biaxial Plastic Buckling of Tubular Structures used as an Energy Absorber," ASME J. Appl. Mech., 74, pp. 638–635, the novel idea, which aims to enhance the strength properties of materials, is extended for studying the biaxial plastic dynamic buckling behavior of circular shells. It can be assumed that changes in local deformation mechanisms, which reflect this enhancement in the strength properties, are mainly governed by the loading path complexity. The question of whether the performance of dynamic axially crushed tubes could be further improved by using the developed device (the absorption par compression-torsion plastique (ACTP)) generating a biaxial loading path (combined compression and torsion) from a uniaxial loading. A key point emerging from this study is that the structure impact response (i.e., the plastic flow mechanism and the absorbed energy) is influenced by the loading rate coupled with the biaxial loading complexity. In this study, three different metallic circular shells made from copper, aluminum, and mild steel, having distinct geometrical parameters, are extensively investigated. The obtained results show that the higher the biaxial loading complexity provided by the ACTP applied, the greater the energy absorbed by the copper, aluminum, and mild-steel structures. Thus, it is easy to demonstrate that the enhancement in the energy absorption, notably in the case of aluminum, is higher than 150%, in favor of the most complicated loading path (i.e., biaxial 45 deg case) compared to the classical uniaxial case. Moreover, the deformation mode for the tested materials is slightly sensitive to the torsion amplitude in dynamic loading, contrary to the quasistatic one. [DOI: 10.1115/1.2839686]

Keywords: plastic buckling, combined compression-torsion dynamic loading, energy absorption

1 Introduction

The collapse of thin-walled circular tubular structures has been largely used as a passive safety concept already adopted since a few decades on various vehicles. Hence, several recent studies conducted under dynamic loading (Refs. [1–13]), reveal that the crushing process is strongly influenced by structural and material aspects, strain rate, and inertia effects. These are the most important factors in structural crashworthiness design. The dynamic axial crushing of the cylindrical tubes represents an efficient energy absorption system, which depends strongly on the above mentioned key parameters. As a matter of fact, several experimental and theoretical studies point out that the dynamic plastic buckling is a complex phenomenon due to the coupling between the inertia effect and plastic properties of the collapsed material. Consequently, the knowledge of the dynamic material phenomena involved in these highly nonlinear dynamic structural problems of the circular shells is crucial for correctly predicting their mean collapse loads (F_{av}) and the corresponding energy absorption capacities. Three types of plastic instability can be often generated [4,5,7]: (i) Euler mode (total bending of the structure), (ii) dynamic plastic buckling (buckling over the entire length of the cylinder with a moderate radial displacement), and, finally, (iii) dynamic progressive buckling (successive folding process). Of particular interest in this study is the latter buckling type. It takes place at relatively low impact velocities (lower than 30 m/s), where the inertia effects are negligible [14], and for a ratio of the

mass of the striker and the mass of the specimen higher than 14. Hence, the plastic flow is always controlled by sequential compression and bending phases, which constitute a complex load/unload path. Thus, the phenomenon of dynamic buckling proves a large sensitivity to the loading path, particularly in the plastic zone [6].

Recently, many research programs have been focused on the axial crushing of metallic structures with different sections, mainly circular and square, using thin-walled high-strength steel sections [13,15]. Note that the objective of such researches is to design a cross section, which minimizes the amount of forces transferred to the integrity of occupants and deforms regularly during a vehicular crush. Hence, motivated by a high crashworthiness performance, the appropriate determination of the most important criteria, a superior strength-to-weight ratio and a low cost-to-weight ratio, is persistently aimed. For different geometries, most of the current developments are based, in general, on materials of high-performance structures with high strength, high modulus, low weight, and good toughness properties. Furthermore, a great deal of research and development showed that several other solutions such as composite materials and cellular materials (honeycombs, metallic foams, etc.) [16–18] are considered to improve the energy absorption capacities.

Being interested in the same type of systems, the present investigation is an extension of the original idea developed in Ref. [19], in which the authors illustrate that the increase in the energy absorption, under quasi-static biaxial loading path (combined compression and torsion), attains up to 35% compared to the classical uniaxial case for the copper circular shells. The developed idea focuses on the possibility of change in the material strength properties even under dynamic loading via the loading path complexity concept. In fact, Baleh and Abdul-Latif [19] proposed a spe-

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cific mechanical assembly (the absorption par compression-torsion plastique (ACTP)) capable of transforming a uniaxial external loading into a biaxial one. Consequently, the plastic buckling becomes more and more complicated since a shear component is added to these sequential phases of compression and bending under dynamic loading; i.e., three different strains (compression, bending, and shear) are simultaneously applied with more complex load/unload conditions. Provoked by the ACTP, several degrees of complexity related to biaxial loading paths can be also created within the loaded circular shells during their collapses. Indeed, without intervening on the nature and on the geometry of the deformed tubes, additional energy absorption is generally captured.

The principal objective of this experimental work is to study the effect of the loading rate coupled with the combined biaxial loading condition on the progressive dynamic buckling response for three metallic structures (made from copper, aluminum, and mild steel) having different dimensions (i.e., different values of $\eta = R_m/t$ and $\lambda = R_m/L$ (R_m : mean radius, L : initial length, and t : thickness)). Using the ACTP, the behavior of the biaxially crushed materials is studied, demonstrating mainly the dependence of the plastic buckling on the specimen geometry, loading path complexity, and its rate. The energy absorption capacity of each system under different biaxial loading paths is determined and then analyzed. Moreover, the strength property enhancement phenomenon of the tested materials is obviously affected by the loading path complexity [19]. Accordingly, the higher the biaxial loading complexity applied, the greater the energy absorbed in the copper, aluminum, and mild-steel cases for a given structure. It is noteworthy that the mild steel, which does not illustrate a noticeable sensitivity to this biaxial loading path under a quasistatic condition, shows, however, a reasonable sensitivity to dynamic loading type and its rate.

The main finding reveals that the maximum enhancement in the absorbed energy is higher than 150% in the case of aluminum in favor of the most complicated loading path vis-à-vis a classical uniaxial case. Contrary to the quasistatic loading, the deformation mode for the employed materials is slightly sensitive to the torsion amplitude in the dynamic condition.

2 Experimental Program

In order to improve more the energy absorption capacity, the solution developed recently via the biaxial loading path has clearly demonstrated its efficiency. Hence, the proposed device, the ACTP, has been used to test several metallic circular shells made from copper, aluminum, and mild steel [19]. As a result of the loading path complexity, a change in the strength properties of the materials is obviously observed within the loaded structure, almost certainly provoked at the dislocation level. In fact, the crushed material undergoes a highly complicated loading path, giving simultaneously three different compressive, bending, and torsional strains.

In this investigation, dynamic biaxial loading paths generated by the ACTP are therefore considered as a new aspect, which complicates more and more the biaxial buckling mechanism. Based on the observation given in Ref. [19], only the biaxial *proportional (integral) loading* situation is employed in this study. It involves a complicated loading condition during which the compression and torsion components are simultaneously applied.

2.1 Tested Materials. In this study, three metallic materials are investigated. These are commercial hardened copper (tensile yield stress: 310 MPa), annealed aluminum alloy (tensile yield stress: 150 MPa), and annealed mild steel (tensile yield stress: 220 MPa), respectively designated according to French standard as NFA 51120 and AFNOR A506411, A50-451 (6060), and NFA 49 330 378 504523 NBK122, having an excellent ductility.

The employed nonwelded circular shells have the following dimensions: Two internal diameters (d) are chosen, 30 mm and

Table 1 Geometrical parameters of the used circular shells

Used material	Copper		Aluminium	Mild steel
Parameter η	15.5	19.5	15.5	19.5
Parameter λ	0.11	0.14	0.11	0.14

38 mm with 1 mm thickness (t), leading respectively to the radial geometrical ratios ($\eta = R_m/t$) of 15.5 and 19.5, where R_m is the mean shell radius. Two longitudinal ratios ($\lambda = R_m/L = 0.11$ and 0.14) are chosen using an initial shell length of 136 mm.

All the specimens are crushed dynamically under compressive loading. They are not subjected either to heat treatment or to special machining operation. The used circular shells with their ratios are summarized in Table 1.

2.2 Description of the Absorption par Compression-Torsion Plastique². The ACTP device is a simple mechanical assembly, which transforms an external uniaxial compression load into a biaxial combined compression-torsion one (Fig. 1). It allows a substantial increase in the energy absorption in comparison with a uniaxial plastic buckling. An additional resistance in the deformed shells is needed due principally to the torsional component (generated by the ACTP), in parallel to the compression one.

Moreover, the device can generate several rates of change of torsional component by means of three distinct propeller inclination angles of 30 deg, 37 deg, and 45 deg for studying the effect of this significant parameter on the collapse operation.

The developed device (Fig. 1) is constituted from a tempered steel hollow interchangeable cylindrical body (1), on which four parallel helicoid grooves are machined. These grooves are characterized by an inclination angle. They are intended to receive a crosspiece (2), provided with four pivots and to guide it in its movement of descent by inculcating a rotational movement. Hence, the two principal parts of this apparatus form a slide-helicoid connection. This mechanism permits to transform an initial external load of uniaxial nature into a biaxial combined compression-torsion one. In order to minimize the friction in the contact zone between the grooves and the crosspiece (2), the crosspiece pivots are equipped with bronze rollers (3). Since it is considerably difficult to evaluate the friction effect on the deformation operation, its effect is therefore neglected in this work.

The crushed structure (9) is mutually dependent on the crosspiece and cylindrical body by means of a mechanical tube extremities fixation system. For each extremity, the system is made principally from two hard steel disks (11). Two half-conical shells (10) and a clip (13), over which these conical surfaces are machined and assembled in opposition attached to the disk (12), maintain the necessary tightening pressure in locking both crushed tube extremities. Therefore, during its biaxial deformation, the specimen (9) is totally conditioned by the crosspiece in its movements of descent and rotation.

2.3 Impact Apparatus. All the experimental tests are carried out using a dynamic drop mass bench of a maximum impact velocity of 10 m/s and of a maximum kinetic energy of 2.5 kJ. It is equipped with a dynamic load cell of 20 tons, a 5000 g accelerometer, and a laser beam displacement transducer (series M5L of international Bullier) for a measurement bracket of 100 mm. These instruments are connected to a rapid acquisition chain (1 MHz), which ensures the simultaneous recording of these experimental data: force, acceleration, and displacement.

To assure the synchronization of the acquisition of these essential data, two photocells are used. The positioning of these photocells is important and depends on the striker position just before

²Patent no. WO 2005090822.

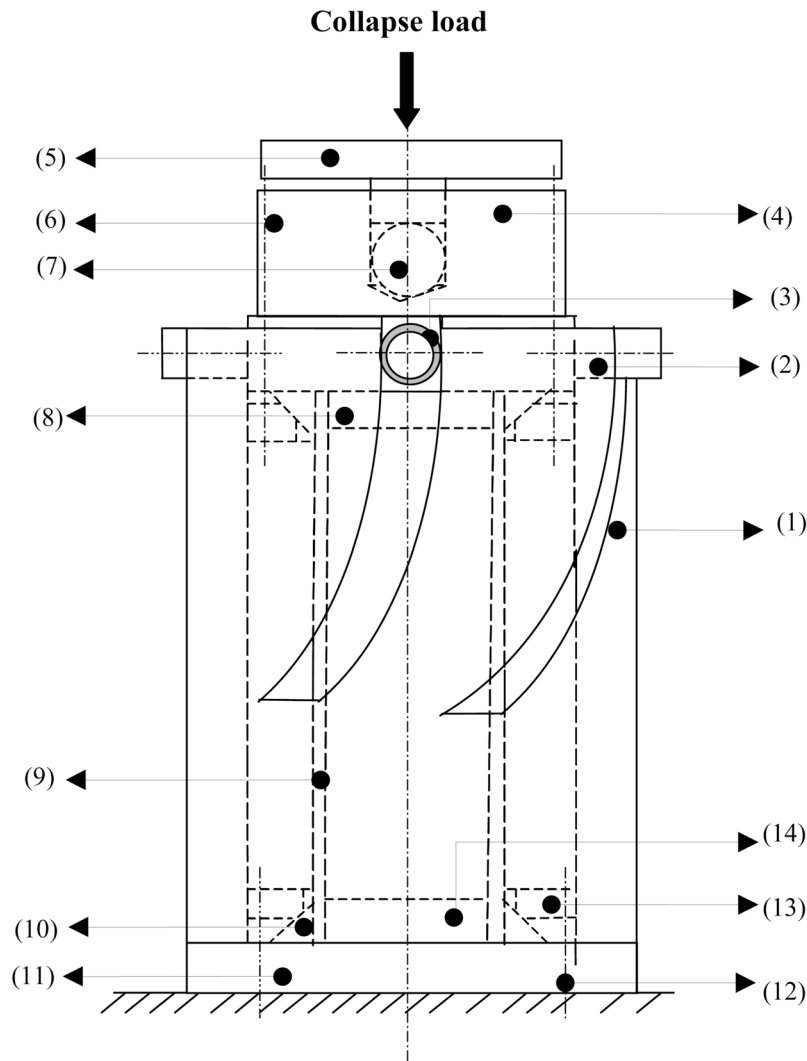
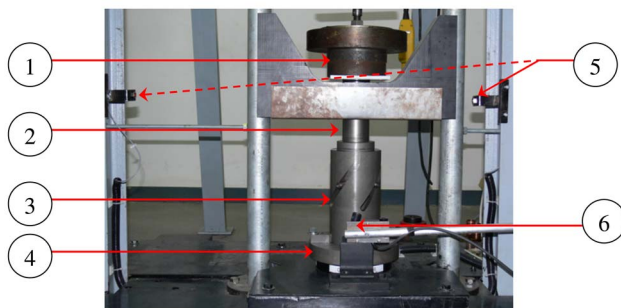


Fig. 1 Brief view of the ACTP device: (1) cylindrical body; (2) crosspiece; (3) roller; (4) intermediate cylinder; (5) receiving disk; (6) higher tightening screw; (7) centering ball; (8) higher disk; (9) specimen; (10) lower conical half-shells; (11) basic disk; (12) lower tightening screw; (13) lower conical clip; (14) lower disk.

the impact (Fig. 2). It is important to underline that their position represents one of the principal difficulties related to the synchronization of the collected experimental data.

Tests are conducted under an initial impact velocity of about



(1) drop mass, (2) striker, (3) the ACTP, (4) dynamic load cell, (5) photocells, (6) displacement transducer

Fig. 2 Presentation of the used measurement instrumentations of the drop mass bench

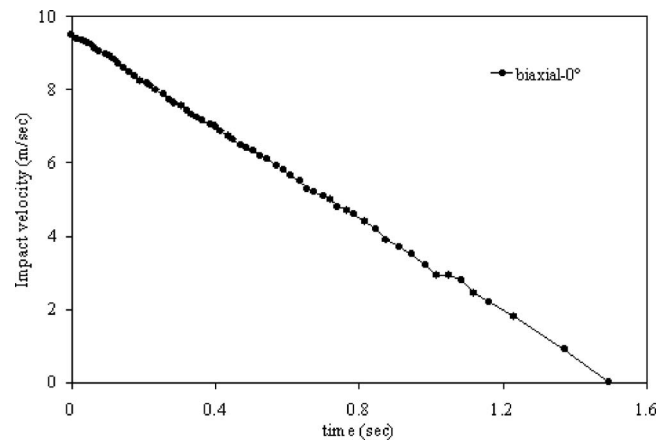


Fig. 3 Variation of the impact velocity during the crushing process

Table 2 Deformation modes and the mean and peak collapse loads for different circular shells having different geometrical parameters loaded under different dynamic loading complexities

		Copper				Aluminium		Mild steel	
Geometrical parameters		Impact velocity (9.5 m/s)							
		15.5		19.5		15.5		19.5	
η	λ	0.11		0.14		0.11		0.14	
Loading path type		Deformation mode	F_{av} and F_{max} (kN)	Deformation mode	F_{av} and F_{max} (kN)	Deformation mode	F_{av} and F_{max} (kN)	Deformation mode	F_{av} and F_{max} (kN)
Uniaxial loading	Dy free ends	DM	19.8–49	DM	21.8–56	DXM	11–41.2	DM	27.5–94
	Dy biaxial 0 deg ^a	AM	21.2–58	AM	22.3–6	AM	11–34.8	AM	29.1–96
Biaxial loading	Dy biaxial 30 deg	AXM	21.5–60	AM	24.8–64	XM	15.2–41.2	DXM	29.5–94
	Dy biaxial 7 deg	XM	22–64	AM	23–73	XM	15.5–53	DXM	31.3–95
	Dy biaxial 5 deg	XM	24.2–78	AM	26.8–82	DM	26–57.2	DM	31.6–00

^aUniaxial fixed-end loading type.

9.5 m/s using either a mass of 45.5 kg (for the copper and mild-steel tubes) or 27 kg (for the aluminum tubes). As a typical example, the rate of change of the impact velocity during the crushing process is obviously illustrated in Fig. 3.

2.4 Experimental Procedure. Each employed circular shell should be first mechanically assembled inside the ACTP, then it is loaded between the platen and the striker of the dynamic drop mass bench under an initial impact velocity of about 9.5 m/s at room temperature. Having its instrumentation system, the impact apparatus over which the ACTP is fixed is connected to a rapid acquisition chain to record the force, displacement, and acceleration during the collapse process.

In order to ensure the accuracy of experimental results, each test is repeated twice under the same experimental conditions (applied velocity and temperature). If the differences between the two responses exceed 3%, then another test should be performed.

3 Experimental Results and Discussion

Three different modes of deformation are frequently generated during tests: diamond mode (DM), axisymmetric mode (AM), and mixed mode (XM). For this last mode, it is necessary to distinguish two nuances: the AXM and the DXM corresponding to the XMs with axisymmetric predominance and with diamond dominance, respectively.

Extracted from a recent work [19], some data conducted under quasistatic loading are used for the sake of comparison. The load-deflection and energy absorption evolutions during the crushing process are described in Figs. 5, 6, 9, and 12.

The notations Dy and Qs correspond to the dynamic and quasistatic loading condition, respectively. Table 2 demonstrates, for each crushed specimen, the most repetitive deformation modes obtained under the dynamic load and the mean (F_{av}) and peak (F_{max}) collapse loads. It is well known that the impact velocity changes progressively with the crushing distance (Fig. 3); there-

fore, to determine objectively the dynamic mean collapse load for the employed metals, a crushed distance of 40 mm is used during which all recorded forces are integrated. The choice of such a distance will be discussed subsequently.

3.1 Uniaxial Dynamic Plastic Buckling

3.1.1 Deformation Modes. An examination of the tested circular shells at the end of the uniaxial collapse (Fig. 4) reveals that their dynamic progressive plastic buckling takes place initially with the formation of a wrinkle at each end simultaneously before invading the rest of the structure from only one side. This is different from the quasistatic loading condition, where the deformation is unilateral with one intact end. A first analysis of Table 2 shows the dependence of the deformation mode on the geometrical parameters and on the imposed boundary condition (free or fixed-end condition), rather than of the nature of the material, as shown in Ref. [19].

The aluminum and copper shells, characterized by the same geometry ($\lambda=0.11$ and $\eta=15.5$), deform with the DM in the free-end case even for the copper and mild shell steel having a bigger geometry ($\lambda=0.14$ and $\eta=19.5$). Nonetheless, for these bigger structures, the AM is essentially recorded in the biaxial 0 deg case (Fig. 4(b) and Table 2). Moreover, under quasistatic loading condition, the plastic instability generates globally the same deformation mode, except for the mild steel where the deformation mode is always of DM type [19]. However, this corroborates, in general, correctly the conclusions reported by several authors (e.g., Refs. [20,21]).

3.1.2 Buckling Loads. With regard to the applied loads (F_{av} and F_{max}), Table 2 points out a significant effect of the coupling among the geometrical parameters, employed boundary conditions, and loading rate on the values of these loads for the three employed materials. This effect appears principally at the beginning of the crushing process (Fig. 5). In comparison, the quasi-



Fig. 4 Deformation modes of the three used structures under two boundary conditions: (a) free ends (DM); (b) fixed ends (AM)

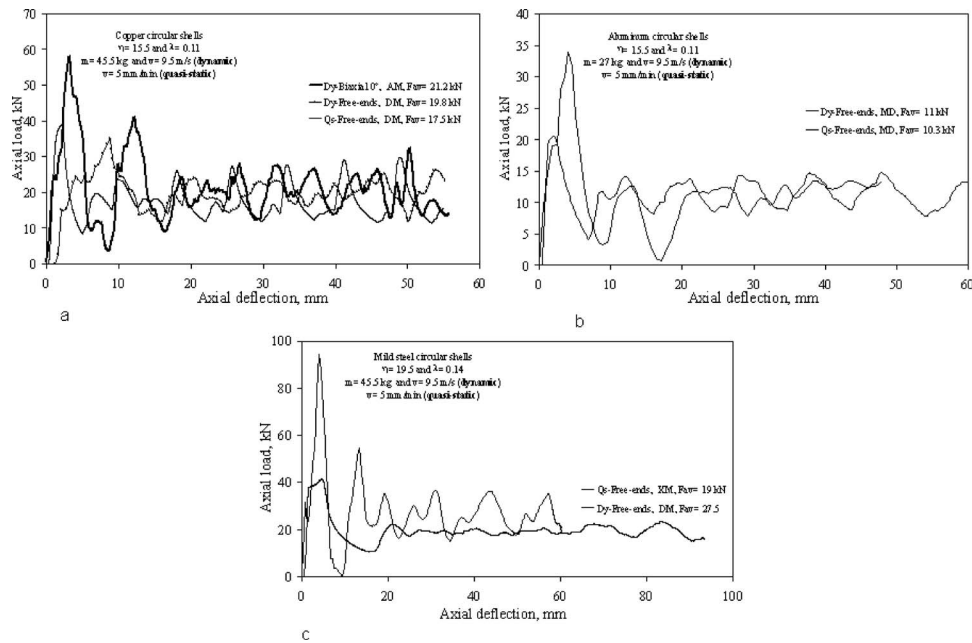


Fig. 5 Plot of crushing load evolution versus axial deflection under uniaxial dynamic and quasistatic loading conditions for the (a) copper, (b) aluminum, and (c) mild-steel circular shells

static and dynamic load-deflection curves have a rather similar form, especially at the end of the collapse process where the two loading rates become analogous due to the progressive reduction in dynamic velocity. However, the most remarkable aspect is the enhancement in the mean and peak collapse loads, whatever the material and the impact mass are. Under quasistatic loading, it has been especially shown [22] that the $F_{\max}$ does not vary with the used boundary conditions (i.e., free and fixed ends). However, a substantial variation in the $F_{\max}$ is observed in the fixed-end case loaded dynamically: varying from 39.4 kN to 58 kN (for copper with $\eta=15.5$), 20.6 kN to 34.8 kN (for aluminum), and

40.6 kN to 96 kN (for mild steel). This means that these metals show an obvious sensitivity to the used dynamic loading rate. Likewise, the response of these metals shows a reasonable increase in their dynamic mean collapse loads (F_{av}) with respect to the quasistatic ones. In fact, for a boundary condition of the free-end type, these variations are from 17 kN to 19.8 kN (for copper with $\eta=15.5$), 10.3 kN to 11 kN (for aluminum), and 21.1 kN to 27.3 kN (for mild steel). This is due principally to the progressive decrease in loading rate toward the quasistatic condition during the crushing process. As previously shown under qua-

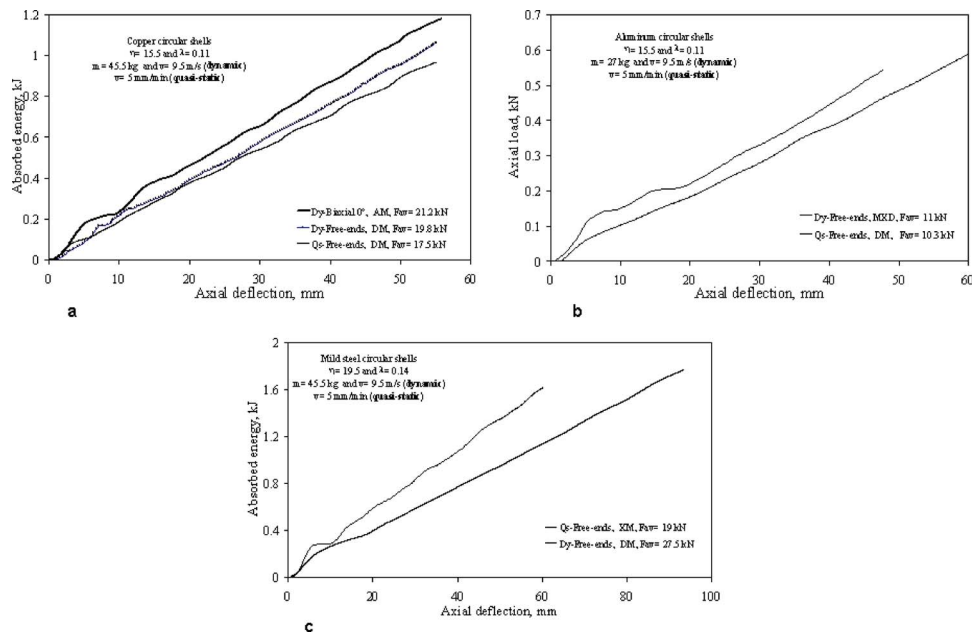


Fig. 6 Energy absorption evolution versus axial deflection under uniaxial dynamic and quasistatic loading condition for the (a) copper structures, (b) aluminum, and (c) mild-steel structures

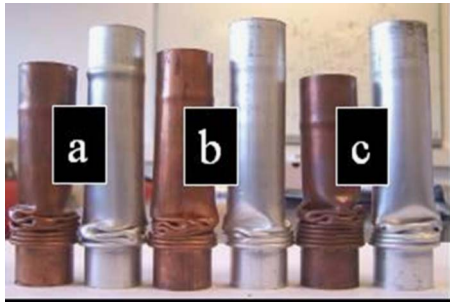


Fig. 7 Typical examples of the deformation modes for the copper and aluminum specimens having $\eta=15.5$ under different dynamic loading complexities: (a) biaxial 45 deg, (b) biaxial 37 deg, and (c) biaxial 30 deg

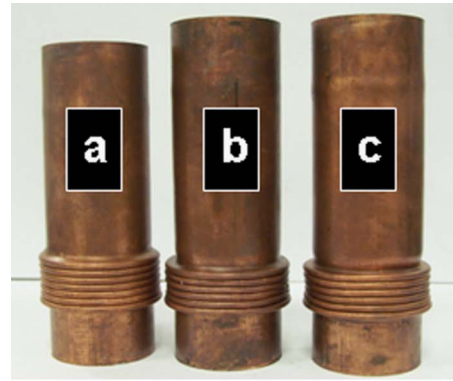
sistatic loading, the fixed-end (biaxial 0 deg) case has always a higher value than that of free-end one for these metals and their structures loaded dynamically (Table 2).

3.1.3 Energy Absorption. As far as the energy absorption is concerned, Fig. 6 illustrates a clear enhancement in dissipating the kinetic energy. Hence, enhancements in the absorbed energies for an axial deflection ($\delta=40$ mm) are about 10%, 18%, and 38% for copper, aluminum, and mild-steel shells, respectively. It is important to underline that the mild-steel shells illustrate an obvious sensitivity to the dynamic strain rates (Fig. 6(c)). However, this material does not demonstrate any sensitivity to the quasistatic strain rate [19]. Moreover, the tests performed on the copper structures show that the dynamic crushing with the biaxial 0 deg case dissipates more energy than the free-end one (Fig. 6(a)). Figures 6(b) and 6(c) present an additional absorption of energy, compared to that of quasistatic equivalent one. Thus, an increase of 40% in the energy absorption (i.e., additional energy) is clearly noticed in the case of aluminum and mild steel for an axial deflection of $\delta=14.5$ mm. As a result, the strain rate sensitivity of the material has a principal effect on the peak collapse load. Such an increase confirms the increase in the resistance of aluminum and mild steel to the plastic deformation (Figs. 5(b) and 5(c)). This resistance decreases, however, progressively in the course of the crushing process due to the decrease in the loading rate.

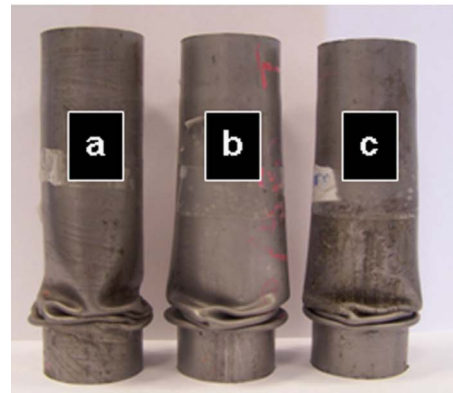
3.2 Biaxial Dynamic Plastic Buckling

3.2.1 Deformation Modes. Table 2 summarizes the dynamic mechanical behavior of the different circular shells crushed biaxially. The recorded deformation mode is almost of XM type for these shells having the geometry ($\eta=15.5$ and $\lambda=0.11$). As expected, the diamond part of the XM becomes more significant with the increase in the loading path complexity and substituting the AM systematically. It is also noticed that the biaxial 45 deg case engenders a change in deformation mode for the aluminum shells, generating therefore the DM instead of the XM. This could be due to its mechanical behavior compared to that of copper (Fig. 7). On the other hand, it is recognized that the AM takes place only in the copper tubes having $\eta=19.5$ and $\lambda=0.14$ (see Fig. 8(a)).

This shows, in a formal manner, that the geometrical parameters (in particular, η) represent predominant factors, with respect to the lateral perturbation generated by the torsional component. In fact, under biaxial compression-torsion loading, the torsional component always provokes a certain deviation related to the coaxiality between the “axial fibers” of the deformed circular shell and the loading axis. The experimental results confirmed that at the beginning of the entry of torsion in action, this practically cannot disturb the deformation mode; i.e., the deformation type is mainly controlled by the two parameters η and λ . However, with the progressive augmentation in the torsional component effect,



a



b

Fig. 8 Deformation modes under different dynamic loading complexities: (a) biaxial 45 deg, (b) biaxial 37 deg, and (c) biaxial 30 deg for specimens having $\eta=19.5$. (a) Copper. (b) Mild steel.

these parameters cannot, hereafter, be the only principal factors in determining the deformation mode. Accordingly, a competition phenomenon takes place among the geometrical parameters (η and λ), the torsional component effect, and, notably, its rate of change. Hence, the tangential disturbance becomes more significant with collapse progression especially with the increase of the loading path complexity, giving consequently a higher tangential disturbance. It is well known that when the η has a value less than 15, the AM becomes generally the predominant deformation mode for the circular shells [20]. However, under a biaxial combined load, it is observed that for the copper circular shells with $\eta=19.5$, the violation of this coaxiality requires further lateral disturbance. Thus, the deformation mode for such structures is of AM type, and it is not the case for the structures having a weaker η ($\eta=15.5$), as demonstrated in Figs. 7 and 8(a). In the mild-steel case, Fig. 8(b) and Table 2 show clearly that these shells with $\eta=19.5$ and $\lambda=0.14$ generate the DXM. This is due to the lateral perturbation effect whatever the inclination angle is, except for the angle 0 deg (biaxial 0 deg) where the deformation mode is of AM type. Note that no analysis can be given which explains why a particular mode shape is adopted by the given structures for this material. Nonetheless, it seems that the interpretation of such a structural behavior could be governed by the material behavior.

It is finally found that the mode of deformation for the employed materials under uniaxial loading is, in general, independent of the loading rate (quasistatic or dynamic). Nevertheless, the increase in the torsional component supports the appearance of the DM quite systematically, particularly in the biaxial 45 deg case.

3.2.2 Buckling Loads. The load-deflection curves of different dynamic biaxial loading situations are demonstrated in Fig. 9.

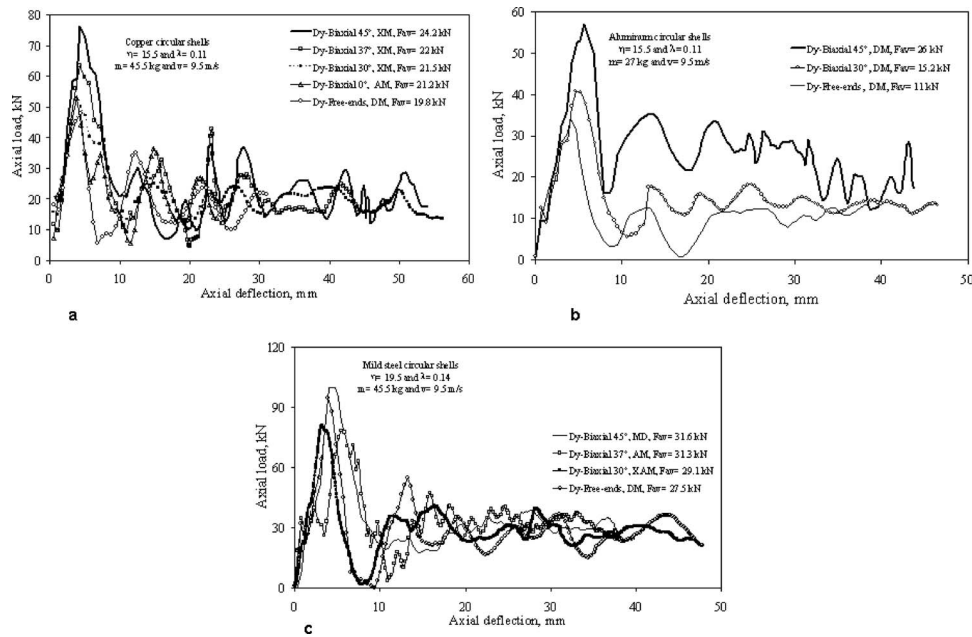


Fig. 9 Evolution of collapse loads versus the axial deflection in different biaxial dynamic loading cases for the (a) copper, (b) aluminum, and (c) mild-steel shells

Their evolutions take similar forms with respect to the same shells loaded quasistatically, given in Ref. [19]. These curves are characterized primarily by a noticeable increase in the peak collapse load (F_{max}) particularly during the first plastic flow phase compared to the uniaxial case for the three metals. In fact, depending proportionally on the loading path complexity, such loads undergo a considerable amplification (Fig. 9) even for different values of η as in the copper case. This leads, for example, to enhancements in the F_{max} of 59% and 46% for the two sections having $\eta = 15.5$ and 19.5, respectively (Table 2). Beyond this peak load, the curves are characterized by a load decrease and then a series of fluctuations about a mean postbuckling load, the peaks and troughs being directly related to the formation of buckles and folding at various buckling levels. An interaction effect between the loading rate and the emergence of the torsional component on the material behavior takes place. Then, the dynamic velocity decreases naturally and progressively with crushing continuation to join roughly the quasistatic state. For this reason and to give a maximum of objectivity to determine the mean collapse load for each material and loading condition, several calculations are performed in order to determine this value using different axial deflections varying from 20 mm to 50 mm. It is finally concluded that these mean collapse loads are determined by using an axial deflection of 40 mm. This choice represents a better compromise, as illustrated clearly in Fig. 10.

It is the biaxial 45 deg loading situation which generates always the strongest F_{av} (24.2 kN for $\eta = 15.5$ and 26.8 kN with $\eta = 19.5$) for the copper shells. Moreover, the magnitudes of the F_{av} are always more significant in the biaxial loading than in the uniaxial one (Fig. 9(a)).

On the other hand, even with a low impact velocity (at the end of the crushing test), the aluminum specimens demonstrate a remarkable sensitivity not only to the strain rate but also, in a more impressive manner, to the loading path complexity effect (Fig. 9(b)). Consequently, the F_{av} for the three given loading situations are 11 kN (free ends), 15.2 kN (biaxial 30 deg), and 26 kN (biaxial 45 deg), giving a maximum increase of 136% in the biaxial 45 deg case. As a matter of fact, one can interpret this behavior by the fact that such a material changes its local behavior (could be at the dislocation level) with respect to the strain rate coupled with

loading path complexity factor.

The crushing behavior of the mild-steel structures is described in Fig. 9(c) with four different levels of complexity. The corresponding F_{av} evolves with almost the same trend as in the case of copper shells (Table 2). Contrary to the results given in Ref. [19], under quasistatic loading, the dynamic buckling behavior of mild steel shows a rather significant sensitivity to the loading path nature. Indeed, a certain modification in the overall plastic flow mechanism occurs. Moreover, the initial plasticity peak evolution translates a certain sensitivity of this material to the loading path complexity (Table 2 and Fig. 9(c)).

In order to thoroughly understand why the F_{av} and the energy absorbed increase proportionally with the loading complexity, especially under dynamic conditions, let us analyze this effect in Fig. 11, which displays the crushing load evolution for each fold during the collapse process for the two extreme cases, the free ends and the biaxial 45 deg. The data in this figure demonstrate clearly that the mean value of each fold is generally affected by the loading path complexity.

The influence of the loading rate and its complexity on the

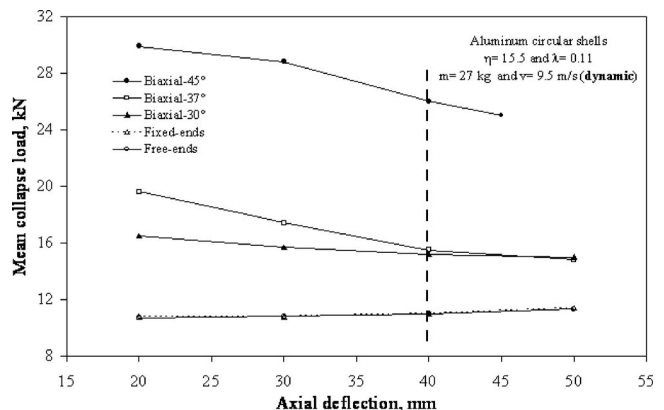


Fig. 10 Determination of the mean collapse loads with different loading complexities using different axial deflections

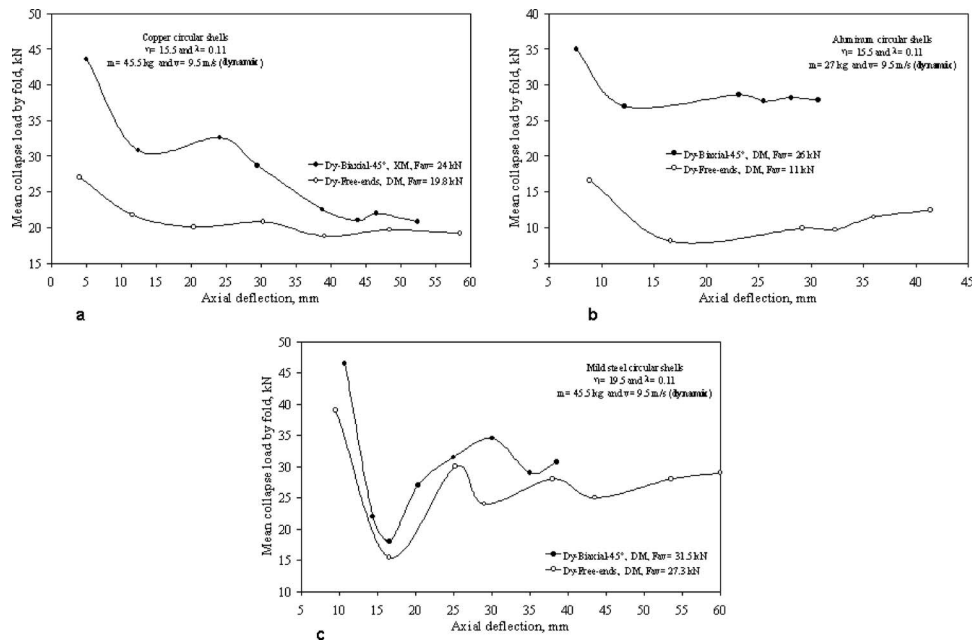


Fig. 11 Variation of the mean load for each fold occurring during shell collapse of (a) copper, (b) aluminum, and (c) mild-steel shells under two loading paths

crushing behavior is largely dependent on the used metals. For the copper shells (Fig. 11(a)), the overall plastic flow is affected by these factors. In fact, for an axial deflection of almost $\delta = 47$ mm, six buckles are observed in the free-end case against seven for the biaxial 45 deg case. It seems that the difference between these loading conditions related to the mean collapse load by fold is almost due to changes in the overall plastic flow, together with the material strength properties. On the other hand, in the case of mild-steel structures and for $\delta = 37$ mm, five buckles are recorded in the free-end case against eight for the biaxial 45 deg (Fig. 11(c)). Since the increase in the mean collapse load is not highly pronounced, one can therefore consider that this takes place al-

most via only the change in the overall plastic flow.

In this work, the most significant result is related to the aluminum shell behavior. Indeed, Fig. 11(b) illustrates that a considerable enhancement in the mean collapse load by fold is obviously induced. Actually, this increase is involved by an interaction between the loading path complexity and its rate. Note that using the same biaxial loading configuration and geometrical parameters (η and λ), but under quasistatic loading rate, this material behavior demonstrates a modest sensitivity to the loading path complexity [19]. However, under dynamic loading conditions, a change in the strength properties of the material can be provoked by local physical modifications, rather at the dislocation level, therefore leading

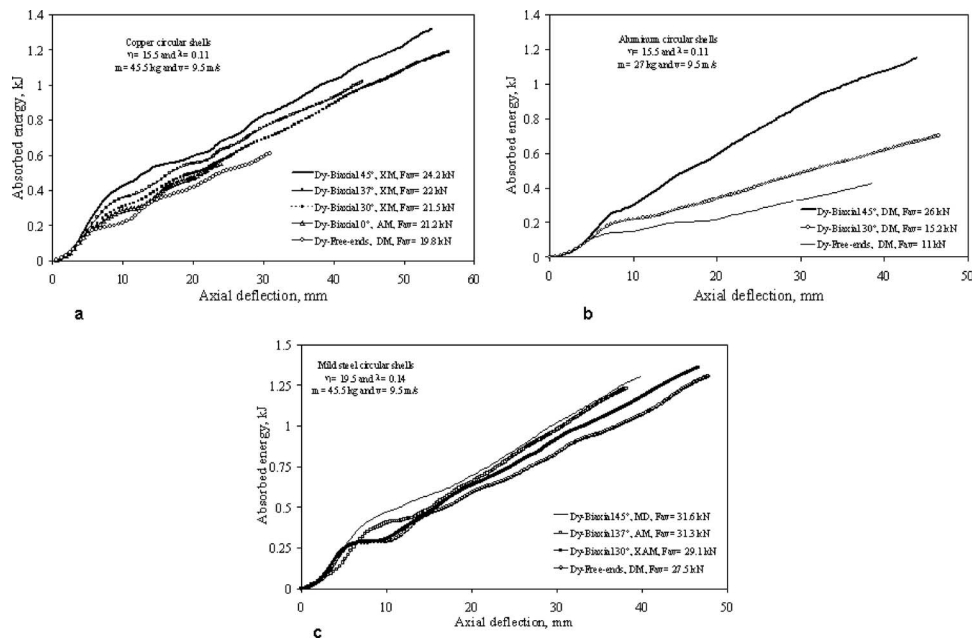


Fig. 12 Energy absorption evolution versus axial deflection under different dynamic loading complexities for the (a) copper structures, (b) aluminum, and (c) mild-steel shells

to an enhancement in the work hardening of the material. In a more precise manner, the load enhancement is mainly governed by a change in plastic collapse properties and, secondarily, by a change in the number of buckles (i.e., overall plastic flow) in the biaxial 45 deg case compared to the free-end one.

Concerning the heating effect, an increase in the temperature is evidently observed notably within the plastic hinges. Note that in this work, such an increase is not experimentally measured. Hence, the thermal effect is considered neither qualitatively nor quantitatively to estimate its effect on the collapse process.

3.2.3 Energy Absorption. The energy absorption is described in Fig. 12. Figure 12(a) demonstrates the energy absorption for the copper structures ($\eta=15.5$ and $\lambda=0.11$) under five loading paths (three biaxial loadings with different complexities and two uniaxial loadings). It reveals that after an axial displacement of approximately 10 mm, the evolution of the energy absorbed undergoes a deceleration. Likewise, in the biaxial loading cases, whatever the angle of torsion is, the absorbed energy increases remarkably for the biaxial 45 deg. Beyond a displacement of 20 mm, these evolutions become practically monotonous and similar. It is worth emphasizing that the energy absorbed, as expected, is more significant in biaxial than in uniaxial and increases proportionally with the inclination angle. For the three materials, the biaxial 45 deg has the highest capacity to absorb the energy than the other cases. In the case of aluminum, Fig. 12(b) shows a significant enhancement in the energy absorption as a result of the loading path complexity. The estimated energies for a crushing distance of $\delta=40$ mm are 0.42 kJ for the free ends, 0.62 kJ in the case of biaxial 30 deg and 1.07 kJ for the biaxial 45 deg. The increase in the absorbed energy under the biaxial 45 deg is higher than 150% in comparison with the traditional free-end buckling.

As a comparison between the quasistatic and dynamic shell behaviors, the energy absorption histograms are presented in Fig. 13 for an axial deflection of $\delta=30$ mm in the case of copper and aluminum shells. For the copper structures, the dynamic biaxial 37 deg and 45 deg loadings absorb more energy than the quasistatic ones for the same structural configuration (Fig. 13(a)). Once again, this highlights the loading rate effect on the material behavior. In addition, the energy absorption capacity for the aluminum shells (Fig. 13(b)) demonstrates a substantial improvement under dynamic biaxial loading conditions, whereas the same structures show a moderate sensitivity to the loading complexity for the same structures under quasistatic loading. Hence, it is obviously found that the maximum enhancement is higher than 160% in favor of the biaxial 45 deg load vis-à-vis a classical uniaxial case for an axial deflection of $\delta=30$ mm.

4 Concluding Remarks

The objective of this work is to enhance the strength properties of materials under dynamic loading conditions. Hence, the key point is to study the effect of the loading rate coupled with the biaxial loading condition (combined compression and torsion) on the progressive dynamic buckling response for three metallic structures (made from copper, aluminum, and mild steel) having different dimensions. It is intriguing to note that three parameters control, in general, the crushing process under biaxial loading conditions: specimen geometry (λ , η), loading complexity, and its rate. Using the original device (the ACTP), several degrees of complexity of the biaxial loading paths are created within the loaded circular shells during their collapses via three inclination angles (30 deg, 37 deg and 45 deg) with the integral biaxial loading situation. Under a dynamic loading path complexity, three different strains (compression, bending, and shear) are simultaneously applied with more complex load/unload conditions. This leads, *a priori*, to induce local physical phenomena responsible for changes in strength properties (enhancement in the work hardening), notably in the case of aluminum shells and also in the overall plastic flow (change in the number of buckles for a given

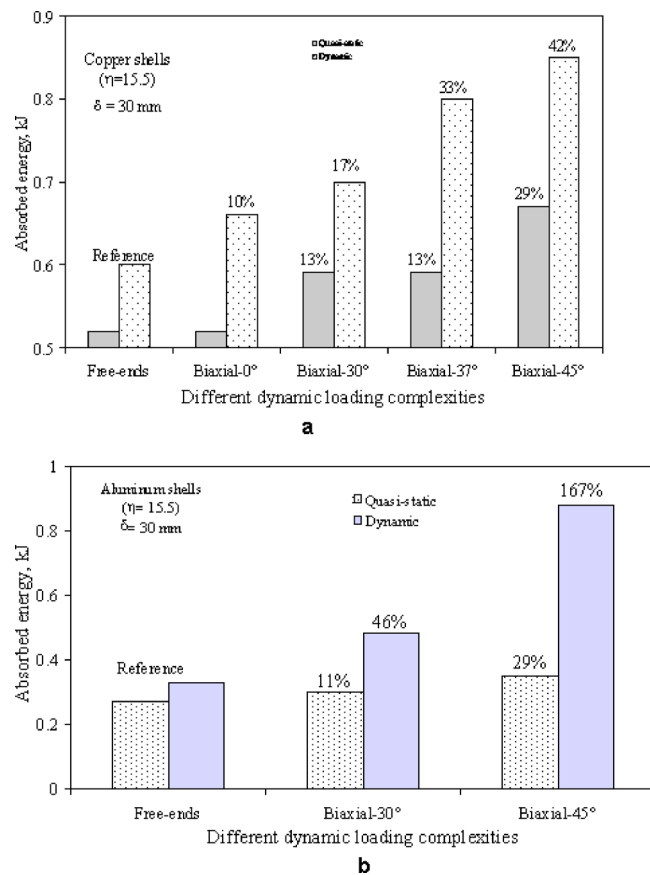


Fig. 13 Histograms of the energy absorbed under different dynamic loading complexities for an axial deflection of $\delta=30$ mm for the (a) copper and (b) aluminum structures

axial deflection). Furthermore, the mild steel, which does not illustrate a noticeable sensitivity to the quasistatic biaxial loading conditions [19], shows, however, a reasonable sensitivity to loading path complexity and its rate.

The main conclusions are

- (1) For the uniaxial case, there is a similarity of the deformation modes recorded under dynamic and quasistatic loadings.
- (2) The used materials (copper and, especially, aluminum and mild steel) show an obvious sensitivity to the dynamic loading compared to the quasistatic one.
- (3) Under biaxial loading, the higher the inclination angle and the more complex loading applied, the more severe the rates of change of torsional component and, consequently, the greater the mean collapse load and the corresponding absorbed energy. Therefore, the biaxial 45 deg is considered as the most significant case, giving the highest F_{av} as well as the energy absorbed. It is therefore recognized that the enhancement in the energy absorption for the aluminum shells is higher than 150%;
- (4) Under dynamic biaxial loading, the deformation mode for the materials is moderately sensitive to the amplitude of torsion, contrary to the quasistatic one.
- (5) Coupling of the studied parameters (geometry, type of material, loading path complexity, and its rate) with the heat effect in controlling the structure behavior needs a thorough investigation.
- (6) A microstructural study, particularly within the plastic hinges, now becomes a primordial issue in order to interpret in a coherent way the enhancement in the mean

collapse load and the absorbed energy. This makes it possible to forecast very interesting scenarios caused by the imposed experimental conditions.

Nomenclature

- $\eta = R_m/t$ = radial geometrical parameter
 $\lambda = R_m/L$ = longitudinal geometrical parameter
 R_m = mean radius of the shell
 L = initial length of the shell
 t = thickness of the shell
 d = internal diameter of the shell
 D_y = dynamic loading condition
 Q_s = quasistatic loading condition
 F_{av} = mean collapse load
 F_{max} = peak collapse load
 biaxial 30 deg,
 37 deg, and 45 deg = combined biaxial compression and torsion of the three inclination angles (30 deg, 37 deg, and 45 deg)

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Modal Properties of Planetary Gears With an Elastic Continuum Ring Gear

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The distinctive modal properties of equally spaced planetary gears with elastic ring gears are studied through perturbation and a candidate mode method. All eigenfunctions fall into one of four mode types whose structured properties are derived analytically. Two perturbations are used to obtain closed-form expressions of all the eigenfunctions. In the discrete planetary perturbation, the unperturbed system is a discrete planetary gear with a rigid ring. The stiffness of the ring is perturbed from infinite to a finite number. In the elastic ring perturbation, the unperturbed system is an elastic ring supported by the ring-planet mesh springs; the sun, planet and carrier motions are treated as small perturbations. A subsequent candidate mode method analysis proves the perturbation results and removes any reliance on perturbation parameters being small. All vibration modes are classified into rotational, translational, planet, and purely ring modes. The well defined properties of each type of mode are analytically determined. All modal properties are verified numerically. [DOI: 10.1115/1.2839892]

1 Introduction

Planetary gears are widely used in automotive and aerospace transmissions due to the advantages, such as compactness, high torque/weight ratio, low bearing load, and high transmission ratio. In practical systems where planetary gear vibration is a key concern, ring gear elastic deformation is significant. This is especially true for planetary gears with thin rims, including those used in aerospace applications. The free vibration of planetary gears with equally spaced planets has typically been studied by treating all the planetary gear components as rigid bodies [1–7]. Lin and Parker [5] established a lumped parameter model that includes both transverse and torsional motion. The modal properties were obtained analytically, and the vibration modes are classified into rotational, translational, and planet modes. In the present paper, these modes are called *discrete* rotational, translational, and planet modes. Lin and Parker used this discrete model to study natural frequency and vibration mode sensitivity [8], natural frequency veering [9], and parametric instability caused by changing contact conditions at the multiple tooth meshes [10].

This study analytically addresses the dynamics of planetary gears having elastic ring gears. An elastic-discrete model is developed, where the ring gear is modeled as an elastic body while all other gears are represented as rigid bodies. Modal properties are derived in detail using eigenvalue perturbation and a candidate mode method. Two unperturbed systems are considered to form a complete representation for all modes. This yields closed-form expressions for all the eigenfunctions and a systematic characterization of planetary gears' highly structured modal properties. All vibration modes are classified in detail into four different types according to their unique characteristics. These perturbation results are proved by a mathematically rigorous approach where vibration modes having the form revealed by perturbation are assumed and then shown to satisfy all equations of the elastic-discrete eigenvalue problem. This builds a base for subsequent

studies, such as dynamic response, parametric instability, and contact loss nonlinearity, all of which commonly use modal expansion methods.

2 Modeling and Equations of Motion

An elastic-discrete model of a planetary gear is shown in Fig. 1. All gear meshes are represented by linear springs. The sun, carrier, and planets are considered as rigid bodies, while the ring gear is modeled as a thin elastic body. The bearings and supports of the sun, carrier, and planets are modeled as two perpendicular springs of equal stiffness. The bearings and supports of the ring gear are represented as an elastic foundation with uniform radial and tangential distributed stiffnesses per unit length k_{rbs} and k_{tus} , respectively. The planets are identical and equally spaced. All ring-planet mesh stiffnesses are equal (k_{rp}), and all sun-planet mesh stiffnesses are equal (k_{sp}), where k_{rp} and k_{sp} are averages over a mesh cycle. The angular speeds are assumed to be small, so gyroscopic effects are neglected.

The coordinates are shown in Fig. 1. The deformations of the sun and carrier $\mathbf{p}_j = [x_j \ y_j \ u_j]^T$, $j=s, c$ are described relative to the fixed basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$; the tangential displacement of the ring is $u(\theta, t)$; the ring radial deflection is determined from the inextensibility condition $w = -\partial u / \partial \theta$ [11]; and the deflections of the planets are $\mathbf{p}_n = [\xi_n \ \eta_n \ u_n]^T$, $n=1, \dots, N$. The symbol u_j denotes rotational (or tangential) deflection (rotation in radians times the gear base radii r_s, r_r, r_p or radius of the carrier r_c).

The equations of motion for the sun and carrier are the same as those in the discrete model [5], while the equations of motion for the ring and planets change. The equation of motion for the elastic ring gear is [11]

$$M_e \ddot{u} + k_{\text{bend}} L_1 u + k_{rp} L_2 u + k_{rp} \sum_{n=1}^N L_3^n (\xi_n \sin \alpha_r - \eta_n \cos \alpha_r - u_n) = 0 \quad (1)$$

$$M_e = \rho R \left(1 - \frac{\partial^2}{\partial \theta^2} \right), \quad k_{\text{bend}} = \frac{EJ}{R^3(1 - \nu^2)},$$

$$L_1 = - \left(\frac{\partial^6}{\partial \theta^6} + 2 \frac{\partial^4}{\partial \theta^4} + \frac{\partial^2}{\partial \theta^2} \right)$$

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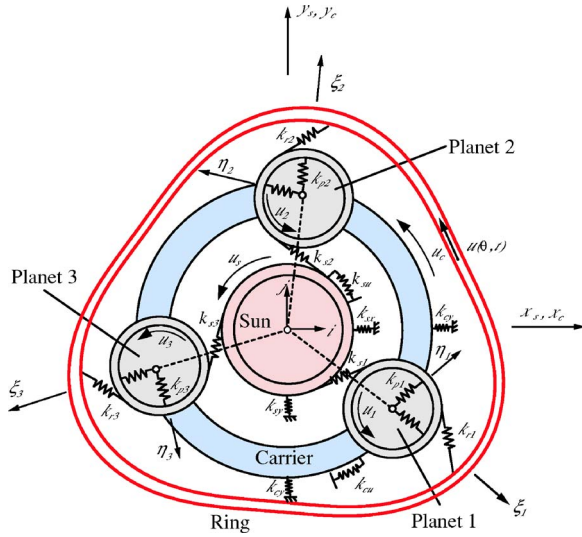


Fig. 1 Elastic-discrete model of a planetary gear and corresponding system coordinates. The distributed springs around the ring circumference are not shown.

$$L_2 = - \sum_{n=1}^N \left[\left(\sin^2 \alpha_r \frac{\partial^2}{\partial \theta^2} - \cos^2 \alpha_r \right) \delta(\theta - \psi_n) + \left(\sin \alpha_r \frac{\partial}{\partial \theta} + \cos \alpha_r \right) \sin \alpha_r \frac{\partial \delta(\theta - \psi_n)}{\partial \theta} \right] + \left(k_{\text{rus}} R - k_{\text{rbs}} R \frac{\partial^2}{\partial \theta^2} \right) / k_{\text{rp}} \quad (2)$$

$$L_3^n = \cos \alpha_r \delta(\theta - \psi_n) - \sin \alpha_r \frac{\partial \delta(\theta - \psi_n)}{\partial \theta} \quad (2)$$

where k_{bend} is the ring bending stiffness (see the Nomenclature). L_1 , L_2 , and L_3 are dimensionless operators. The first two terms of Eq. (1) represent the in-plane vibration of a free ring; the last two terms incorporate the effects of gear meshes and elastic supports.

Separation of the ring rigid body motions from the elastic deformation $v(\theta, t)$ is achieved with the expansion

$$u(\theta, t) = v(\theta, t) + U_1(t)e^{i\theta} + U_{-1}(t)e^{-i\theta} + U_0(t) = \sum_{m=\pm 2}^{\pm \infty} V_m(t)e^{im\theta} + U_1(t)e^{i\theta} + U_{-1}(t)e^{-i\theta} + U_0(t) \quad (3)$$

Thus, v is orthogonal to the rigid body motions

$$\int_0^{2\pi} v d\theta = 0, \quad \int_0^{2\pi} v e^{i\theta} d\theta = 0, \quad \int_0^{2\pi} v e^{-i\theta} d\theta = 0 \quad (4)$$

Substituting Eq. (3) into Eq. (1) and forming the inner product of the result with $e^{im\theta}$ yield the discretized equations of motion. Comparison of the equations for the rigid ring motions U_1, U_{-1}, U_0 to the equations of motion for a rigid ring planetary gear model with variables $\mathbf{p}_r = (x_r, y_r, u_r)^T$ [5] yields the relations

$$x_r = -i(U_1 - U_{-1}), \quad y_r = U_1 + U_{-1}, \quad u_r = U_0 \cos \alpha_r, \quad I_r = m_r R^2 \quad (5)$$

The true moment of inertia expression for a ring is $I_{r,\text{true}} = m_r(r_2^2 + r_1^2)/2$, where r_1 and r_2 are the inner and outer radii of the ring gear, respectively. The difference between I_r and $I_{r,\text{true}}$ is small when the ring is thin.

We introduce the following dimensionless quantities:

$$\tilde{v} = \frac{v}{R}, \quad \tau = \frac{t}{T}, \quad T = \sqrt{\frac{m_r}{k_{\text{rp}}}}, \quad \tilde{k}_i = \frac{k_i}{k_{\text{rp}}}, \quad i = r, c, s, p, \text{rp}, \text{sp}, \text{bend}, \quad \tilde{k}_{\text{rbs}} = \frac{k_{\text{rbs}} R}{k_{\text{rp}}} \quad (6)$$

$$\tilde{k}_{\text{rus}} = \frac{k_{\text{rus}} R}{k_{\text{rp}}}, \quad \tilde{m}_j = \frac{m_j}{m_r}, \quad \tilde{I}_j = \frac{I_j}{m_r R_j^2}, \quad j = r, c, s, n \quad (7)$$

In what follows, the $\sim$ on all variables is omitted, and the equations of motion remain the same except that k_{rp} is replaced by 1, M_e is replaced by $1/2\pi(1 - (\partial^2/\partial\theta^2))$, and $k_{\text{rbs}}R$, $k_{\text{rus}}R$ are replaced by k_{rbs} , k_{rus} .

The displacement of the whole system is separated into $v(\theta, \tau)$ and $\mathbf{q}(\tau)$. v is the elastic deformation of the ring gear, and $\mathbf{q}$ is a vector of the deflections for the discrete elements including the ring rigid body motions,

$$\mathbf{q} = [\underbrace{x_r, y_r, u_r}_{\mathbf{p}_r}, \underbrace{x_c, y_c, u_c}_{\mathbf{p}_c}, \underbrace{x_s, y_s, u_s}_{\mathbf{p}_s}, \underbrace{\xi_1, \eta_1, u_1}_{\mathbf{p}_1}, \dots, \underbrace{\xi_N, \eta_N, u_N}_{\mathbf{p}_N}]^T \quad (8)$$

The dimensionless equations of motion and the associated eigenvalue problem in extended operator form are

$$M\ddot{\mathbf{a}} + K\mathbf{a} = 0 \quad (9)$$

$$-\omega^2 M\mathbf{a} + K\mathbf{a} = 0 \quad (10)$$

where $\mathbf{a} = [v, \mathbf{q}^T]^T$ is referred to as an *extended variable*, ω is the natural frequency, and M , K are extended stiffness and inertia operators defined by their action on elements of the space of extended variables according to

$$M\mathbf{a} = \begin{bmatrix} (v - \partial^2 v / \partial \theta^2) / (2\pi) \\ \mathbf{M}\mathbf{q} \end{bmatrix}, \quad K\mathbf{a} = \begin{bmatrix} (k_{\text{bend}} L_1 + L_2)v + L_3\mathbf{q} \\ L_4 v + \mathbf{K}\mathbf{q} \end{bmatrix} \quad (11)$$

$$L_3\mathbf{q} = \sum_{n=1}^N \left[\cos \alpha_r \delta(\theta - \psi_n) - \sin \alpha_r \frac{\partial \delta(\theta - \psi_n)}{\partial \theta} \right] \delta_n \quad (12)$$

$$L_4 v = \left[\sum_{n=1}^N (\mathbf{b}_r^T \chi|_{\theta=\psi_n}) \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{b}_p^T \chi|_{\theta=\psi_1} \quad \dots \quad \mathbf{b}_p^T \chi|_{\theta=\psi_N} \right]^T, \quad (13)$$

$$\chi = \frac{\partial v}{\partial \theta} \sin \alpha_r + v \cos \alpha_r \quad (13)$$

$$\mathbf{b}_r = [-\sin \psi_m \quad \cos \psi_m \quad 1]^T, \quad \mathbf{b}_p = [\sin \alpha_r, \quad -\cos \alpha_r, \quad -1]^T \quad (14)$$

$$\delta_n = -x_r \sin \psi_m + y_r \cos \psi_m + u_r + \xi_n \sin \alpha_r - \eta_n \cos \alpha_r - u_n \quad (15)$$

M and K are self-adjoint with the inner product $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = \int_0^{2\pi} v_1 \bar{v}_2 d\theta + \mathbf{q}_1^T \bar{\mathbf{q}}_2$, where an overbar denotes complex conjugate. $\mathbf{M}$ and $\mathbf{K}$ (see Appendix for details) are the dimensionless mass and stiffness matrices for planetary gears based on a discrete model. Their dimensional forms are identical to the mass and stiffness matrices in Ref. [5] with the only difference in $\mathbf{M}_r$ and $\mathbf{K}_{\text{rb}}$ as

$$\mathbf{M}_r = \text{diag}(1, 1, 1/\cos^2 \alpha_r), \quad \mathbf{K}_{\text{rb}} = \pi \text{diag}(k_{\text{rbs}} + k_{\text{rus}}, k_{\text{rbs}} + k_{\text{rus}}, 2k_{\text{rus}}/\cos^2 \alpha_r) \quad (16)$$

Expansion of Eq. (10) into $N+4$ groups of equations associated with the individual components yields

Table 1 Dimensional parameters and dimensionless natural frequencies of a planetary gear with six equally spaced planets. The designations R, T, P, and PR denote rotational, translational, planet, and purely ring modes.

Inertias (kg)	$I_r/r_s^2=8.891, I_c/r_c^2=6.000, I_s/r_s^2=2.500, I_p/r_p^2=2.000$
Masses (kg)	$m_r=7.350, m_c=5.430, m_s=0.400, m_p=1.000$
Stiffnesses (N/m)	$k_{rp}=k_{sp}=10^8, k_{rbs}=k_{rus}=0, k_{bend}=5 \times 10^6, k_s=k_{su}=5 \times 10^7, k_c=k_{cu}=5 \times 10^{11}, k_p=10^9$
Pressure angle (deg)	$\alpha_r=\alpha_s=24.60$
Dimensionless natural frequencies	$\omega_1=0.1520$ (R), $\omega_{2,3}=0.1871$ (T), $\omega_{4,5}=0.6472$ (P), $\omega_6=1.0227$ (P), $\omega_{7,8}=1.0231$ (T), $\omega_9=1.1009$ (R), $\omega_{10,11}=1.1695$ (P), $\omega_{12}=1.6971$ (PR), $\omega_{13}=1.8549$ (P), $\omega_{14}=1.9161$ (R)

$$-\frac{\omega^2}{2\pi}\left(1-\frac{\partial^2}{\partial\theta^2}\right)v+k_{bend}L_1v+L_2v+L_3\mathbf{q}=0 \quad (17)$$

$$-\omega^2\mathbf{M}_r\mathbf{p}_r+\left(\mathbf{K}_{rb}+\sum_n\mathbf{K}_{r1}^n\right)\mathbf{p}_r+\sum_n\mathbf{K}_{r2}^n\mathbf{p}_n+\sum_n(\mathbf{b}_r\chi|_{\theta=\psi_n})=0 \quad (18)$$

$$-\omega^2\mathbf{M}_c\mathbf{p}_c+\left(\mathbf{K}_{cb}+\sum_n\mathbf{K}_{c1}^n\right)\mathbf{p}_c+\sum_n\mathbf{K}_{c2}^n\mathbf{p}_n=0 \quad (19)$$

$$-\omega^2\mathbf{M}_s\mathbf{p}_s+\left(\mathbf{K}_{sb}+\sum_n\mathbf{K}_{s1}^n\right)\mathbf{p}_s+\sum_n\mathbf{K}_{s2}^n\mathbf{p}_n=0 \quad (20)$$

$$-\omega^2\mathbf{M}_p\mathbf{p}_n+(\mathbf{K}_{c2}^n)^T\mathbf{p}_c+(\mathbf{K}_{r2}^n)^T\mathbf{p}_r+(\mathbf{K}_{s2}^n)^T\mathbf{p}_s+\mathbf{K}_{pp}\mathbf{p}_n+\mathbf{b}_p\chi|_{\theta=\psi_n}=0, \quad n=1, \dots, N \quad (21)$$

Equation (10) is cast entirely in discrete form with modal expansion of v as

$$v(\theta, \tau)=\sum_{m=\pm 2}^{\pm JN} V_m(\tau)e^{im\theta} \quad (22)$$

where $J \geq 1$ is an integer. The basis functions $e^{im\theta}$ are complete, Eq. (22) converges, and J is arbitrarily large. Thus, the error in Eq. (22) can be made as small as desired. No restriction is put on J in what follows, so the findings apply to the continuum ring model without any limitation introduced by the expansion (22).

A discretized model results from substitution of Eq. (22) into Eqs. (17)–(21) and then forming the inner product of Eqs. (17), (18), and (21) with $e^{ip\theta}$. Numerical experiments on the discretized equations confirm that ring elastic deformation alters the natural frequencies and vibration modes compared to the lumped parameter model and introduces additional natural frequencies associated with modes dominated by ring elastic deformation. The numerical solutions indicate that all vibration modes of this elastic-discrete model are classified into four types: rotational, translational, planet, and purely ring modes.

For example, a planetary gear with six equally spaced planets is analyzed with $J=3$ in Eq. (22). The system parameters and the dimensionless natural frequencies are listed in Table 1. The natural frequencies in Table 1 include all four mode types: ω_1, ω_9 , and ω_{14} are for rotational modes; $\omega_{2,3}$ and $\omega_{7,8}$ are for translational modes; $\omega_{4,5}$ and $\omega_{10,11}$ are for degenerate planet modes (type 2) and ω_6, ω_{13} are for distinct planet modes (type 3); ω_{12} is for a purely ring mode.

Figure 2(a) shows the vibration mode of a rotational mode (ω_1). From the numerical simulations, a rotational mode has the following characteristics: (a) The discrete elements $\mathbf{q}$ have the same properties as a discrete rotational mode, where the translations of the sun, carrier, and ring rigid motion are zero, and all planets have identical deflections; (b) the associated natural frequency is distinct; (c) the elastic deformation of the ring contains only $jN, j=1, 2, \dots, J$ nodal diameter components.

Figure 2(b) shows the vibration mode of a translational mode ($\omega_{2,3}$). A translational mode has the following characteristics: (a) The discrete elements $\mathbf{q}$ have the same properties as a discrete translational mode, where the rotations of the sun, carrier, and ring rigid motion are zero, and the deflections of the planets are related by a rotation matrix; (b) the associated natural frequency is repeated with multiplicity 2; (c) the elastic deformation of the ring contains only $jN \pm 1$ nodal diameter components, where j is any nonzero integer satisfying $jN \pm 1 \in \{-JN, -JN+1, \dots, JN\}$ (a condition imposed by the $\pm JN$ limits in Eq. (22)).

Planet modes are classified into two subtypes according to the degeneracy of the natural frequencies. For odd N , all planet modes are degenerate, as are the majority of planet modes for even N . Degenerate planet modes have the following characteristics: (a) The discrete elements $\mathbf{q}$ have the same properties as a discrete planet mode, where the deflections of the sun, carrier, and ring rigid motion are zero, and the deflections of the planets are scalar multiples of the first planet's deflection; (b) the associated natural frequency is repeated with multiplicity 2; (c) each mode is associated with a particular $s \in \{2, 3, \dots, \text{int}((N-1)/2)\}$. For that particular s , the elastic deformation of the ring contains only $jN \pm s$ nodal diameter components, where j is any integer satisfying $jN \pm s \in \{-JN, -JN+1, \dots, JN\}$. Figure 2(c) shows a degenerate planet mode ($\omega_{4,5}$) where the two nodal diameter component is the dominant ring deformation. For even N , the remaining planet modes have distinct natural frequencies. Their discrete elements behave as in (a) above, but their elastic ring deflection contains only $jN+N/2$ nodal diameter components, where j is any integer satisfying $jN \pm N/2 \in \{-JN, \dots, JN\}$ (see Fig. 2(d) for a distinct planet mode).

Thus, planet modes are classified into $\text{int}(N/2)-1$ subtypes according to the ring nodal diameter components they contain. Planet modes having $jN \pm s$ nodal diameter components are named type s planet modes. Each planet mode belongs to a unique type. For the example where $N=6$, two types exist: the degenerate planet modes are type 2 ($s=2, \dots, \text{int}((N-1)/2)$), and the distinct planet modes are type 3 ($s=N/2$), which only exist for even N . There are no planet modes outside of these two types for $N=6$. Table 2 summarizes the number of degenerate/distinct planet modes and their types for varying numbers of planets.

Figure 2(e) shows a purely ring mode (ω_{12}). A purely ring mode has the following characteristics: (a) The discrete elements $\mathbf{q}$ are all zero; (b) the natural frequency is distinct; (c) the elastic deformation of the ring contains only a single nodal diameter component.

In this example ($N=6, J=3$), $3N+2JN+7=61$ eigensolutions are obtained numerically: $J+6=9$ rotational modes, $4J+10=22$ translational modes, $(2JN-7J)+(3N-9)=24$ planet modes divided as $2J+3=9$ degenerate pairs and $J+3=6$ distinct modes, and $2J=6$ purely ring modes.

The remainder of this paper analytically proves that these properties (natural frequency multiplicity, modal properties, and the number of each type of mode) hold for general planetary gears.

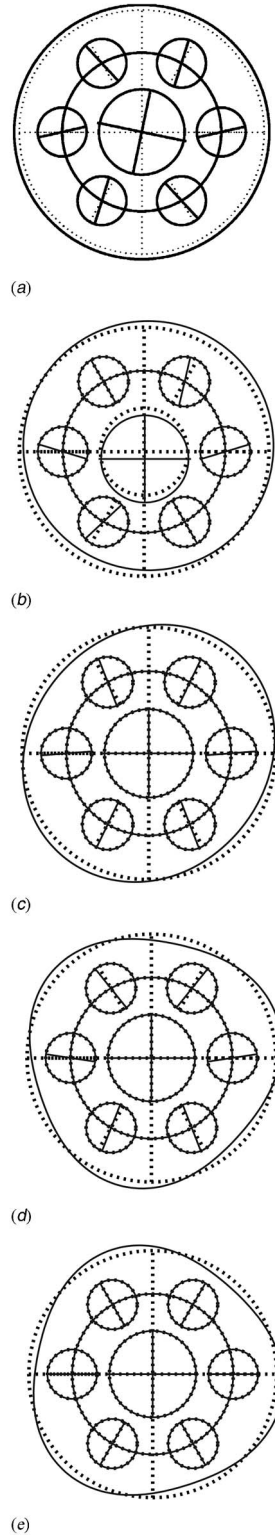


Fig. 2 Typical modes of a planetary gear. The system parameters are given in Table 1. Distinct planet modes as in (d) only exist for an even number of planets.

3 Perturbation Analysis

To find all natural frequencies and vibration modes of the elastic-discrete model of a planetary gear, two perturbations are

used for different ranges of parameters. For the chosen nondimensional variables, the stiffness of the ring-planet mesh is always unity ($k_{rp} \equiv 1$). The first perturbation is termed *discrete planetary perturbation* (DPP), with the unperturbed system being a discrete planetary gear having a nearly rigid ring gear where the bending stiffness is $O(1/\varepsilon)$ while the stiffnesses of all remaining meshes/supports are $O(1)$. The small quantity ε is the ring bending compliance. The opposite case of DPP is *elastic ring perturbation* (ERP). In this case, the bending stiffness is $O(1)$ and the stiffnesses of the remaining meshes/supports (except $k_{rp} \equiv 1$) are $O(1/\varepsilon)$. The unperturbed system for the ERP is an elastic ring having multiple springs with the elimination of the rigid body motions. The attached springs represent the ring-planet gear meshes. The combined eigensolutions from the DPP and ERP form a complete set of eigensolutions for planetary gears having elastic rings without any redundancy (as proved in a subsequent candidate mode method solution). This process leads to analytical results that mathematically expose the system's highly structured modal properties.

3.1 Discrete Planetary Perturbation. In DPP, the ring bending stiffness is much larger than the mesh and bearing stiffnesses. The mesh and bearing stiffnesses are $O(1)$, while the ring bending stiffness $k_{\text{bend}} = 1/\varepsilon$, where ε is a small parameter. The eigenvalue problem in extended operator form is

$$-\omega^2 \mathbf{M} \mathbf{a} + \hat{\mathbf{K}} \mathbf{a} = \begin{bmatrix} -\omega^2(v - \partial^2 v / \partial \theta^2) / (2\pi) \\ -\omega^2 \mathbf{M} \mathbf{q} \end{bmatrix} + \begin{bmatrix} L_1 v / \varepsilon \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} L_2 v + L_3 \mathbf{q} \\ L_4 v + \mathbf{K} \mathbf{q} \end{bmatrix} = \mathbf{0} \quad (23)$$

where $\mathbf{M}$ and $\hat{\mathbf{K}}$ are self-adjoint operators. The eigensolutions of Eq. (23) are represented as

$$\mathbf{a} = \mathbf{a}^0 + \varepsilon \mathbf{a}^1 + O(\varepsilon^2), \quad \omega^2 = \omega_0^2 + \varepsilon \mu + O(\varepsilon^2),$$

$$\mathbf{a}^0 = \begin{bmatrix} v^0 \\ \mathbf{q}^0 \end{bmatrix}, \quad \mathbf{a}^1 = \begin{bmatrix} v^1 \\ \mathbf{q}^1 \end{bmatrix} \quad (24)$$

Substitution of Eq. (24) into Eq. (23) gives the perturbation equations. The perturbation equation of order ε^{-1} is $L_1 v^0 = 0$. L_1 is positive definite, giving

$$v^0 = 0 \quad (25)$$

Substitution of Eq. (25) into the remaining perturbation equations yields

$$-\omega_0^2 \mathbf{M} \mathbf{q}^0 + \mathbf{K} \mathbf{q}^0 = \mathbf{0} \quad (26)$$

$$L_1 v^1 = -L_3 \mathbf{q}^0, \quad -\omega_0^2 \mathbf{M} \mathbf{q}^1 + \mathbf{K} \mathbf{q}^1 = \mu \mathbf{M} \mathbf{q}^0 - L_4 v^1 \quad (27)$$

Equation (26) is the eigenvalue problem for a discrete (rigid ring) planetary gear model [5]. From (25) and (26), the unperturbed eigenfunction is

$$\mathbf{a}^0 = \begin{bmatrix} 0 \\ \mathbf{q}^0 \end{bmatrix} \quad (28)$$

The structured properties of the discrete model unperturbed eigensolutions are proven analytically in Ref. [5], where the discrete system vibration modes $\mathbf{q}^0$ are classified into rotational, translational, and planet modes. In this study, they are called *discrete* rotational, translational, and planet modes. In the elastic-discrete model, similar mode types are found; they are called rotational, translational, and planet modes. The different mode types are considered separately.

Common to each mode type, v^1 is solved from the first equation of Eq. (27) by expanding v^1 as $v^1 = \sum_{m=\pm 2}^{\pm N} V_m^1 e^{im\theta}$, multiplying Eq. (27) by $e^{-im\theta}$, and integrating from 0 to 2π . This yields

Table 2 Number of planet modes in different subtypes for different number of planets N , where $\times$ denotes *not applicable*

Planet mode category	Number of planets, N						
	4	5	6	7	8	9	10
Distinct planet modes	$J+3$	0	$J+3$	0	$J+3$	0	$J+3$
Degenerate planet modes	0	$2(2J+3)$	$2(2J+3)$	$4(2J+3)$	$4(2J+3)$	$6(2J+3)$	$6(2J+3)$
Type 2 planet modes	$J+3$	$2(2J+3)$	$2(2J+3)$	$2(2J+3)$	$2(2J+3)$	$2(2J+3)$	$2(2J+3)$
Type 3 planet modes	$\times$	$\times$	$J+3$	$2(2J+3)$	$2(2J+3)$	$2(2J+3)$	$2(2J+3)$
Type 4 planet modes	$\times$	$\times$	$\times$	$\times$	$J+3$	$2(2J+3)$	$2(2J+3)$
Type 5 planet modes	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$J+3$

$$V_m^1 = -\frac{\cos \alpha_r - im \sin \alpha_r}{2\pi m^2(m^2 - 1)^2} \sum_{n=1}^N \delta_n^0 e^{-im\psi_n} \quad (29)$$

where δ_n^0 is the n th ring-planet mesh deflection without considering the elastic deformation of the ring gear, as given by Eq. (15).

According to Eq. (29), $V_{-m}^1 = \bar{V}_m^1$.

3.1.1 Rotational Modes. When the unperturbed eigenfunction $\mathbf{q}^0$ from Eq. (26) is a discrete rotational mode, the translational motions of the sun, carrier, and ring are zero and all the planets have the same deflections [5]

$$\mathbf{q}^0 = [0 \quad 0 \quad u_r^0 \quad 0 \quad 0 \quad u_c^0 \quad 0 \quad 0 \quad u_s^0 \quad \xi_1^0 \quad \eta_1^0 \quad u_1^0 \quad \dots \quad \xi_1^0 \quad \eta_1^0 \quad u_1^0]^T \quad (30)$$

In the absence of any rigid constraints on any degrees of freedom (e.g., fixed carrier rotation), six such modes exist, each having a distinct natural frequency. Application of these properties to Eq. (29) yields

$$V_m^1 = -\frac{\cos \alpha_r - im \sin \alpha_r}{2\pi m^2(m^2 - 1)^2} \delta_{r1}^0 \sum_{n=1}^N e^{-im\psi_n}, \quad (31)$$

$$\delta_{r1}^0 = u_r^0 + \xi_1^0 \sin \alpha_r - \eta_1^0 \cos \alpha_r - u_1^0$$

Because the planets are equally spaced with $\psi_n = 2\pi(n-1)/N$, the identity $\sum_{n=1}^N e^{-im\psi_n} = 0$ holds for $m \neq jN$, where j is an arbitrary nonzero integer. Thus, for $\mathbf{q}^0$ being a discrete rotational mode, the elastic deformation of the ring in the perturbed system contains only the jN nodal diameter components

$$V_m^1 = -\frac{\cos \alpha_r - im \sin \alpha_r}{2\pi m^2(m^2 - 1)^2} N \delta_{r1}^0, \quad m = \pm N, \dots, \pm jN \quad (32)$$

The eigenvalue perturbation μ is determined by the solvability condition of the second of Eq. (27) as (with $\langle \mathbf{M}\mathbf{q}^0, \mathbf{q}^0 \rangle = 1$)

$$\mu = \langle L_4 v^1, \mathbf{q}^0 \rangle = -\frac{N^2(\delta_{r1}^0)^2 \sum_{m=jN}^{\pm jN} \gamma_m}{\pi}, \quad \gamma_m = \frac{\cos^2 \alpha_r + m^2 \sin^2 \alpha_r}{m^2(m^2 - 1)^2} \quad (33)$$

A candidate solution of the second of Eq. (27) is proposed as

$$\mathbf{q}^1 = [0 \quad 0 \quad u_r^1 \quad 0 \quad 0 \quad u_c^1 \quad 0 \quad 0 \quad u_s^1 \quad \xi_1^1 \quad \eta_1^1 \quad u_1^1 \quad \dots \quad \xi_1^1 \quad \eta_1^1 \quad u_1^1]^T \quad (34)$$

Note that $\mathbf{q}^1$ has the same form as $\mathbf{q}^0$. Use of Eq. (34) and the known discrete rotational mode properties reduces Eq. (27) to

$$(2\pi k_{rus}/\cos^2 \alpha_r + N - \omega_0^2 I_r) u_r^1 + N \delta_{r1}^1 = \mu I_r u_r^0 - \frac{N}{\pi} \delta_{r1}^0 \sum_{m=jN}^{\pm jN} \gamma_m \quad (35)$$

$$(k_{cu} + Nk_p - \omega_0^2 I_c) u_c^1 - Nk_p \eta_1^1 = \mu I_c u_c^0 \quad (36)$$

$$(k_{su} + Nk_{sp} - \omega_0^2 I_s) u_s^1 - Nk_{sp} (u_1^1 - \xi_1^1 \sin \alpha_s + \eta_1^1 \cos \alpha_s) = \mu I_s u_s^0 \quad (37)$$

$$N[(\mathbf{K}_{c2}^1)^T \mathbf{p}_c^1 + (\mathbf{K}_{r2}^1)^T \mathbf{p}_r^1 + (\mathbf{K}_{s2}^1)^T \mathbf{p}_s^1 + (\mathbf{K}_{pp} - \omega_0^2 \mathbf{M}_p) \mathbf{p}_1^1] = \mu N \mathbf{M}_p \mathbf{p}_1^0 - \mathbf{b}_p \frac{N}{\pi} \sum_{m=jN}^{\pm jN} \gamma_m \quad (38)$$

Expressing Eqs. (35)–(38) in matrix form yields the 6×6 linear system,

$$\mathbf{A}_{\text{rot}} \mathbf{p}_{\text{rot}}^1 = \mathbf{b}_{\text{rot}} \quad (39)$$

$$\mathbf{p}_{\text{rot}}^1 = [u_r^1 \quad u_c^1 \quad u_s^1 \quad \xi_1^1 \quad \eta_1^1 \quad u_1^1]^T \quad (40)$$

$$\mathbf{b}_{\text{rot}} = \mu \mathbf{M}_{\text{rot}} \mathbf{p}_{\text{rot}}^0 - \frac{N}{\pi} \delta_{r1}^0 \sum_{m=jN}^{\pm jN} \gamma_m [1 \quad 0 \quad 0 \quad \mathbf{b}_p^T]^T,$$

$$\mathbf{M}_{\text{rot}} = \text{diag}(I_r, I_c, I_s, N\mathbf{M}_p) \quad (41)$$

One can show that the solvability condition of Eq. (39) is identical to Eq. (33), so it is satisfied. This guarantees that the solution of the second of Eq. (27) has the assumed form Eq. (34). The normalization condition $\langle \mathbf{q}^1, \mathbf{M}\mathbf{q}^0 \rangle = 0$ becomes $\langle \mathbf{p}_{\text{rot}}^1, \mathbf{M}_{\text{rot}} \mathbf{p}_{\text{rot}}^0 \rangle = 0$ in this problem. This and Eq. (39) yield

$$\begin{bmatrix} \mathbf{A}_{\text{rot}} \\ (\mathbf{M}_{\text{rot}} \mathbf{p}_{\text{rot}}^0)^T \end{bmatrix} \mathbf{p}_{\text{rot}}^1 = \begin{bmatrix} \mathbf{b}_{\text{rot}} \\ 0 \end{bmatrix} \Rightarrow \hat{\mathbf{A}}_{\text{rot}} \mathbf{p}_{\text{rot}}^1 = \hat{\mathbf{b}}_{\text{rot}} \quad (42)$$

The solution of Eq. (42) is $\mathbf{p}_{\text{rot}}^1 = (\hat{\mathbf{A}}_{\text{rot}}^T \hat{\mathbf{A}}_{\text{rot}})^{-1} \hat{\mathbf{A}}_{\text{rot}}^T \hat{\mathbf{b}}_{\text{rot}}$. This completes the solution for $\mathbf{q}^1$ in Eq. (27).

Collecting results, we have six eigenfunctions $\mathbf{a}$ in Eq. (24) with the form

$$\mathbf{a} = \begin{bmatrix} \sum_{m=jN}^{\pm jN} V_m^1 e^{im\theta} \\ \mathbf{q}^0 + \varepsilon \mathbf{q}^1 \end{bmatrix} \quad (43)$$

The discrete elements of the planetary gear (including the ring rigid body motion) deflect as in the discrete rotational modes described in Ref. [5]. The elastic ring deflection contains only the jN nodal diameter components. The natural frequencies of these modes are distinct.

3.1.2 Translational Modes. When the unperturbed eigenfunction from Eq. (26) is a discrete translational mode, the eigenvalues are repeated with multiplicity 2 and the rotational motions of the carrier, sun and ring are zero [5]. The pair of degenerate vibration modes $\hat{\mathbf{q}}^0$ and $\hat{\mathbf{q}}^0$ satisfy

$$\hat{\mathbf{q}}^0 = [\hat{\mathbf{p}}_r^0 \quad \hat{\mathbf{p}}_c^0 \quad \hat{\mathbf{p}}_s^0 \quad \hat{\mathbf{p}}_1^0 \quad \dots \quad \hat{\mathbf{p}}_N^0]^T, \quad \hat{\mathbf{q}}^0 = [\hat{\mathbf{p}}_r^0 \quad \hat{\mathbf{p}}_c^0 \quad \hat{\mathbf{p}}_s^0 \quad \hat{\mathbf{p}}_1^0 \quad \dots \quad \hat{\mathbf{p}}_N^0]^T \quad (44)$$

$$\hat{\mathbf{p}}_j^0 = [x_j^0 \ y_j^0 \ 0]^T, \quad \hat{\mathbf{p}}_j^0 = [y_j^0 \ -x_j^0 \ 0]^T, \quad j = r, c, s \quad (45)$$

When the planets are located at $\psi_n = 2\pi(n-1)/N$, the n th planet displacements $\hat{\mathbf{p}}_n^0$, $\hat{\mathbf{p}}_n^0$ are related as

$$\begin{bmatrix} \hat{\mathbf{p}}_n^0 \\ \hat{\mathbf{p}}_n^0 \end{bmatrix} = \begin{bmatrix} \cos \psi_n \mathbf{I} & \sin \psi_n \mathbf{I} \\ -\sin \psi_n \mathbf{I} & \cos \psi_n \mathbf{I} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{p}}_1^0 \\ \hat{\mathbf{p}}_1^0 \end{bmatrix}, \quad n = 1, 2, \dots, N \quad (46)$$

where $\mathbf{I}$ is a 3×3 identity matrix. Six such eigensolution pairs exist.

The degenerate unperturbed eigenvalue ω_0^2 of multiplicity 2 in Eq. (24) has two orthonormal, unperturbed eigenfunctions $\hat{\mathbf{a}}^0$ and $\hat{\mathbf{a}}^0$ of the extended operator form (10). As a consequence, the unperturbed eigenfunction $\mathbf{a}^0$ is a linear combination of $\hat{\mathbf{a}}^0$ and $\hat{\mathbf{a}}^0$,

$$\mathbf{a}^0 = c_1 \hat{\mathbf{a}}^0 + c_2 \hat{\mathbf{a}}^0, \quad \hat{\mathbf{a}}^0 = \begin{bmatrix} 0 \\ \hat{\mathbf{q}}^0 \end{bmatrix}, \quad \hat{\mathbf{a}}^0 = \begin{bmatrix} 0 \\ \hat{\mathbf{q}}^0 \end{bmatrix} \quad (47)$$

where c_1 and c_2 are constants. Analogous to the procedure for the rotational mode, use of the discrete translational mode properties reduces Eq. (29) to

$$V_m = -\frac{\cos \alpha_r - im \sin \alpha_r}{4\pi m^2(m^2 - 1)^2} \left[\sum_{n=1}^N e^{-i(m-1)\psi_n} + \bar{v} \sum_{n=1}^N e^{-i(m+1)\psi_n} \right],$$

$$v = (c_1 + ic_2)(A_1 - iA_2)$$

$$A_1 = y_r^0 \cos \alpha_r - x_r^0 \sin \alpha_r + \hat{\xi}_1^0 \sin \alpha_r - \hat{\eta}_1^0 \cos \alpha_r - \hat{u}_1^0$$

$$A_2 = -y_r^0 \sin \alpha_r - x_r^0 \cos \alpha_r + \hat{\xi}_1^0 \sin \alpha_r - \hat{\eta}_1^0 \cos \alpha_r - \hat{u}_1^0 \quad (48)$$

where $\bar{v}$ is the complex conjugate of v . $\sum_{n=1}^N e^{-i(m-1)\psi_n}$ being zero requires $m \neq jN+1$, where j is an arbitrary integer; $\sum_{n=1}^N e^{-i(m+1)\psi_n}$ being zero requires $m \neq jN-1$. Thus, V_m^1 vanishes if and only if $m \neq jN \pm 1$. This yields the following rule: The elastic deformation of the ring for elastic translational modes contains only $jN \pm 1$ nodal diameter components.

The solvability conditions of the second equation of Eq. (27) form a 2×2 algebraic eigenvalue problem $\mathbf{D}_r \mathbf{c} = \mu \mathbf{c}$, where $\mathbf{c} = (c_1, c_2)^T$. $\mathbf{D}_r$ is diagonal with the repeated eigenvalues

$$\mu_1 = \mu_2 = -\frac{N^2(A_1^2 + A_2^2)}{4\pi} \sum_{m=jN+1} \gamma_m \quad (49)$$

where here (and in all subsequent summations) j is an integer such that m takes only values within the range specified in Eq. (22), i.e., $-jN \leq m \leq jN$ and $m \neq -1, 0, 1$. Thus, the eigenvalues for the elastic ring model remain degenerate and c_1, c_2 are indeterminate.

The eigenfunction perturbation is proposed as $\mathbf{q}^1 = c_1 \hat{\mathbf{q}}^1 + c_2 \hat{\mathbf{q}}^1$, where $\hat{\mathbf{q}}^1$ and $\hat{\mathbf{q}}^1$ are a pair of vectors having the same properties (44)–(46) as the discrete translational modes. Substitution of $\mathbf{q}^1$ into Eq. (27) yields a set of simplified equations that, if satisfied, ensures Eq. (27) is satisfied for any c_1 and c_2 . The perturbation equations from Eq. (27) for the sun, carrier, and ring rigid motion reduce to the six equations (50)–(55), and the perturbation equations for all the planets reduce to Eqs. (56) and (57)

$$\begin{aligned} & \left(\pi k_{rbs} + \pi k_{rus} + \frac{N}{2} - \omega_0^2 m_r \right) x_r^1 - \frac{N}{2} (\hat{\sigma}_r^1 \sin \alpha_r + \hat{\sigma}_r^1 \cos \alpha_r) \\ & = \mu m_r x_r^0 + \beta_1 \end{aligned} \quad (50)$$

$$\begin{aligned} & \left(\pi k_{rbs} + \pi k_{rus} + \frac{N}{2} - \omega_0^2 m_r \right) y_r^1 + \frac{N}{2} (\hat{\sigma}_r^1 \cos \alpha_r - \hat{\sigma}_r^1 \sin \alpha_r) \\ & = \mu m_r y_r^0 + \beta_2 \end{aligned} \quad (51)$$

$$(k_c + Nk_{pn} - \omega_0^2 m_c) x_c^1 + \frac{N}{2} k_p (-\hat{\xi}_1^1 + \hat{\eta}_1^1) = \mu m_c x_c^0 \quad (52)$$

$$(k_c + Nk_{pn} - \omega_0^2 m_c) y_c^1 + \frac{N}{2} k_p (-\hat{\xi}_1^1 - \hat{\eta}_1^1) = \mu m_c y_c^0 \quad (53)$$

$$\left(k_s + \frac{N}{2} k_{sp} - \omega_0^2 m_s \right) x_s^1 + \frac{N}{2} k_{sp} (-\hat{\sigma}_s^1 \sin \alpha_s + \hat{\sigma}_s^1 \cos \alpha_s) = \mu m_s x_s^0 \quad (54)$$

$$\left(k_s + \frac{N}{2} k_{sp} - \omega_0^2 m_s \right) y_s^1 + \frac{N}{2} k_{sp} (-\hat{\sigma}_s^1 \cos \alpha_s - \hat{\sigma}_s^1 \sin \alpha_s) = \mu m_s y_s^0 \quad (55)$$

$$k_{pn} \hat{\mathbf{p}}_c^1 + \mathbf{K}_{r4}^1 \hat{\mathbf{p}}_r^1 + \mathbf{K}_{s4}^1 \hat{\mathbf{p}}_s^1 + (\mathbf{K}_{pp} - \omega_0^2 \mathbf{M}_p) \hat{\mathbf{p}}_1^1 = \mu \mathbf{M}_p \hat{\mathbf{p}}_1^0 + \beta_3 \mathbf{b}_p \quad (56)$$

$$k_{pn} \hat{\mathbf{p}}_c^1 + \mathbf{K}_{r4}^1 \hat{\mathbf{p}}_r^1 + \mathbf{K}_{s4}^1 \hat{\mathbf{p}}_s^1 + (\mathbf{K}_{pp} - \omega_0^2 \mathbf{M}_p) \hat{\mathbf{p}}_1^1 = \mu \mathbf{M}_p \hat{\mathbf{p}}_1^0 + \beta_4 \mathbf{b}_p \quad (57)$$

where

$$\hat{\sigma}_r^1 = \hat{\xi}_1^1 \sin \alpha_r - \hat{\eta}_1^1 \cos \alpha_r - \hat{u}_1^1, \quad \hat{\sigma}_r^1 = \hat{\xi}_1^1 \sin \alpha_r - \hat{\eta}_1^1 \cos \alpha_r - \hat{u}_1^1 \quad (58)$$

$$\hat{\sigma}_s^1 = -\hat{\xi}_1^1 \sin \alpha_s - \hat{\eta}_1^1 \cos \alpha_s + \hat{u}_1^1, \quad \hat{\sigma}_s^1 = -\hat{\xi}_1^1 \sin \alpha_s - \hat{\eta}_1^1 \cos \alpha_s + \hat{u}_1^1 \quad (59)$$

$$\beta_1 = \frac{-N^2(A_1 \sin \alpha_r + A_2 \cos \alpha_r)}{4\pi} \sum_{m=jN+1} \gamma_m,$$

$$\beta_2 = \frac{N^2(A_1 \cos \alpha_r - A_2 \sin \alpha_r)}{4\pi} \sum_{m=jN+1} \gamma_m \quad (60)$$

$$\beta_3 = \frac{NA_1}{2\pi} \sum_{m=jN+1} \gamma_m, \quad \beta_4 = \frac{NA_2}{2\pi} \sum_{m=jN+1} \gamma_m \quad (61)$$

$$\mathbf{K}_{r4}^1 = \begin{bmatrix} -\sin^2 \alpha_r & \sin \alpha_r \cos \alpha_r & 0 \\ \sin \alpha_r \cos \alpha_r & \cos^2 \alpha_r & 0 \\ \sin \alpha_r & \cos \alpha_r & 0 \end{bmatrix},$$

$$\mathbf{K}_{s4}^1 = k_{sp} \begin{bmatrix} -\sin^2 \alpha_s & -\sin \alpha_s \cos \alpha_s & 0 \\ -\sin \alpha_s \cos \alpha_s & -\cos^2 \alpha_s & 0 \\ \sin \alpha_s & \cos \alpha_s & 0 \end{bmatrix}$$

$\hat{\sigma}_r^1$ and $\hat{\sigma}_s^1$ are the deflections of the first planet in the direction of the lines of action for the ring-planet and sun-planet meshes, respectively. The superscript 1 denotes the first-order perturbation.

Expressing Eqs. (50)–(57) in matrix form after multiplying Eqs. (56) and (57) by $N/2$, the second equation of Eq. (27) reduces to the 12×12 linear system,

$$\mathbf{A}_{tm} \mathbf{p}_{tm}^1 = \mathbf{b}_{tm} \quad (62)$$

$$\mathbf{p}_{tm}^1 = [x_r^1 \ y_r^1 \ x_c^1 \ y_c^1 \ x_s^1 \ y_s^1 \ \hat{\xi}_1^1 \ \hat{\eta}_1^1 \ \hat{u}_1^1 \ \hat{\xi}_1^1 \ \hat{\eta}_1^1 \ \hat{u}_1^1]^T \quad (63)$$

$$\mathbf{b}_{tm} = \mu \mathbf{M}_{tm} \mathbf{p}_{tm}^0 + [\beta_1 \ \beta_2 \ 0 \ 0 \ 0 \ 0 \ \beta_3 \mathbf{b}_p^T \ \beta_4 \mathbf{b}_p^T]^T \quad (64)$$

$$\mathbf{M}_{tm} = \text{diag} \left(1, 1, m_c, m_c, m_s, m_s, \frac{N}{2} \mathbf{M}_p, \frac{N}{2} \mathbf{M}_p \right) \quad (65)$$

One can show that the solvability condition of Eq. (62) is identical to Eq. (49), so it is satisfied. Thus, Eq. (27) is satisfied for the given $\mathbf{q}^1$ independent of c_1 and c_2 (which remain indeterminate),

and the perturbation $\mathbf{q}^1$ has the same form as $\mathbf{q}^0$.

In summary, there are six degenerate pairs of eigenfunctions $\mathbf{a}$ in Eq. (24) with the form

$$\mathbf{a} = \begin{bmatrix} \varepsilon \sum_{m=jN+1} V_m^1 e^{im\theta} + \text{c.c.} \\ \mathbf{q}^0 + \varepsilon \mathbf{q}^1 \end{bmatrix} = \begin{bmatrix} \varepsilon \sum_{m=jN \pm 1} V_m^1 e^{im\theta} \\ \mathbf{q}^0 + \varepsilon \mathbf{q}^1 \end{bmatrix} \quad (66)$$

Note that terms associated with $m=jN-1$ in Eq. (66) are the complex conjugate terms for $m=jN+1$. The discrete elements of the planetary gear (including the ring rigid body motion) deflect as in the translational modes described in Ref. [5]. The elastic ring deflection contains only the $jN \pm 1$ nodal diameter components. The natural frequencies of these modes are degenerate.

3.1.3 Planet Modes. For $N \geq 4$, the unperturbed system has three unperturbed eigenvalues associated with the discrete planet modes, and each of them is degenerate with multiplicity $N-3$. For these modes, the sun, carrier, and rigid ring motions are zero. The deflections of the planets are proportional with $\mathbf{p}_n^0 = w_n^l \mathbf{p}_1^0$, where the $N-3$ sets of coefficients satisfy [5]

$$\sum_{n=1}^N w_n^l = 0, \quad \sum_{n=1}^N w_n^l \cos \psi_n = 0, \quad \sum_{n=1}^N w_n^l \sin \psi_n = 0, \quad l = 1, \dots, N-3 \quad (67)$$

When N is odd, the $N-3$ solutions of Eq. (67) are

$$w_n^{2s-3} = \cos s\psi_n, \quad w_n^{2s-2} = \sin s\psi_n, \quad s = 2, \dots, \frac{N-1}{2} \quad (68)$$

When N is even, the $N-3$ solutions of Eq. (67) consist of Eq. (68) for $s=2, \dots, \text{int}(N-1/2)$ and the additional solution

$$w_n^{N-3} = \cos \frac{N}{2} \psi_n \quad (69)$$

A general discrete planet mode of the unperturbed system is the linear combination

$$\mathbf{q}^0 = \sum_{l=1}^{N-3} d_l \mathbf{q}_l^0$$

with

$$\mathbf{q}_l^0 = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad w_1^l \mathbf{p}_1^0 \quad \dots \quad w_{N-3}^l \mathbf{p}_1^0]^T \quad (70)$$

With this mode, reduction of Eq. (29) yields the elastic deformation of the ring as

$$V_m^1 = -\sigma_r^0 \frac{\cos \alpha_r - im \sin \alpha_r}{2\pi m^2(m^2-1)^2} \sum_{l=1}^{N-3} \left(d_l \sum_{n=1}^N w_n^l e^{-im\psi_n} \right), \quad \sigma_r^0 = \xi_1^0 \sin \alpha_r - \eta_1^0 \cos \alpha_r - u_1^0 \quad (71)$$

The $N-3$ solvability conditions for the second of Eq. (27) give

$$\mathbf{D}_p \mathbf{d} = \mu \mathbf{d}, \quad \mathbf{d} = [d_1 \quad d_2 \quad \dots \quad d_{N-3}]^T \quad (72)$$

$$\mathbf{D}_p = [D_{ij}]_{(N-3) \times (N-3)} = -\frac{(\sigma_r^0)^2}{2\pi} \left[\sum_{m=\pm 2}^{\pm jN} \gamma_m \left(\sum_{n=1}^N w_n^l e^{-im\psi_n} \right) \left(\sum_{n=1}^N w_n^j e^{im\psi_n} \right) \right]_{(N-3) \times (N-3)} \quad (73)$$

Although the elements D_{ij} of $\mathbf{D}_p$ appear complicated, use of the solutions (68) and (69) simplifies them. $\mathbf{D}_p$ is diagonal, yielding closed-form expressions for μ . When N is odd, the first-order eigenvalue perturbations are

$$\mu_{2s-3} = \mu_{2s-2} = -\frac{N^2(\sigma_r^0)^2}{4\pi} \sum_{m=jN+s} \gamma_m, \quad s = 2, \dots, \frac{N-1}{2} \quad (74)$$

When N is even, Eq. (74) holds for $s=2, \dots, (N/2)-1$, and the remaining eigenvalue perturbation is

$$\mu_{N-3} = -\frac{N^2(\sigma_r^0)^2}{4\pi} \sum_{m=jN+(N/2)} \gamma_m \quad (75)$$

For each of the three unperturbed discrete planet modes with multiplicity $N-3$, the corresponding perturbed eigenfunctions evolve into $\text{int}((N-3)/2)$ pairs of degenerate planet modes for arbitrary N and one additional distinct planet mode for even N .

For degenerate planet modes with natural frequency perturbation from Eq. (74), the unperturbed eigenfunction is a linear combination of two instead of $N-3$ modes in Eq. (70). According to this and Eq. (68), Eq. (71) reduces to

$$V_m^1 = -\sigma_r^0 \frac{\cos \alpha_r - im \sin \alpha_r}{2\pi m^2(m^2-1)^2} \sum_{n=1}^N (d_{2s-3} \cos s\psi_n e^{-im\psi_n} + d_{2s-2} \sin s\psi_n e^{-im\psi_n}) \quad (76)$$

V_m^1 is zero when $m \neq jN \pm s$. This yields a rule governing the nodal diameter components of the ring modal deflections for a mode with given $s \in \{2, \dots, \text{int}((N-1)/2)\}$: The elastic deformation of the ring for degenerate planet modes contains only $jN \pm s$ nodal diameter components. The nonzero nodal diameter components are

$$V_m^1 = -N\sigma_r^0 \frac{\cos \alpha_r - im \sin \alpha_r}{4\pi m^2(m^2-1)^2} (d_{2s-3} - id_{2s-2}), \quad m = jN + s \quad (77)$$

$$V_m^1 = -N\sigma_r^0 \frac{\cos \alpha_r - im \sin \alpha_r}{4\pi m^2(m^2-1)^2} (d_{2s-3} + id_{2s-2}), \quad m = jN - s \quad (78)$$

For distinct planet modes whose natural frequency is $\omega^2 = \omega_0^2 + \varepsilon \mu_{N/2}$ (exist only for even N), Eq. (71) reduces to

$$V_m^1 = -\sigma_r^0 \frac{\cos \alpha_r - im \sin \alpha_r}{2\pi m^2(m^2-1)^2} \sum_{n=1}^N \cos\left(\frac{N}{2} \psi_n\right) e^{-im\psi_n} = -N\sigma_r^0 \frac{\cos \alpha_r - im \sin \alpha_r}{4\pi m^2(m^2-1)^2} \quad \text{for all } m = jN \pm \frac{N}{2} \quad (79)$$

Terms in the first expression for V_m^1 in Eq. (79) vanish for $m \neq jN \pm (N/2)$. Accordingly, the perturbed eigenfunction contains only $jN \pm (N/2)$ nodal diameter components ($s=N/2$).

For the degenerate eigensolution $\mu_{2s-3} = \mu_{2s-2}$ with specified $s \in \{2, 3, \dots, \text{int}((N-1)/2)\}$, the eigenfunction perturbation $\mathbf{q}^1$ is proposed as the linear combination:

$$\mathbf{q}^1 = d_{2s-3} \hat{\mathbf{q}}^1 + d_{2s-2} \hat{\mathbf{q}}^1 \quad (80)$$

$$\hat{\mathbf{q}}^1 = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad z_1^1(\mathbf{p}_1^1)^T \quad \dots \quad z_N^1(\mathbf{p}_1^1)^T]^T,$$

$$\hat{\mathbf{q}}^1 = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad z_1^2(\mathbf{p}_1^1)^T \quad \dots \quad z_N^2(\mathbf{p}_1^1)^T]^T \quad (81)$$

where $\hat{\mathbf{q}}^1$ and $\hat{\mathbf{q}}^1$ have the same form as the discrete planet modes in Eq. (70). Substituting this form of $\mathbf{q}^1$ into Eq. (27), the equations associated with the sun, carrier, and ring rigid motions lead to three equations identical to Eq. (67) except $w_n^l \rightarrow z_n^l$, but here $l=1, 2$. The solutions for z_n^1, z_n^2 are

$$z_n^1 = \cos s\psi_n, \quad z_n^2 = \sin s\psi_n, \quad n = 1, 2, \dots, N \quad (82)$$

The remaining equations of Eq. (27) (the ones associated with deflections of the planets) yield

$$(\mathbf{K}_{pp} - \omega_0^2 \mathbf{M}_p) \mathbf{p}_1^1 = \mu_{2s-3} \mathbf{M}_p \mathbf{p}_1^0 + \frac{N\sigma_r}{2\pi} \mathbf{b}_p \sum_{m=jN+s} \gamma_m \quad (83)$$

One can show that the solvability condition of Eq. (83) is identical to Eqs. (74) and (75). Thus, $\mathbf{p}_1^1$ is solved from Eq. (83), which, with Eq. (82), completes the solution for $\mathbf{q}^1$. This ensures $\mathbf{q}^1$ has the structure of a discrete planet mode.

For the distinct eigenvalue μ_{N-3} in Eq. (75), one can similarly show that the eigenfunction perturbation $\mathbf{q}^1$ has the form of a discrete planet mode.

In summary, for each $s \in \{2, 3, \dots, \text{int}((N-1)/2)\}$ there are three degenerate pairs of eigenfunctions $\mathbf{a}_s$ in Eq. (24) with the form

$$\mathbf{a}_s = \begin{bmatrix} \varepsilon \sum_{m=jN+s} V_m^1 e^{im\theta} + \text{c.c.} \\ \mathbf{q}^0 + \varepsilon \mathbf{q}^1 \end{bmatrix} \quad (84)$$

For even N , an additional three distinct eigenfunctions are present with the form of Eq. (84) and $s=N/2$. The discrete elements of the planetary gear (including the ring rigid body motion) deflect as in the planet modes described in Ref. [5]. The elastic ring deflection contains only the $jN \pm s$ nodal diameter components.

3.2 Elastic Ring Perturbation. ERP is the complementary case of DPP. The stiffness of the ring-planet mesh is unity in both cases (from Eq. (6)). In ERP, the ring bending stiffness is $O(1)$, while in DPP it is $O(1/\varepsilon)$; the stiffnesses of all the remaining meshes/bearings are $O(1/\varepsilon)$, while in DPP they are $O(1)$. The perturbation parameter is defined by $\varepsilon = 1/k_{sp}$. A perturbation process similar to Eqs. (24)–(27) yields the perturbation equations for $\mathbf{a}^0$

$$-\omega_0^2(1 + \partial^2/\partial\theta^2)v^0/(2\pi) + k_{\text{bend}}L_1v^0 + L_2v^0 + L_3\mathbf{q}^0 = 0 \quad (85)$$

$$\mathbf{K}_{rb}\mathbf{p}_r^0 = \mathbf{0}, \quad \mathbf{K}_{cb}\mathbf{p}_c^0 + \sum_n \mathbf{K}_{c2}\mathbf{p}_n^0 = \mathbf{0} \quad (86)$$

$$\left(\mathbf{K}_{sb} + \sum_n \mathbf{K}_{s1}^n \right) \mathbf{p}_s^0 + \sum_n \mathbf{K}_{s2}^n \mathbf{p}_n^0 = \mathbf{0},$$

$$(\mathbf{K}_{c2}^n)^T \mathbf{p}_c^0 + (\mathbf{K}_{s2}^n)^T \mathbf{p}_s^0 + \mathbf{K}_{pp}\mathbf{p}_n^0 = \mathbf{0} \quad (87)$$

Equations (86) and (87) form a problem as $\mathbf{A}\mathbf{q}^0 = \mathbf{0}$. One can prove that $\mathbf{A}$ is positive definite so $\mathbf{q}^0 = \mathbf{0}$. Accordingly, the last item in Eq. (85) vanishes, so the unperturbed system is an elastic ring having equally spaced spring supports with elimination of the three rigid body motions as indicated in Eq. (4). The unperturbed eigenfunction is

$$\mathbf{a}_0 = \begin{bmatrix} v^0 \\ \mathbf{0} \end{bmatrix}^T \quad (88)$$

Equations (28) and (88) are the unperturbed eigenfunctions from DPP and ERP, respectively. Together they form a nonoverlapping, complete (in the mathematical sense) basis for the linear space of extended variables $\mathbf{a} = [v, \mathbf{q}^T]^T$. This suggests that the set of perturbed eigenfunctions from DPP and ERP forms a complete set of vibration modes for planetary gears having elastic ring gears. This conclusion is made rigorous subsequently.

The perturbation equations for $\mathbf{a}^1$ are

$$\mathbf{K}_{rb}\mathbf{p}_r^1 = - \sum_n \mathbf{b}_r \left(\frac{\partial v^0}{\partial \theta} \sin \alpha_r + v^0 \cos \alpha_r \right) \bigg|_{\theta=\psi_n} \quad (89)$$

$$\mathbf{K}_{cb}\mathbf{p}_c^1 + \sum_n \mathbf{K}_{c2}^n \mathbf{p}_n^1 = \mathbf{0}, \quad \left(\mathbf{K}_{sb} + \sum_n \mathbf{K}_{s1}^n \right) \mathbf{p}_s^1 + \sum_n \mathbf{K}_{s2}^n \mathbf{p}_n^1 = \mathbf{0} \quad (90)$$

$$(\mathbf{K}_{c2}^n)^T \mathbf{p}_c^1 + (\mathbf{K}_{s2}^n)^T \mathbf{p}_s^1 + \mathbf{K}_{pp}\mathbf{p}_n^1 = - \mathbf{b}_p \left(\frac{\partial v^0}{\partial \theta} \sin \alpha_r + v^0 \cos \alpha_r \right) \bigg|_{\theta=\psi_n} \quad (91)$$

$$-\frac{\omega_0^2}{2\pi} \left(1 + \frac{\partial^2}{\partial \theta^2} \right) v^1 + k_{\text{bend}}L_1v^1 + L_2v^1 = \frac{\mu}{2\pi} \left(v^0 + \frac{\partial^2 v^0}{\partial \theta^2} \right) - L_3\mathbf{q}^1 \quad (92)$$

We draw on the modal properties of a ring on a general elastic foundation as determined analytically in Ref. [11], where the modal expressions for rings having equally spaced springs are given. In the unperturbed problem (85), each spring is oriented with an angle of $\pi/2 - \alpha_r$ to the radial direction. With elimination of the ring rigid body motions, the ring deflection is represented as Eq. (22). Thus, $2JN-2$ unperturbed modes exist. For a free ring with no supports, all the natural frequencies are degenerate with multiplicity two. When the ring has equally spaced springs, some natural frequencies split and the others remain degenerate. The unperturbed modes of the ERP are classified into four types based on the nodal diameter components they contain: Type 0, Type 1, Type s , and single nodal diameter component modes [11].

For brevity, only Type 0 modes are considered. They are linear combinations of the jN nodal diameter components, $v^0 = \sum_{j=1}^J V_{jN}^d \cos jN\theta$. Such a mode exists for each of the J values of $d=N, 2N, \dots, JN$, where d indicates the dominant nodal diameter component. Substitution of this expression for v^0 into Eq. (89) yields

$$\mathbf{K}_{rb}\mathbf{p}_r^1 = \begin{bmatrix} 0 & 0 & -N \cos \alpha_r \sum_{m=jN}^{j=1, \dots, J} V_m^d \end{bmatrix}^T \quad (93)$$

Because $\mathbf{K}_{rb}$ is diagonal, the first two elements of $\mathbf{p}_r^1$ corresponding to ring rigid translations are zero, which is the same as for a discrete rotational mode. Similar analysis of Eqs. (90) and (91) for the sun, carrier, and planets reveals that $\mathbf{q}^1$ has the form (30) of a discrete rotational mode. The eigenvalue perturbation μ is obtained from the solvability condition of Eq. (92). Following lengthy algebra, the solution v^1 of Eq. (92) has the same form as v^0 . These results show that the perturbed eigenfunction has the properties of a rotational mode as defined earlier.

Similar processes show that when the unperturbed mode v^0 is of Type 1 from Ref. [11], the perturbed mode of the elastic-discrete model is a translational mode. When the unperturbed mode v^0 is of Type s from Ref. [11], the perturbed mode is a planet mode. When the unperturbed mode is a single nodal diameter component mode, the perturbed mode is a purely ring mode.

Thus, every unperturbed ERP mode evolves into one of the four modal categories of the elastic-discrete model. The same is true for DPP. The numbers of modes obtained from each of DPP and ERP are $3N+9$ and $2JN-2$, respectively. The total number of eigenfunctions obtained from the perturbation analyses is $3N + 2JN + 7$, which equals the number of degrees of freedom for arbitrary J in Eq. (22). The modal property classification from perturbation analysis exactly matches the properties of the numerical results in Fig. 2, and Tables 1 and 2. Evidently, all modes have been included and categorized from the two perturbations.

4 Candidate Mode Method

The foregoing perturbation analysis derives the modal properties by combining two perturbation problems, each having a different perturbation parameter and unperturbed problem. The method appeals to physical reasoning where the elastic-discrete system modes are seen to evolve from known simpler systems. A plausible argument given above heuristically concludes that this approach captures all modes of the general system. Nevertheless, perturbation is inherently linked to small values of the perturbation parameter, and the use of two separate perturbation problems

to conclude that all modes are accounted for is not rigorous mathematically. Guided by the foregoing perturbation results, this section derives the general elastic-discrete system modal properties in a rigorous way that is free from any reliance on a small parameter. This alternate derivation assumes eigensolutions having the properties of the four mode types from perturbation and then confirms that such eigensolutions satisfy the eigenvalue problem. An accounting at the end ensures this approach captures all possible vibration modes.

A candidate rotational mode has the ring deflection

$$v_{\text{rot}} = \sum_{j=1}^J V_{jN} \cos jN\theta \quad (94)$$

and discrete element deflection $\mathbf{q}_{\text{rot}}$ having the form (30). Substituting Eq. (94) into Eq. (17), multiplying by $\cos l\theta$, and integrating from 0 to 2π yield

$$\begin{aligned} -\frac{1+l^2}{2}\omega^2 V_l + \frac{c_l}{2}V_l + N \cos^2 \alpha_r \sum_{j=1}^J V_{jN} + N \sigma_r \cos \alpha_r &= 0, \\ l = N, 2N, \dots, JN & \\ c_l = 2\pi k_{\text{bend}} l^2 (l^2 - 1)^2 + 2\pi k_{\text{rus}} + 2\pi l^2 k_{\text{rbs}}, & \end{aligned} \quad (95)$$

$$\sigma_r = \xi_1 \sin \alpha_r - \eta_1 \cos \alpha_r - u_1 \quad (96)$$

Use of the assumed modal properties to reduce Eq. (18) yields only one equation for the ring rigid motion,

$$(2\pi k_{\text{rus}}/\cos^2 \alpha_r + N - \omega^2/\cos^2 \alpha_r)u_r + N \sigma_r + \cos \alpha_r \sum_{j=1}^J V_{jN} = 0 \quad (97)$$

The remaining equations in Eq. (18) vanish. Similarly, Eqs. (19) and (20) reduce to

$$(k_{\text{cu}} + Nk_p - \omega^2 I_c)u_c - Nk_p \eta_1 = 0 \quad (98)$$

$$(k_{\text{su}} + Nk_{\text{sp}} - \omega^2 I_s)u_s + Nk_{\text{sp}}(-\xi_1 \sin \alpha_s - \eta_1 \cos \alpha_s + u_1) = 0 \quad (99)$$

With the assumed modal form and algebraic manipulation, Eq. (21) becomes

$$\begin{aligned} (\mathbf{K}_{c2}^1)^T \mathbf{p}_c + (\mathbf{K}_{r2}^1)^T \mathbf{p}_r + (\mathbf{K}_{s2}^1)^T \mathbf{p}_s + (\mathbf{K}_{\text{pp}} - \omega^2 \mathbf{M}_p) \mathbf{p}_1 \\ + \mathbf{b}_p \cos \alpha_r \sum_{j=1}^J V_{jN} = \mathbf{0} \end{aligned} \quad (100)$$

Equations (95)–(100) form a reduced eigenvalue problem of order $J+6$ with the eigenvector $(V_N, \dots, V_{JN}, u_r, u_c, u_s, \xi_1, \eta_1, u_1)^T$. In general, the eigenvalues are all distinct (except for especially chosen parameters). From the eigenvectors of the reduced eigenvalue problem, $J+6$ rotational modes of the full system are constructed from Eq. (94) and $\mathbf{q}_{\text{rot}}$.

A pair of candidate translational modes is

$$\hat{\mathbf{a}} = \left[\sum_{m=jN+1} V_m e^{im\theta} + \text{c.c.}, \hat{\mathbf{q}}_{\text{trn}}^T \right]^T, \quad \hat{\mathbf{a}} = \left[\sum_{m=jN+1} iV_m e^{im\theta} + \text{c.c.}, \hat{\mathbf{q}}_{\text{trn}}^T \right]^T \quad (101)$$

where $\hat{\mathbf{q}}_{\text{trn}}, \hat{\mathbf{q}}_{\text{trn}}^T$ are a pair of discrete translational modes having the same form, as described in Eqs. (44)–(46). (Recall the note below Eq. (49) regarding allowable values of m .) Guided by the perturbation solution in Eq. (48), V_m is expressed as $V_m = (\cos \alpha_r - im \sin \alpha_r)U_m$, where U_m is complex.

Substituting $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}$ into Eq. (17), multiplying by $e^{-il\theta}$, and integrating from 0 to 2π yield the equations governing U_m . When $l=jN+1$, there are $2J-1$ equations,

$$\begin{aligned} -(1+l^2)\omega^2 U_l + c_l U_l + \frac{N}{2}(A_1 - iA_2) \\ + N \sum_{m=jN+1} (\cos^2 \alpha_r + m^2 \sin^2 \alpha_r)U_m = 0 \end{aligned} \quad (102)$$

where A_1 and A_2 have the form in Eq. (48), and c_l is defined in Eq. (96). When $l=jN-1$, the following $2J-1$ equations result

$$\begin{aligned} -(1+l^2)\omega^2 U_l + c_l U_l + \frac{N}{2}(A_1 + iA_2) \\ + N \sum_{m=jN-1} (\cos^2 \alpha_r + m^2 \sin^2 \alpha_r)U_m = 0 \end{aligned} \quad (103)$$

For other values of l , the resulting equations from Eq. (17) vanish. For each l in Eq. (102), there is a corresponding $-l$ in Eq. (103) whose equation is the complex conjugate of Eq. (102). Thus, Eqs. (103) and (102) are equivalent. One obtains $4J-2$ real equations because U_l in Eq. (102) is complex. Substitution of $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}$ into Eq. (18)–(21) generates an additional 12 real equations similar to Eqs. (50)–(57) with the elimination of superscripts 0 or 1, substitution of $\mu=0$, and replacement of $\beta_1, \beta_2, \beta_3, \beta_4$ by $\beta_5, \beta_6, \beta_7, \beta_8$, respectively,

$$\begin{bmatrix} \beta_5 \\ \beta_6 \end{bmatrix} = \frac{Ne^{-i\alpha_r}}{2} \sum_{m=jN+1} (\cos^2 \alpha_r + m^2 \sin^2 \alpha_r)U_m \cdot \begin{bmatrix} i \\ -1 \end{bmatrix} + \text{c.c.} \quad (104)$$

$$\begin{bmatrix} \beta_7 \\ \beta_8 \end{bmatrix} = \sum_{m=jN+1} (\cos^2 \alpha_r + m^2 \sin^2 \alpha_r)U_m \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} + \text{c.c.} \quad (105)$$

The resulting $4J+10$ real equations form a reduced order eigenvalue problem. Because $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}$ are interchangeable, all eigensolutions of the reduced order problem must occur as degenerate eigenvalues with multiplicity 2. With these eigensolutions, $2J+5$ pairs of degenerate translational modes are constructed from Eq. (101).

A pair of candidate planet modes for a selected $s \in \{2, 3, \dots, \text{int}((N-1)/2)\}$ is

$$\mathbf{a}_{s1} = \left[\sum_{m=jN+s} V_m e^{im\theta} + \text{c.c.}, \hat{\mathbf{q}}_{\text{plt},s}^T \right]^T \quad (106)$$

$$\mathbf{a}_{s2} = \left[\sum_{m=jN+s} iV_m e^{im\theta} + \text{c.c.}, \hat{\mathbf{q}}_{\text{plt},s}^T \right]^T \quad (107)$$

$$\hat{\mathbf{q}}_{\text{plt},s}^T = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \cos s\psi_1 \mathbf{p}_1^T \quad \dots \quad \cos s\psi_N \mathbf{p}_1^T],$$

$$\hat{\mathbf{q}}_{\text{plt},s}^T = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \sin s\psi_1 \mathbf{p}_1^T \quad \dots \quad \sin s\psi_N \mathbf{p}_1^T] \quad (108)$$

where $\hat{\mathbf{q}}_{\text{plt},s}, \hat{\mathbf{q}}_{\text{plt},s}^T$ are a pair of discrete planet modes having the same form, as described in Eq. (70). The linear combination $d_{2s-3}\mathbf{a}_{s1} + d_{2s-2}\mathbf{a}_{s2}$ gives the elastic deformation of the ring in the form

$$v = \sum_{m=jN+s} V_m (d_{2s-3} + id_{2s-2})e^{im\theta} + \text{c.c.} \quad (109)$$

Comparing Eq. (109) to the perturbation solution (77) and (78) suggests that the V_m in Eqs. (106) and (107) can be written as $V_m = (\cos \alpha_r - im \sin \alpha_r)U_m$ with real U_m and $U_{-m} = U_m$. This is adopted in the candidate modes (106) and (107).

Substituting $\mathbf{a}_{s1}$ into Eq. (17), multiplying by $e^{-il\theta}$, and integrating from 0 to 2π yield the equations for U_m . When $l=jN+s$, there are $2J$ equations,

$$-(1+l^2)\omega^2 U_l + c_l U_l + \frac{N}{2}\sigma_r + N \sum_{m=jN+s} (\cos^2 \alpha_r + m^2 \sin^2 \alpha_r) U_m = 0 \quad (110)$$

where σ_r is defined in Eq. (71). When $l=jN-s$, $2J$ equations result that are equivalent to Eq. (110). For other values of l , the resulting equations vanish. With the properties of $\mathbf{a}_{s1}$, Eqs. (18)–(20) are satisfied. Substitution of $\mathbf{a}_{s1}$ into Eq. (21) yields the same equation for each n ,

$$(\mathbf{K}_{pp} - \omega^2 \mathbf{M}_p) \mathbf{p}_1 = -2 \sum_{m=jN+s} (\cos^2 \alpha_r + m^2 \sin^2 \alpha_r) U_m \quad (111)$$

The resulting $2J+3$ equations in Eqs. (110) and (111) form a reduced order eigenvalue problem with $2J+3$ eigensolutions. Substitution of $\mathbf{a}_{s2}$ into Eq. (10) yields the same $2J+3$ order eigenvalue problem. Therefore, each of the $2J+3$ eigensolutions corresponds to a pair of planet modes. Thus, for each s , $2J+3$ pairs of degenerate modes are constructed from Eqs. (106)–(108). When N is odd, there are $(N-3)/2$ different values of $s \in \{2, 3, \dots, N-1/2\}$, so $(N-3)(2J+3)/2$ degenerate pairs of planet modes are constructed from Eqs. (106)–(108). When N is even, there are $N/2-2$ different values of $s \in \{2, 3, \dots, N/2-1\}$, so $(N/2-2) \times (2J+3)$ degenerate pairs of planet modes are similarly constructed.

When N is even, besides the degenerate planet modes, there are additional distinct planet modes. They have the same form as the degenerate planet mode in Eq. (106) with $s=N/2$. With some algebraic manipulation of Eq. (106), the distinct planet modes have the form

$$\mathbf{a} = \left[\sum_{m=jN+(N/2)}^{j=0, \dots, J-1} V_m \cos m\theta \quad \mathbf{q}_{\text{plt}, N/2}^T \right]^T,$$

$$\mathbf{q}_{\text{plt}, N/2}^T = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{p}_1^T \quad -\mathbf{p}_1^T \quad \dots \quad \mathbf{p}_1^T \quad -\mathbf{p}_1^T] \quad (112)$$

where V_m is real. A similar reduction as above yields a $J+3$ order eigenvalue problem for the V_m and $\mathbf{p}_1$ with eigenvalue ω^2 from Eqs. (17) and (21). This gives $J+3$ planet modes with distinct eigenvalues from Eq. (112). Totally, for even N , there are $(2JN-7J)+(3N-9)$ planet modes constructed from Eqs. (106)–(108) and (112). Table 2 summarizes the different numbers and types of planet modes.

The final mode type is that of purely ring modes having the (not normalized) form

$$\mathbf{a} = [(\cos \alpha_r \sin m\theta - m \sin \alpha_r \cos m\theta) V_m \quad \mathbf{0}]^T, \quad m = \begin{cases} jN, & j = 1, \dots, J \quad \text{for odd or even } N \\ jN + N/2, & j = 0, \dots, J-1 \quad \text{for even } N \end{cases} \quad (113)$$

In such modes, only the ring gear deforms, and the ring has nodes at all ring-planet mesh locations. All purely ring mode natural frequencies are distinct. Note that a purely ring mode with $m=jN$ in Eq. (113) has the same structure as a rotational mode except that many elements of the rotational mode are zero. Similarly, a purely ring mode with $m=jN+N/2$ in Eq. (113) has the same structure as a distinct planet mode. Rotational modes and purely ring modes with $m=jN$ emerge as split modes of the degenerate eigensolution pairs of a free ring; distinct planet modes and purely ring modes with $m=jN+N/2$ similarly emerge as split modes for even N [11].

Substitution of Eq. (113) into Eq. (17), multiplication by $e^{-il\theta}$, and integration from 0 to 2π yield the following J order diagonal eigenvalue problem for odd N ($2J$ order for even N) with eigenvalue ω^2 ,

$$[-(1+l^2)\omega^2 + c_l] V_l = 0,$$

$$l = \begin{cases} jN, & j = 1, \dots, J \quad \text{for odd or even } N \\ jN + N/2, & j = 0, \dots, J-1 \quad \text{for even } N \end{cases} \quad (114)$$

where c_l is from Eq. (96). The remaining equations (18)–(21) vanish for the $\mathbf{a}$ in Eq. (113). According to Eq. (114), the closed-form natural frequencies expressions are $\omega^2 = c_l / (1+l^2)$, where c_l depends on the ring bending stiffness (k_{bend}) and the distributed stiffnesses around the ring circumference (k_{rus} and k_{rbs}). Thus, the natural frequencies of purely ring modes are independent of mesh stiffnesses (k_{sp} and k_{rp}). This can also be explained through the gear mesh deflections. The general expressions of sun-planet and ring-planet mesh deflections (δ_{sn} and δ_{rn}) are

$$\delta_{sn} = y_s \cos \psi_{sn} - x_s \sin \psi_{sn} - \xi_n \sin \alpha_s - \eta_n \cos \alpha_s + u_s + u_n \quad (115)$$

$$\delta_{rn} = \left(v \cos \alpha_r + \frac{\partial v}{\partial \theta} \sin \alpha_r \right) \bigg|_{\theta=\psi_n} - x_r \sin \psi_{rn} + y_r \cos \psi_{rn} + u_r + \xi_n \sin \alpha_r - \eta_n \cos \alpha_r - u_n \quad (116)$$

Substitution of Eq. (113) into Eqs. (115) and (116) ensures that both the sun-planet and ring-planet mesh deformations are zero.

Overall, four types of modes are identified. For odd N , the numbers of modes for rotational, translational, planet, and purely ring modes are $J+6$, $4J+10$, $(2JN-6J)+(3N-9)$, and J , respectively. For even N , they are $J+6$, $4J+10$, $(2JN-7J)+(3N-9)$, and $2J$, respectively. While the numbers of planet and purely ring modes are different for odd and even numbers of planets, the total number of modes is $(2J+3)N+7$ for either odd or even N . This total equals the total degrees of freedom with $v(\theta, t)$ from Eq. (22) and J arbitrarily large. Thus, all modes have been categorized.

Furthermore, the numbers of rotational and translational modes are independent of the number of planets N . Changing the number of planets N , while retaining the same J in the ring deformation expansion (22), only changes the numbers of planet modes and purely ring modes. Table 2 lists how the number of planets N affects the number of degenerate and distinct planet modes, and it specifies the planet mode type breakdown for each N . If N increases by 1, the total number degrees of freedom increases by $2J+3$ as the total degrees of freedom is $(2J+3)N+7$. If N increases by 1 from odd to even, $J+3$ of the additional modes are distinct planet modes, and the remaining J additional modes are purely ring modes (the total number of purely ring modes becomes $2J$). If N increases by 1 from even to odd, all $2J+3$ additional modes are planet modes; furthermore, J purely ring modes change into planet modes. Therefore, the number of planet modes increases by $3J+3$, and the number of purely ring modes decreases by J .

The natural frequency multiplicities and all modal properties from the candidate mode method match the numerical solution in Tables 1 and 2 and Fig. 2 (as well as the perturbation results) for arbitrary N and J .

5 Conclusions

The distinctive modal properties of planetary gears having equally spaced planets and an elastic continuum ring gear are derived using perturbation analysis and proved using a candidate mode method. The main conclusions are the following:

1. All vibration modes of equally spaced planetary gears having an elastic ring gear are classified into rotational, translational, planet, and purely ring modes. For each mode type, the deflections of each planetary gear component, including

the elastic ring, are derived in closed form. In addition, the number of each mode type and multiplicity of the natural frequencies are determined.

2. The modal deflection properties of the sun, carrier, and planets for rotational, translational, and planet modes are the same as for the discrete model, while the deformation of the ring gear is governed by simple analytical rules dictating which nodal diameter components are present in each mode type.
3. Rotational modes contain jN nodal diameter ring deformation components, while the sun, carrier, and ring rigid motions have only rotational motion. All planets have the same displacement. The natural frequencies are distinct.
4. Translational modes contain $jN \pm 1$ nodal diameter ring deformation components, while the sun, carrier, and ring rigid motions have only translational motion. The deflections of individual planets are related by a rotation matrix. The natural frequencies have multiplicity 2.
5. Planet modes contain $jN \pm s$ nodal diameter ring deformation components, where s is one of $2, 3, \dots, \text{int}(N/2)$. The translation and rotation of the sun, carrier, and rigid ring are zero, and the deflections of the planets are proportional to each other. Most of these natural frequencies have multiplicity 2, but some natural frequencies are distinct for an even number of planets.
6. A purely ring mode has only a single nodal diameter ring deformation component. The deflections of all the discrete elements, including the ring rigid motion, are zero. The natural frequencies are distinct.
7. Changing the number of planets N does not affect the number of rotational and translational modes. How the vibration modes are distributed between purely ring modes and planet modes with the addition of a planet depends on whether N changes from odd to even or *vice versa*.

Nomenclature

Superscripts 0 and 1 of $\mathbf{a}, \mathbf{q}, v$ denote the unperturbed and the first-order perturbation eigenfunctions, respectively. Subscripts c, r, s, n denote the carrier, ring, sun, and the n th planet.

- α_r = ring-planet pressure angle
- α_s = sun-planet pressure angle
- ψ_n = location of the n th planet
- $\psi_{rn} = \psi_n + \alpha_r$
- $\psi_{sn} = \psi_n - \alpha_s$
- ν = Poisson's ratio
- ρ = mass density per unit length
- E = Young's modulus
- J = area moment of inertia
- N = number of planets
- R = neutral radius of the ring gear
- k_j, k_{ju} = translational and rotational stiffness of supports/bearing for the carrier and sun, $j=c, s$
- k_{bend} = ring bending stiffness
- k_{rp} = ring-planet mesh stiffness
- k_{sp} = sun-planet mesh stiffness
- $k_{\text{rbs}}, k_{\text{rus}}$ = radial, tangential distributed ring elastic foundation stiffnesses
- I_j = mass moment of inertia for the ring, carrier and sun, $j=r, c, s$
- m_j = mass of the ring, carrier and sun, $j=r, c, s$
- r_j = base radius for the ring and sun, $j=r, c, s$
- r_1, r_2 = inner, outer radii of the ring gear
- u, w = ring tangential, radial deflections
- v = ring elastic tangential deflection
- x_j, y_j, u_j = translational and rotational displacements of the ring, sun and carrier, $j=r, c, s$

ξ_n, η_n, u_n = radial, tangential, and rotational displacements of the n th planet

Appendix: Nondimensional Matrices $\mathbf{M}$ and $\mathbf{K}$

$$\mathbf{M} = \text{diag}(\mathbf{M}_r, \mathbf{M}_c, \mathbf{M}_s, \mathbf{M}_1, \dots, \mathbf{M}_N)$$

$$\mathbf{M}_j = \text{diag}(m_j, m_j, I_j/r_j^2), \quad j = c, s, 1, \dots, N,$$

$$\mathbf{M}_r = \text{diag}(1, 1, 1/\cos^2 \alpha_r)$$

$$\mathbf{K} = \begin{bmatrix} \sum \mathbf{K}_{r1}^n + \mathbf{K}_{rb} & & & \mathbf{K}_{r2}^1 & \cdots & \mathbf{K}_{r2}^N \\ & \sum \mathbf{K}_{c1}^n + \mathbf{K}_{cb} & & \mathbf{K}_{c2}^1 & \cdots & \mathbf{K}_{c2}^N \\ & & \sum \mathbf{K}_{s1}^n + \mathbf{K}_{sb} & \mathbf{K}_{s2}^1 & \cdots & \mathbf{K}_{s2}^N \\ & & & \mathbf{K}_{pp}^1 & & \\ \text{symmetric} & & & & \ddots & \\ & & & & & \mathbf{K}_{pp}^N \end{bmatrix}$$

$$\mathbf{K}_{jb} = \text{diag}(k_{jx}, k_{jy}, k_{ju}), \quad j = c, s,$$

$$\mathbf{K}_{rb} = \pi \text{diag}(k_{\text{rbs}} + k_{\text{rus}}, k_{\text{rbs}}, 2k_{\text{rus}}/\cos^2 \alpha_r)$$

k_{rbs} and k_{rus} are uniform radial and tangential distributed stiffnesses, respectively. k_{ju}^* is torsional stiffness with units $F-L/\text{rad}$ and $k_{ju} = k_{ju}^*/r_j^2$ with units F/L .

$$\mathbf{K}_{pp}^n = \mathbf{K}_{r3}^n + \mathbf{K}_{c3}^n + \mathbf{K}_{s3}^n$$

$$\mathbf{K}_{r1}^n = \begin{bmatrix} \sin^2 \psi_{rn} & -\cos \psi_{rn} \sin \psi_{rn} & -\sin \psi_{rn} \\ & \cos^2 \psi_{rn} & \cos \psi_{rn} \\ \text{symmetric} & & 1 \end{bmatrix}$$

$$\mathbf{K}_{r2}^n = \begin{bmatrix} -\sin \psi_{rn} \sin \alpha_r & \sin \psi_{rn} \cos \alpha_r & \sin \psi_{rn} \\ \cos \psi_{rn} \sin \alpha_r & -\cos \psi_{rn} \cos \alpha_r & -\cos \psi_{rn} \\ \sin \alpha_r & -\cos \alpha_r & -1 \end{bmatrix}$$

$$\mathbf{K}_{r3}^n = \begin{bmatrix} \sin^2 \alpha_r & -\cos \alpha_r \sin \alpha_r & \cos \alpha_r \\ & \cos^2 \alpha_r & \cos \alpha_r \\ \text{symmetric} & & 1 \end{bmatrix}$$

$$\mathbf{K}_{c1}^n = k_{pn} \begin{bmatrix} 1 & 0 & -\sin \psi_n \\ & 1 & \cos \psi_n \\ \text{symmetric} & & 1 \end{bmatrix},$$

$$\mathbf{K}_{c2}^n = k_{pn} \begin{bmatrix} -\cos \psi_n & \sin \psi_n & 0 \\ -\sin \psi_n & -\cos \psi_n & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{K}_{c3} = \text{diag}(k_{pn}, k_{pn}, 0)$$

$$\mathbf{K}_{s1}^n = k_{sp} \begin{bmatrix} \sin^2 \psi_{sn} & -\cos \psi_{sn} \sin \psi_{sn} & -\sin \psi_{sn} \\ & \cos^2 \psi_{sn} & \cos \psi_{sn} \\ \text{symmetric} & & 1 \end{bmatrix}$$

$$\mathbf{K}_{s2}^n = k_{sp} \begin{bmatrix} \sin \psi_{sn} \sin \alpha_s & \sin \psi_{sn} \cos \alpha_s & -\sin \psi_{sn} \\ -\cos \psi_{sn} \sin \alpha_s & -\cos \psi_{sn} \cos \alpha_s & \cos \psi_{sn} \\ -\sin \alpha_s & -\cos \alpha_s & 1 \end{bmatrix}$$

$$\mathbf{K}_{s3}^n = k_{sp} \begin{bmatrix} \sin^2 \alpha_s & \cos \alpha_s \sin \alpha_s & -\sin \alpha_s \\ & \cos^2 \alpha_s & -\cos \alpha_s \\ \text{symmetric} & & 1 \end{bmatrix}$$

$$\psi_{sn} = \psi_n - \alpha_s, \quad \psi_{rn} = \psi_n + \alpha_r$$

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A Microcontact Non-Gaussian Surface Roughness Model Accounting for Elastic Recovery

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Most statistical contact analyses assume that asperity height distributions ($g(z^*)$) follow a Gaussian distribution. However, engineered surfaces are frequently non-Gaussian with the type dependent on the material and surface state being evaluated. When two rough surfaces experience contact deformations, the original topography of the surfaces varies with different loads, and the deformed topography of the surfaces after unloading and elastic recovery is quite different from surface contacts under a constant load. A theoretical method is proposed in the present study to discuss the variations of the topography of the surfaces for two contact conditions. The first kind of topography is obtained during the contact of two surfaces under a normal load. The second kind of topography is obtained from a rough contact surface after elastic recovery. The profile of the probability density function is quite sharp and has a large peak value if it is obtained from the surface contacts under a normal load. The profile of the probability density function defined for the contact surface after elastic recovery is quite close to the profile before experiencing contact deformations if the plasticity index is a small value. However, the probability density function for the contact surface after elastic recovery is closer to that shown in the contacts under a normal load if a large initial plasticity index is assumed. How skewness (Sk) and kurtosis (Kt), which are the parameters in the probability density function, are affected by a change in the dimensionless contact load, the initial skewness (the initial kurtosis is fixed in this study) or the initial plasticity index of the rough surface is also discussed on the basis of the topography models mentioned above. The behavior of the contact parameters exhibited in the model of the invariant probability density function is different from the behavior exhibited in the present model.

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Keywords: microcontact, skewness, kurtosis, non-Gaussian probability density function

1 Introduction

All engineered surfaces are microscopically rough, and the contact between two such surfaces is carried by the asperities on the surfaces. This characterization is important to the study of many interfacial phenomena, such as friction, wear, and thermal and electrical contact resistances. Study of the deformation behavior of contact asperities and accurate modeling of rough surfaces is important in order to better understand contact problems.

A significant amount of research has been carried out on the contact of rough surfaces as detailed in review papers [1–3]. Rough surfaces are usually described in one of the following ways: (a) statistical description of rough surfaces with certain asperity shapes; (b) generation of rough surfaces, based on statistical description; (c) digitization of measured real rough surfaces; and (d) characterization using fractal geometry theory. These different roughness representations give rise to different microcontact models. Most of these papers mentioned that the pioneering contribution to this field was made by Greenwood and Williamson (GW) [4], who developed an elastic contact model (the GW model). In actuality, this model was first introduced by Zhuravlev [5], who derived the general integral expressions for the contact load and the contact area, and calculated integrals for the case when the distribution of summits is linear. The GW model is similar to Zhuravlev's model. In the GW model, the hemispherical tipped

asperities are assumed to have equal radii, and most importantly, the asperity heights from a certain mean reference plane follow a probability density function (statistical distribution). With the additional assumption of elastic (Hertzian) microcontacts existing between this rough surface and a rigid smooth surface, Greenwood and Williamson derived relationships for the total contact area and the contact load, expressed as a function of the separation between the flat and the mean summit level. The basic asperity GW model has been extended to study the cases of other contact geometries, such as curved surfaces [6], surfaces with nonuniform radii of curvature of asperity peaks [7,8], two rough surfaces with elliptic paraboloidal asperities [9], and anisotropic surfaces [10]. Although the GW contact model has been widely accepted, Greenwood and Wu [11] made the comment that the definition of a "peak" as a point higher than its neighbors on a profile introduced by the GW model was a mistake, and it should be made clear that peaks or summits are not asperities. The true picture of an asperity goes back to Archard's concept of "protuberances on protuberances" [11].

Traditionally, statistical microcontact models have assumed a symmetric Gaussian distribution for the heights of the surface asperities. However, it has been recognized that many common machining processes produce non-Gaussian surfaces that are characterized by two nondimensional statistical characteristics, skewness and kurtosis, which represent the asymmetry and flatness of the distribution, respectively [12–14]. Typical skewness and kurtosis values for surface roughness produced with various manufacturing processes are given in Ref. [15]. Electrical discharge machine (EDM) and turning operations produce positively skewed surfaces, whereas milling, honing, and grinding produce negatively skewed surfaces. Several research papers have been

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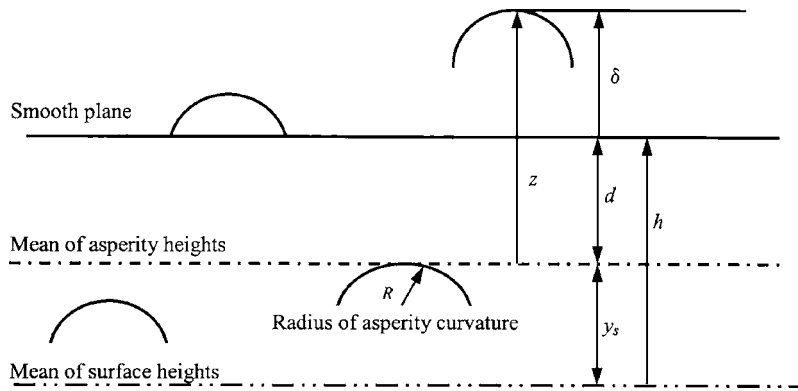


Fig. 1 The schematic diagram of two contact surfaces with deformations

written on the contact of rough surfaces with an asymmetric distribution of asperity heights [16–21]. McCool introduced a two-parameter Weibull probability density function to model the distribution of asperity heights [16], and subsequently normalized it to a one-parameter distribution to account for such asymmetric surfaces [17]. Yu and Polycarpou adopted the Weibull distribution to model asperity heights, and used the elastic-plastic model developed by Chang, Etsion and Bogoy (the CEB model) to calculate the contact load, real area of contact, and the number of contacting asperities [18]. Yu and Polycarpou also extended rough surfaces to include the contact of two rough surfaces in which their distribution of asperity heights can either be symmetric or asymmetric, with the asymmetric case being modeled using the normalized Weibull distribution [19]. Several other studies have proposed other non-Gaussian distributions, like the Yu and Polycarpou one based on Nayak's model, in which the skewness and kurtosis of the asperity height distribution are derived as a function of bandwidth parameter α [20], and Pearson's system of frequency curves to represent asymmetric rough surfaces [21]. According to the above studies, the probability density function of asperity heights before rough surfaces make contact can be either a Gaussian or a non-Gaussian distribution function. The assumption of a Gaussian distribution, when the actual distribution of the asperity heights is non-Gaussian, can lead to incorrect results.

In the present study, the probability density functions of asperity heights for two contact conditions are discussed. The skewness (Sk) and kurtosis (Kt) parameters in the non-Gaussian probability density function are varied with the mean separation of two contact surfaces. These two parameters are evaluated under the condition of either two contact surfaces under a constant load or the rough surface after elastic recovery. When two rough surfaces meet, the original topography of the surfaces is no longer invariant but varied with applied loads. The topographies of a surface obtained from experimental results [22] of different loads show the probability density function of surface asperities to be non-Gaussian distributions. The variations of surface skewness with the mean asperity separation (d^*) have been studied by Chung and Lin [23]. In their study, the relationship between skewness (Sk) and the mean asperity separation (d^*) was derived based on the experimental results shown in the study of Othmani and Kaminsky [22]. Hence, the results obtained by Chung and Lin [23] are suitable for a specific rough surface only. In order to improve the semiempirical limitation of the results by Chung and Lin [23], a method is thus proposed in the present study to obtain the variations of the probability density function of asperity heights with the mean asperity separation by assuming the initial topography to have either a Gaussian or non-Gaussian distribution. The skewness (Sk) and the kurtosis (Kt) are thus varied depending on the dimensionless contact load (F_t^*) and the initial plasticity index of the rough surface. The key concept is that the value of the probability density function of asperity heights is set equal to zero

when asperity heights (z^*) are greater than the mean asperity separation (d^*). These two roughness parameters are determined by the principle that the integration of the probability density function over the range of z^* (the dimensionless asperity height based on the mean of asperity heights) is equal to 1. The probability density function after unloading and further finishing the elastic recovery is quite different from that obtained from surface contacts under a constant load. It is totally dependent on the area magnitude beneath the initial probability density function profile associated with the elastic deformation region. The area in the profile of the probability density function associated with the elastic recovery of surface asperities shows a direct relationship with the initial plasticity index. The theoretical results of the skewness and kurtosis parameters varying with the mean asperity separation and the initial skewness and kurtosis are investigated. The elastic-plastic microcontact behavior of two rough surfaces is developed to investigate the effect of a variable non-Gaussian probability density function of asperity heights (g) on the total contact area and the contact load.

2 Theoretical Analysis of Contact Surfaces

The contact of two rough surfaces (see Fig. 1) can be modeled by a flat smooth surface in contact with a rough surface. z is the height of an asperity measured from the mean surface of summits heights. The asperity interference δ is given as

$$\delta = z - d \quad (1)$$

where d denotes the distance between the smooth plane and the mean surface of summits heights. If the mean radii of curvature of the asperities on Surfaces 1 and 2 are R_1 and R_2 , respectively, the equivalent rough surface can be expressed to have the radius of curvature R satisfying $1/R = (1/R_1) + (1/R_2)$. σ_1 and σ_2 denote the standard deviations of Surfaces 1 and 2, respectively. The standard deviation σ for this equivalent rough surface before contact deformations occur satisfies $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$.

In the present study, three deformation regimes are included in the study of asperity deformations under interference. The theory developed for an asperity is then extended to evaluate the total contact area and the contact load on the base of the apparent area. The following assumptions are made in the present study.

1. There is an isotropic surface for roughness.
2. All asperities have the same radius of curvature near their summits.
3. All asperity heights vary randomly.
4. The interactions between neighboring asperities on the same surface are neglected.
5. Asperity deformations only occur during the contact process; bulk deformation at the substrate level is not considered.

The second assumption is needed because the GW model [4] adopted in this study evaluates the contact parameters.

2.1 Contact Parameters at Elastic and Fully Plastic Deformations. According to the Hertz theory, the elastic contact area A_e , the elastic contact load F_e , and the average contact pressure P_e , produced by a sphere with a radius of R in contact with a flat, smooth plate with an elastic interference δ , are given as [24]

$$A_e = \pi R \delta \quad (2)$$

$$F_e = \frac{4}{3} E R^{1/2} \delta^{3/2} \quad (3)$$

$$P_e = \frac{4}{3} \frac{E}{\pi} \left(\frac{\delta}{R} \right)^{1/2} \quad (4)$$

where E denotes effective Young's modulus of two solid contact surfaces (Surfaces 1 and 2) with Young's moduli E_1 and E_2 and the Poisson ratios ν_1 and ν_2 , respectively. This can be stated as

$$\frac{1}{E} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}$$

In the fully plastic deformation regime, the asperity's contact area A_p , the contact load F_p , and the average contact pressure P_p can be expressed as [24]

$$A_p = 2\pi R \delta \quad (5)$$

$$F_p = H A_p \quad (6)$$

$$P_p = H \quad (7)$$

where H is the hardness of the softer material of the two contact solids.

2.2 Critical Interference and Contact Parameters in the Elastoplastic Deformation Regime. The critical interference δ_c , which marks the transition from the elastic deformation to the elastoplastic deformation, is given by [25]

$$\delta_c = \left(\frac{\pi K H}{2E} \right)^2 R \quad (8)$$

where the maximum contact pressure factor K is related to the Poisson ratio of the softer material [26] expressed as $K=0.454+0.41\nu$.

Kogut and Etsion [27] used a finite element method to solve the elastoplastic contact problem of a single asperity and found that the entire elastoplastic regime extends over interference values in the range $1 \leq \delta/\delta_c \leq 110$, with a distinct transition in the mean contact pressure at $\delta/\delta_c=6$. The asperity's contact area A_{ep} , the contact load F_{ep} , and the average contact pressure P_{ep} in the elastoplastic deformation regime are presented in a dimensionless form as [27]

$$\frac{A_{ep}}{\pi R \delta_c} = a_1 \left(\frac{\delta}{\delta_c} \right)^{b_1} \quad (9)$$

$$\frac{F_{ep}}{2/3 K H \pi R \delta_c} = a_2 \left(\frac{\delta}{\delta_c} \right)^{b_2} \quad (10)$$

$$\frac{P_{ep}}{H/2.8} = a_3 \left(\frac{\delta}{\delta_c} \right)^{b_3} \quad (11)$$

where a_1, b_1, a_2, b_2, a_3 , and b_3 are constants and are summarized in distinct elastoplastic subregimes as follows.

(a) For the first elastoplastic regime ($1 \leq \delta/\delta_c \leq 6$),

$$a_1 = 0.93, \quad b_1 = 1.136, \quad a_2 = 1.03, \quad b_2 = 1.425, \\ a_3 = 1.19, \quad b_3 = 0.289$$

(b) For the second elastoplastic regime ($6 \leq \delta/\delta_c \leq 110$),

$$a_1 = 0.94, \quad b_1 = 1.146, \quad a_2 = 1.40, \quad b_2 = 1.263, \\ a_3 = 1.61, \quad b_3 = 0.117$$

2.3 Contact Load and Total Contact Area. The above analyses developed for the three deformation regimes of one asperity can be applied to model the contact behavior of two rough surfaces. Define N as the number of asperities on a nominal area A_n , then the expected contact number n at a mean asperity separation d is expressed as

$$n = N \int_d^\infty [g(z)]_d dz = \eta A_n \int_d^\infty [g(z)]_d dz \quad (12)$$

where η denotes the area density of asperities and $[g(z)]_d$ represents the probability density function of the asperity heights at a mean asperity separation d . With the base model developed by Kogut and Etsion (the KE model), the following equations for the total real contact area A_t and the contact load F_t for a given mean asperity separation d are obtained using [28]

$$A_t(d) = A_{et}(d) + A_{ept}(d) + A_{pt}(d) \\ = \eta A_n \left[\int_d^{d+\delta_c} A_e [g(z)]_d dz + \int_{d+\delta_c}^{d+\delta_p} A_{ep} [g(z)]_d dz \right. \\ \left. + \int_{d+\delta_p}^\infty A_p [g(z)]_d dz \right] \\ = \eta \pi R A_n \left[\int_d^{d+\delta_c} \delta [g(z)]_d dz \right. \\ \left. + 0.93 \delta_c \int_{d+\delta_c}^{d+6\delta_c} \left(\frac{\delta}{\delta_c} \right)^{1.136} [g(z)]_d dz \right. \\ \left. + 0.94 \delta_c \int_{d+6\delta_c}^{d+\delta_p} \left(\frac{\delta}{\delta_c} \right)^{1.146} [g(z)]_d dz + 2 \int_{d+\delta_p}^\infty \delta [g(z)]_d dz \right] \quad (13)$$

and

$$F_t(d) = F_{et}(d) + F_{ept}(d) + F_{pt}(d) \\ = \eta A_n \left[\int_d^{d+\delta_c} F_e [g(z)]_d dz + \int_{d+\delta_c}^{d+\delta_p} F_{ep} [g(z)]_d dz \right. \\ \left. + \int_{d+\delta_p}^\infty F_p [g(z)]_d dz \right] \\ = \eta A_n E R^{1/2} \left[\frac{4}{3} \int_d^{d+\delta_c} \delta^{3/2} [g(z)]_d dz \right. \\ \left. + 1.373 \delta_c^{3/2} \int_{d+\delta_c}^{d+6\delta_c} \left(\frac{\delta}{\delta_c} \right)^{1.425} [g(z)]_d dz \right. \\ \left. + 1.867 \delta_c^{3/2} \int_{d+6\delta_c}^{d+\delta_p} \left(\frac{\delta}{\delta_c} \right)^{1.263} [g(z)]_d dz \right. \\ \left. + \frac{2\pi H R^{1/2}}{E} \int_{d+\delta_p}^\infty \delta [g(z)]_d dz \right] \quad (14)$$

where δ_p denotes the interference corresponding to the inception of the fully plastic deformation regime.

Equations (13) and (14) may be normalized by dividing A_n and $A_n E$, respectively. Furthermore, all the length parameters and vari-

ables are normalized by dividing them by σ , which is the rms roughness of the surface height distribution. The resulting dimensionless equations are given by

$$\begin{aligned}
 A_t^*(d^*) &= \frac{A_t}{A_n} = \eta \left[\int_{d^*}^{d^*+\delta_c^*} A_e^*[g(z^*)]_{d^*} dz^* \right. \\
 &\quad + \int_{d^*+\delta_c^*}^{d^*+\delta_p^*} A_{ep}^*[g(z^*)]_{d^*} dz^* [g(z)]_0 dz \\
 &\quad \left. + \int_{d^*+\delta_p^*}^{\infty} A_p^*[g(z^*)]_{d^*} dz^* \right] \\
 &= \pi\beta \left[\int_{d^*}^{d^*+\delta_c^*} \delta^*[g(z^*)]_{d^*} dz^* \right. \\
 &\quad + 0.93\delta_c^* \int_{d^*+\delta_c^*}^{d^*+6\delta_c^*} \left(\frac{\delta^*}{\delta_c^*} \right)^{1.136} [g(z^*)]_{d^*} dz^* \\
 &\quad + 0.94\delta_c^* \int_{d^*+6\delta_c^*}^{d^*+\delta_p^*} \left(\frac{\delta^*}{\delta_c^*} \right)^{1.146} [g(z^*)]_{d^*} dz^* \\
 &\quad \left. + 2 \int_{d^*+\delta_p^*}^{\infty} \delta^*[g(z^*)]_{d^*} dz^* \right] \quad (15)
 \end{aligned}$$

and

$$\begin{aligned}
 F_t^*(d^*) &= \frac{F_t}{A_n E} = \frac{\eta}{E} \left[\int_{d^*}^{d^*+\delta_c^*} F_e^*[g(z^*)]_{d^*} dz^* \right. \\
 &\quad \left. + \int_{d^*+\delta_c^*}^{d^*+\delta_p^*} F_{ep}^*[g(z^*)]_{d^*} dz^* + \int_{d^*+\delta_p^*}^{\infty} F_p^*[g(z^*)]_{d^*} dz^* \right] \\
 &= \beta \left(\frac{\sigma}{R} \right)^{1/2} \left[\frac{4}{3} \int_{d^*}^{d^*+\delta_c^*} \delta^{3/2}[g(z^*)]_{d^*} dz^* \right. \\
 &\quad + 1.373\delta_c^{3/2} \int_{d^*+\delta_c^*}^{d^*+6\delta_c^*} \left(\frac{\delta^*}{\delta_c^*} \right)^{1.425} [g(z)]_{d^*} dz \\
 &\quad + 1.867\delta_c^{3/2} \int_{d^*+6\delta_c^*}^{d^*+\delta_p^*} \left(\frac{\delta^*}{\delta_c^*} \right)^{1.263} [g(z^*)]_{d^*} dz^* \\
 &\quad \left. + \frac{2\pi H\beta}{E} \int_{d^*+\delta_p^*}^{\infty} \delta^*[g(z^*)]_{d^*} dz^* \right] \quad (16)
 \end{aligned}$$

where $\beta = \eta\sigma R$; d^* is the dimensionless mean asperity separation based on surface heights.

2.4 Variations of the Non-Gaussian Probability Density Function During the Loading Contact. The topographies obtained from the experimental results of surface contacts at different loads are generally no longer Gaussian distributions. According to the experimental results shown in the study of Othmani and Kaminsky [22], the surface asperities after the contacts of different loads were found to satisfy a probability density function. The equation for the probability density function $g(z^*)$ can be expressed as [29]

$$g(z^*) = y_e \left(1 + \frac{z^*}{B_1} \right)^{m_1} \left(1 - \frac{z^*}{B_2} \right)^{m_2} \quad -B_1 < z^* < B_2 \quad (17)$$

where $z^* (=z/\sigma)$ is defined as the dimensionless asperity height of an undeformed asperity, as measured on the basis of the mean of

asperity heights. In Eq. (17), m_1 , m_2 , B_1 , and B_2 are obtained by solving the following two equations [29]:

$$\frac{(m_1 + 1)}{B_1} = \frac{(m_2 + 1)}{B_2} \quad (18)$$

$$B_1 + B_2 = \frac{1}{2} \{ \text{Sk}^2(r+2)^2 + 16(r+1) \}^{1/2} \quad (19)$$

where Sk in Eq. (19) denotes the skewness, which is the measure of the asymmetry of the profile about the mean line; the r parameter is written as [29]

$$r = \frac{6(Kt - \text{Sk}^2 - 1)}{(6 + 3\text{Sk}^2 - 2Kt)} \quad (20)$$

where the kurtosis Kt in Eq. (20) represents a measure of the flatness of the roughness profile. The m_1 and m_2 values are given by [29]

$$m_1 = \frac{1}{2} \left\{ r - 2 + r(r+2) \left(\frac{\text{Sk}^2}{\text{Sk}^2(r+2)^2 + 16(r+1)} \right)^{1/2} \right\} \quad (21a)$$

$$m_2 = \frac{1}{2} \left\{ r - 2 - r(r+2) \left(\frac{\text{Sk}^2}{\text{Sk}^2(r+2)^2 + 16(r+1)} \right)^{1/2} \right\} \quad (21b)$$

and y_e in Eq. (17) is written as [29]

$$y_e = \frac{1}{(B_1 + B_2)} \frac{(m_1 + 1)^{m_1} (m_2 + 1)^{m_2}}{(m_1 + m_2 + 2)^{m_1 + m_2}} \frac{\Gamma(m_1 + m_2 + 2)}{\Gamma(m_1 + 1)\Gamma(m_2 + 1)} \quad (22)$$

where Γ is the gamma function. Once Sk and Kt are obtained, the Gaussian or non-Gaussian probability density function $g(z^*)$ can be determined. If the skewness and kurtosis are assumed to be zero and 3, respectively [29], Eq. (17) turns out to be a Gaussian distribution. EDMs and turning operations produce positively skewed surfaces, whereas milling, honing, and grinding produce negatively skewed surfaces [15]. Hence, in the present study, it is assumed that the initial value of Sk_0 is varied from 0 to -0.8 and the initial Kt_0 value is fixed to be 3. It should be stressed that unrestricted value of Kt_0 can be chosen as the initial kurtosis, if these two parameters (Sk_0 , Kt_0) are satisfied by the integration of the probability density function over the range of z^* equal to 1. Nevertheless, in the present study, these two parameters are no longer set to be constant value, but are varied with the interference.

The probability density function $g(z^*)$ adopted in the present study, as Eq. (17) shows, is expressed as a function of the coefficients of y_e , B_1 , B_2 , m_1 , and m_2 . These coefficients are further expressed as a function of the skewness Sk and the kurtosis Kt. Therefore, the probability density function can be determined only when the kurtosis and the skewness are available. In the present study, a method is developed such that the Sk and Kt parameters are varied with the mean asperity separation under the condition of either two contact surfaces under a constant load or a rough surface after elastic recovery. In Eq. (17), the upper bound of z^* is B_2 , whereas the lower bound of z^* is $-B_1$. The values of B_1 and B_2 are varied with the mean asperity separation, so they should satisfy the integration of the non-Gaussian probability density function over the entire range of z^* between $-B_1$ and B_2 being equal to 1 at any mean asperity separation. Prior to the occurrence of surface contacts, the initial skewness Sk_0 and the initial kurtosis Kt_0 are given. The r parameter before surface contacts (r_0) can thus be determined by substituting the Sk_0 and Kt_0 values into Eq. (20). Similarly, the m_1 and m_2 values can also be obtained by substituting Sk_0 and Kt_0 into Eqs. (21a) and (21b) respectively, and then these two algebraic equations are coupled together to solve $(B_1)_0$

Table 1 The initial values of B_1 and B_2 in Eq. (15) corresponding to different initial values of skewness and kurtosis

(Sk_0, Kt_0)	$(B_1)_0$	$(B_2)_0$
$(-0.2, 3)$	16.136	6.136
$(-0.4, 3)$	8	3
$(-0.6, 3)$	5.257	1.923
$(-0.8, 3)$	3.86	1.36

and $(B_2)_0$ by employing Eqs. (18) and (19). Obviously, $(-B_1)_0$ and $(B_2)_0$ are the lower and upper bounds of z^* arising at the asperity heights before surface contact. The lower and upper bounds of z^* , $(-B_1)_0$ and $(B_2)_0$, corresponding to the different initial skewness Sk_0 and the initial kurtosis Kt_0 are shown in Table 1.

It should be mentioned that the $(B_1)_0$ and $(B_2)_0$ parameters are strongly dependent on the initial values of skewness (Sk_0) and kurtosis (Kt_0). Since the lower bound of z^* , $(-B_1)_0$, represents the lowest extreme value of surface asperity heights, it has a value always smaller than the mean asperity heights. According to the study of Zhao and Chang [30], the asperity interactions can affect the mean value of asperity heights. The displacement u of the mean surface line defined for asperity heights can be expressed as [30]

$$u = 1.12 \frac{\sqrt{w_l p_m}}{E} \quad (23)$$

where w_l is the local contact load of an asperity and p_m denotes the global mean contact pressure. If the contact surface is steel, Young's modulus $E=113$ GPa and hardness $H=1.96$ GPa. The mean contact pressure reaches fully plastic deformation when $p_m=H$. Under a relatively large load, the u parameter expressed in a dimensionless form is evaluated to have a value of about 10^{-7} , which is still negligibly small compared to z^* when the asperity heights with contact deformations operate in a region larger than the mean separation. Thus, the B_1 value at any mean asperity separation has a value almost equal to $(B_1)_0$ if contact deformations do not reach the lower bound of z^* . However, the upper bound of z^* always varies with the mean asperity separation (d^*). The upper bound (B_2) will be equal to d^* when the mean asperity separation between two contact surfaces is d^* . The probability density function in the z^* values greater than d^* drops to zero. Using Eq. (19), the skewness parameter Sk can be expressed as a function of B_1 , B_2 and Kt and can be written as

$$Sk = \left(\frac{-y \pm \sqrt{y^2 - 4xz_1}}{2x} \right)^{1/2} \quad (24)$$

where

$$x = 36(B_1 + B_2)^2 + 144$$

$$y = 24b(B_1 + B_2)^2 - a^2 - 4ab - 4b^2 - 48a$$

$$z_1 = 4b^2(B_1 + B_2)^2 - 16(ab + b^2)$$

$$a = 6Kt - 6$$

$$b = 6 - 2Kt$$

Therefore, the Sk value can be determined if Kt is given. However, the genuine Kt and Sk values corresponding to a given mean asperity separation are determined by the principle that the integration of the non-Gaussian probability density function over $-B_1 \leq z^* \leq d^*$ must be equal to 1. The method mentioned above can thus be applied to find the non-Gaussian probability density function $g(z^*)$, and the skewness Sk and the kurtosis Kt evaluated

at various mean asperity separations (d^*), no matter what the initial values of the above three parameters are before contact.

2.5 Variations of the Non-Gaussian Probability Density Function After Elastic Recovery. After the removal of the applied load of two contact surfaces, the elastic recovery of the compressed surface might bring out a new asperity height distribution on the rough surface. The non-Gaussian probability density function after unloading and elastic recovery is quite different from that obtained from surface contacts under a constant load. It is totally dependent on the area magnitude beneath the initial probability density function profile associated with the elastic deformation region. This area lies in the region between $z^*=d^*$ and $z^*=d^*+\delta_c^*$, where δ_c^* denotes the critical interference beyond which the elastoplastic deformation occurs. The critical interference δ_c^* of a rough surface is strongly related to the plasticity index. The plasticity index (ψ) in the study of Greenwood and Williamson [4] is defined as

$$\psi = \left(\delta_c^* \frac{\sigma}{\sigma_s} \right)^{-0.5} = \frac{2E}{\pi KH} \left(\frac{\sigma}{R} \right)^{1/2} \left(1 - \frac{3.717 \times 10^{-4}}{\beta^2} \right)^{1/4} \quad (25)$$

where E denotes Young's modulus, H denotes the hardness of a material, K represents the maximum contact pressure factor ($K \approx 0.57$ for steel), σ represents the standard deviation of surface asperities, and R denotes the mean radius of curvature of the initial surface asperities. The behavior of surface roughness can be described as a function of $\beta (= \eta \sigma R)$ and σ/R . These two parameters, adopted in this study, are given in the study of Nuri and Halling [31]. By incorporating these two roughness parameters and the mechanical properties, the initial surface roughnesses can be characterized by the initial plasticity index. The initial plasticity index, therefore, simply reflects the deformation possibility of a rough surface when under contact. A rougher, softer surface, in general, has a higher plasticity index. If $\psi > 1.0$, the surface in the contact is apt to generate elastoplastic deformation. If $\psi < 0.6$, the surface is apt to have elastic deformation. Furthermore, it should be mentioned that the plasticity index is unnecessary in evaluating the contact parameters in the present model. According to Eq. (25), ψ is actually a variable because the σ parameter in the plasticity index is now varying with the mean asperity height evaluated at different separations, rather than a constant value. The area in the profile of the probability density function associated with the elastic recovery of surface asperities has a direct relationship with the plasticity index. The details on the use of this area to determine the new probability density profile after the elastic recovery will be described in Sec. 3. All new roughness parameters, such as the probability density function, the skewness, and the kurtosis after elastic recovery, are achieved only when they are available during surface contact under a constant load. The flowchart used to obtain these new roughness parameters is shown in Fig. 2. In this flowchart, the subscript d in a parameter denotes the behavior arising at surface contacts with a dimensionless mean separation d^* between these two surfaces, whereas the subscript u denotes the behavior arising at the new surface after elastic recovery.

3 Results and Discussion

The probability density function $g(z^*)$ of asperity heights in a non-Gaussian form before contact deformations is determined by the initial skewness Sk_0 and the initial kurtosis Kt_0 only. After contact deformations, the skewness Sk and the kurtosis Kt no longer remain unchanged, but vary with the interference formed by a rigid, smooth surface in contact with a rough surface. Figure 3(a-1) shows the probability density function of asperity heights before any contact deformation. The non-Gaussian $g(z^*)$ profile is obtained by assuming $Sk_0=-0.2$ and $Kt_0=3$. The vertical line of $z^*=1$ is marked as an example to indicate the contact position of the smooth, rigid surface under a normal load. Another vertical

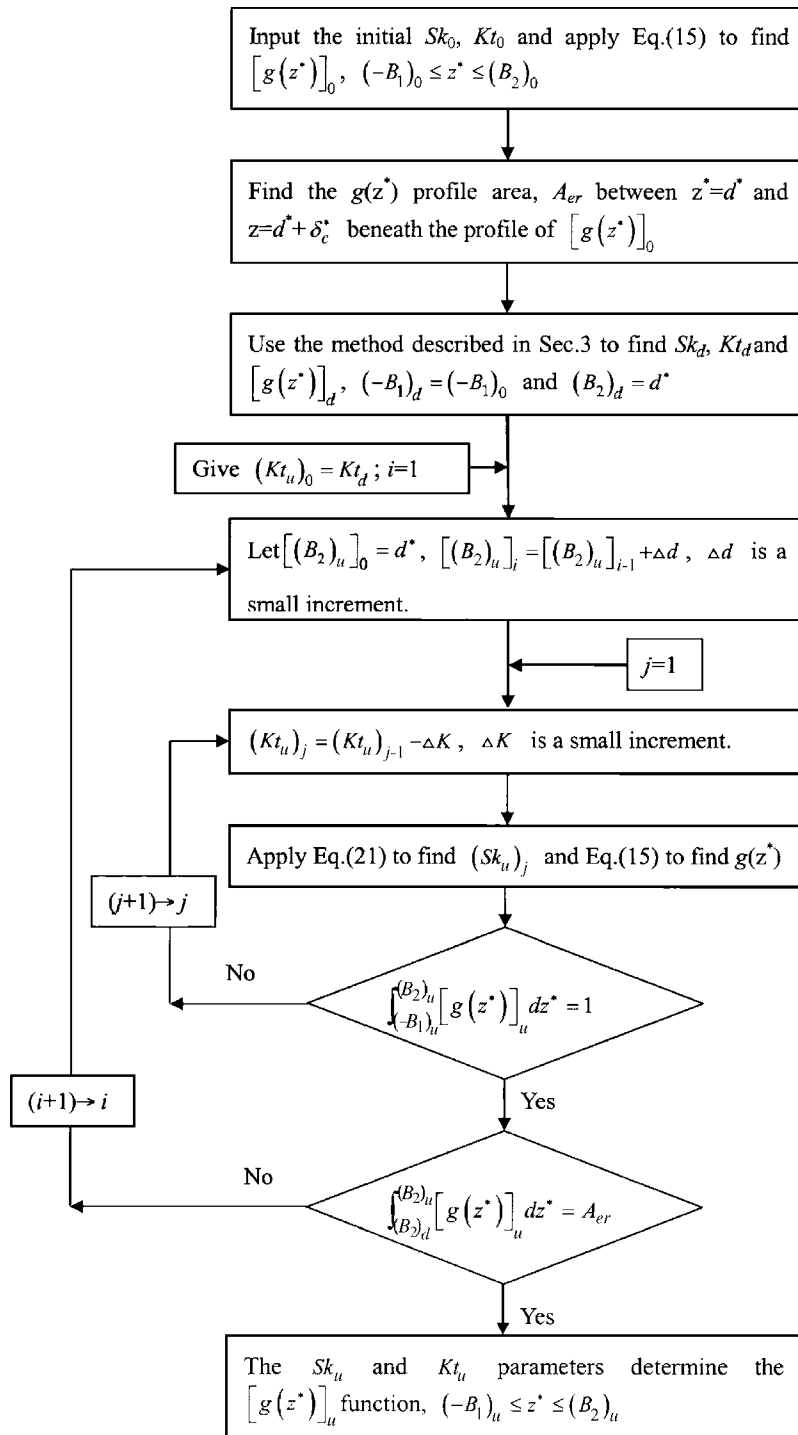


Fig. 2 Flowchart for the numerical analyses of Sk_u and Kt_u

line on the right-hand side of $z^*=1$ is given to mark the border of the elastic and elastoplastic deformation regimes. Therefore, the z^* region between these two vertical lines represents the elastic deformation regime, and the asperity heights associated with this regime are operating under elastic deformation.

The probability density function shown on the right-hand side of the critical interface ($d^*=1$) is then associated with elastoplastic/plastic deformation. The elastic deformation regime in the probability density function $g(z^*)$ is strongly dependent on the initial plasticity index (ψ_0). A small plasticity index implies that the majority of microcontacts are apt to operate in elastic

deformation under a normal load. As Fig. 3(a-1) shows, the elastic deformation regime corresponding to $\psi_0=0.5$ is so wide that the $g(z^*)$ profile for $z^*=1$ is almost entirely located in this regime. The shadow area marked by straight lines beneath the $g(z^*)$ profile as well as in the elastic deformation regime represents the “strain energy” applicable to the elastic recovery. It is stored temporarily when the smooth, rigid surface is pressed against the rough surface at $d^*=1$ under a constant normal load. Under this circumstance, the probability density function is changed to be the profile marked by 3 in Fig. 3(b-1). Since the area beneath the dotted profile (Curve 2) is in a z^* region narrower than that shown in the

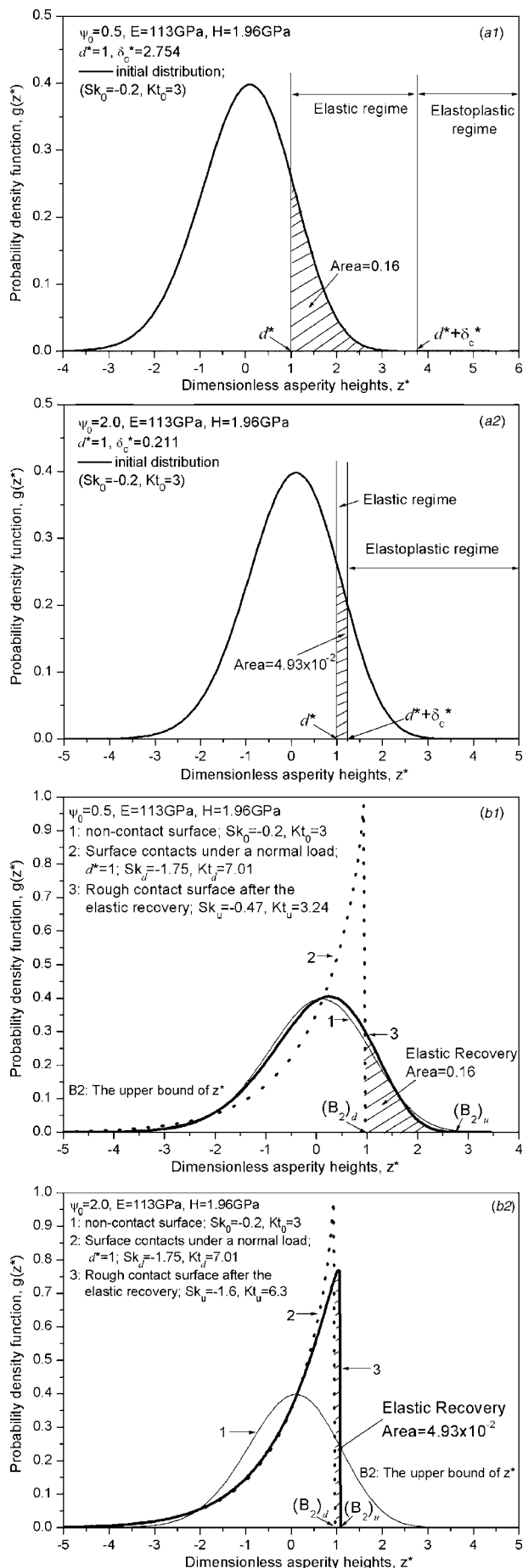


Fig. 3 (a) The probability density functions of surface asperities before contact deformation. The initial plasticity indices are $\psi_0=0.5$ and $\psi_0=2.0$, respectively. The filled-in areas indicate the “strain energy” available for the elastic recovery of the rough surface after finishing the unloading process. (b) Variations of the probability density function for the surface contacts under a normal load and the rough contact surface after elastic recovery. The initial plasticity indices are also 0.5 and 2.0, respectively.

noncontact Profile (Fig. 3(a-1)) the peak value of Curve 2 is significantly elevated because the integration of the area beneath Profile 3 must be equal to 1.

If the normal load is removed from two contact surfaces, the strain energy stored in the specimen is then released in the form of elastic recovery. The probability density function of surface asperities is changed again when the elastic recovery ends; the profile is now marked by 3 in Fig. 3(b-1). The right end of Curve 3 is determined such that the shadow area beneath Curve 3, as well as lying between $z^*=1$, must be equal to the shadow area exhibited in Fig. 3(a-1). If the microcontacts are evaluated with an initial plasticity index (ψ_0) of 0.75, the $g(z^*)$ curve marked by 3 (after elastic recovery) is quite close to the original $g(z^*)$ before any contact although the right end of Curve 3 is shorter compared with that of Curve 1. However, the distinction between Curves 2 and 3 becomes quite significant in either the peak value or the right end of the profile.

If the initial plasticity index (ψ_0) is elevated to 2.0, the regime in the probability density function associated with the elastic deformation regime is significantly reduced (see Fig. 3(a-2)). The regime related to the elastoplastic or the fully plastic regime is, however, enlarged. In Fig. 3(a-2), the shadow area denotes the probability density function of surface asperities operating in the elastic deformation regime. Due to the high initial plasticity index, the shadow area in Fig. 3(a-2) is much smaller than that shown in Fig. 3(a-1).

When the surface asperities are compressed by a rigid, smooth surface with a mean separation $d^*=1$ between them, the profile of the probability density function for the surface asperities under a constant normal load is marked by 2 (dot curve) in Fig. 3(b-2). The profile marked by 3 is, however, the consequence of surface asperities after the elastic recovery. Because of the small shadow area associated with the elastic regime, the profile of the probability density function for the contact surface after the elastic recovery is quite similar to the profile obtained from the contacts under a constant load, except for the profile in the region near their peaks. However, the profile of the probability density function either after the elastic recovery or under a constant load becomes much steeper compared with the $g(z^*)$ profile of a noncontact rough surface. The profile of the $g(z^*)$ function becomes steep and sharp as the d^* value decreases to a sufficiently small value. When $d^*<0.5$, the calculation of integrating $g(z^*)$ is hard to converge unless the error tolerance is enlarged. If the tolerance is loosened, the behavior of the integration of $g(z^*)$ demonstrated in the case of $d^*<0.5$ shows a big difference in its characteristic from that demonstrated in the case of $d^*>0.5$. Therefore, the lower bound of d^* was set to be 0.5 in the present study.

Apart from the initial skewness (Sk_0) and initial kurtosis (Kt_0), the mean separation between two contact surfaces is also an important factor in the variations of $g(z^*)$. Figure 4(a) shows the $g(z^*)$ results evaluated at different separations; the initial skewness is -0.2 and the initial kurtosis is 3. The final skewness and kurtosis evaluated at different separations (d^*) are also given for each profile. These results were predicted using the present model undergoing a constant load. The $g(z^*)$ profile becomes thin and sharp in appearance as the mean separation (d^*) becomes small.

The initial plasticity index should be specified in order to obtain the solution of $g(z^*)$ for the surface asperities after the elastic recovery. The $g(z^*)$ profiles in Fig. 4(b) were obtained after the elastic recovery as well as at different separations; they are the results corresponding to $\psi_0=0.5$, $Sk_0=-0.2$, and $Kt_0=3$. Due to a small value of ψ_0 , the behavior of the elastic deformation is dominant at different mean separations. Therefore, the $g(z^*)$ profiles for the surface asperities after elastic recovery are very similar although they were evaluated at different mean separations. The results in Fig. 4(b) demonstrate that the $g(z^*)$ profile becomes slightly thinner and higher at the peak value when the mean separation (d^*) becomes small. Figure 4(c) shows the $g(z^*)$ profile obtained from the deformed surface after the elastic recovery; the initial plasticity index was elevated to 2.0. The profiles of this figure and those shown in Fig. 4(b) have significant difference. However, the similarities of the profiles shown in Figs. 4(a) and

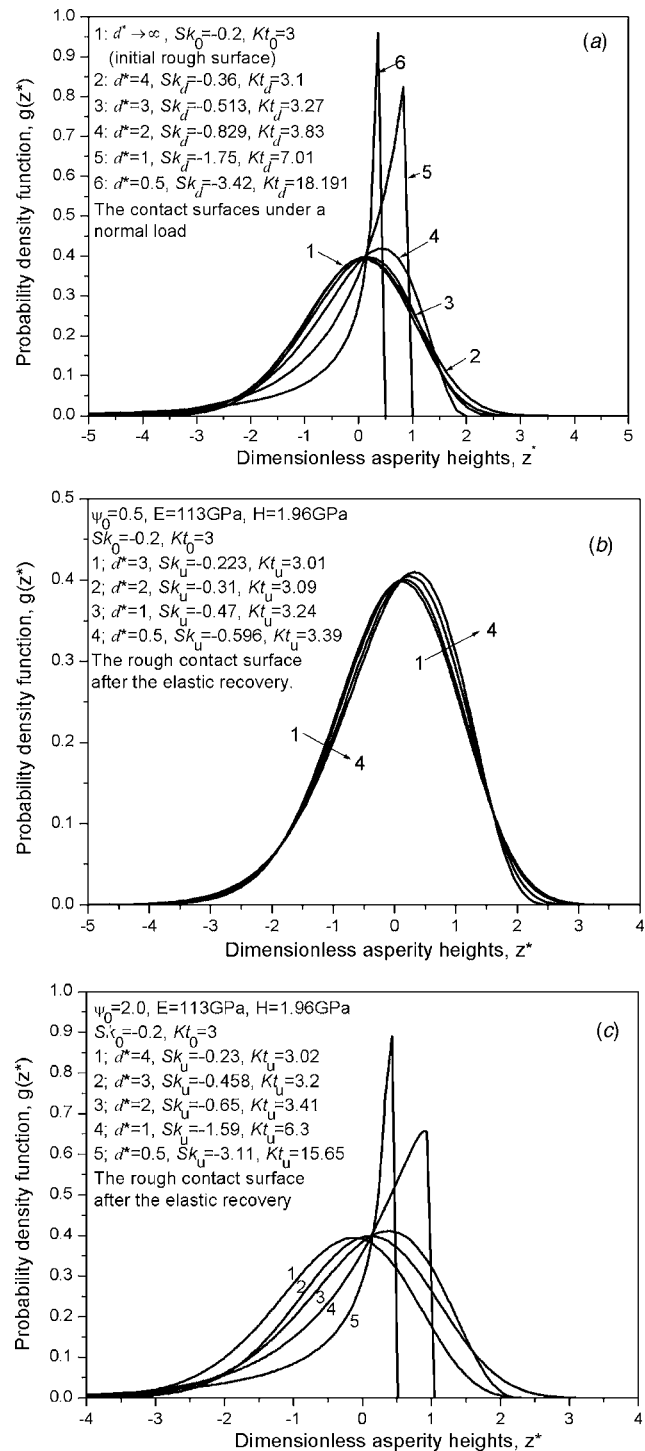


Fig. 4 The probability density functions evaluated at different mean separations for (a) the surface contacts under a normal load; (b) the rough contact surface with a plasticity index of 0.5, which is obtained after elastic recovery; and (c) the rough contact surface with a plasticity index of 2.0, which is obtained after elastic recovery

Fig. 4(b) have significant difference. However, the similarities of the profiles shown in Figs. 4(a) and

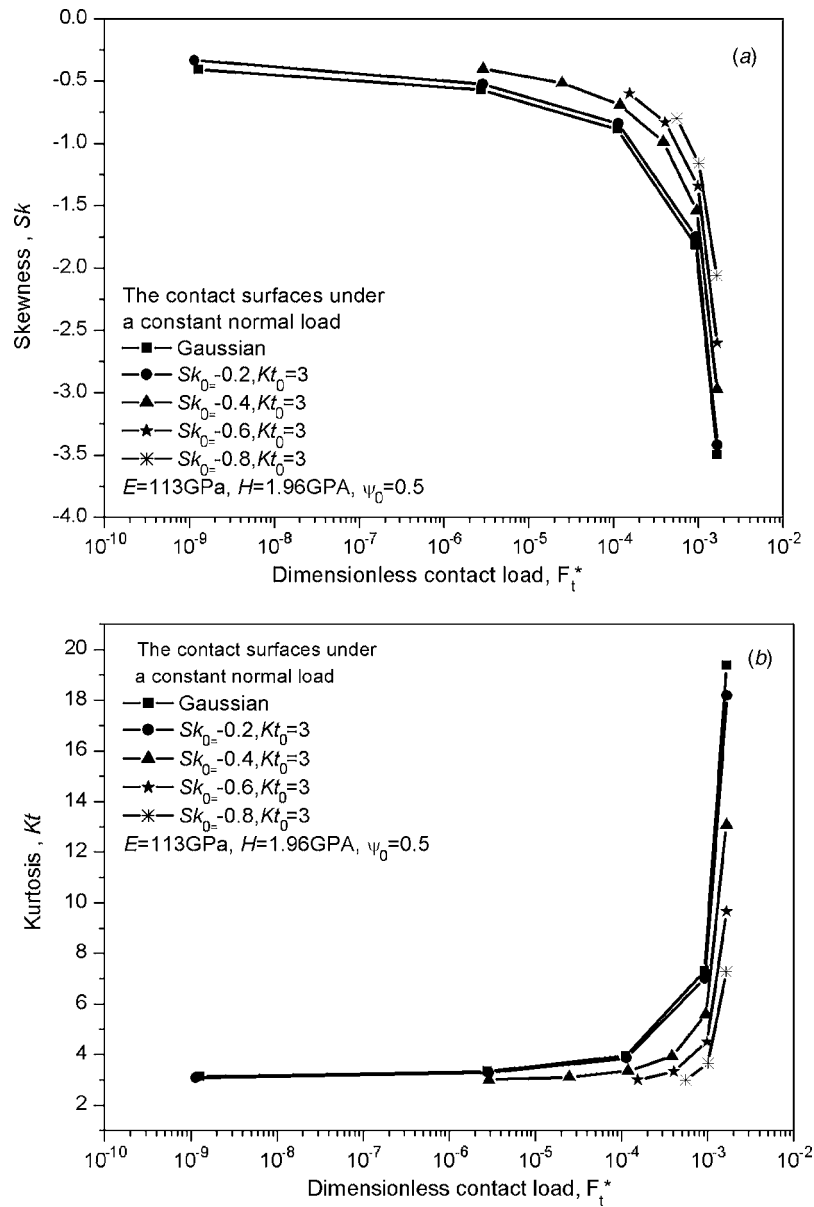


Fig. 5 Variations of (a) the skewness (Sk) and (b) the kurtosis (Kt) with the dimensionless contact load for the contact surfaces under a constant normal load. These data were evaluated by changing the initial skewness (Sk_0) only; the initial kurtosis (Kt_0) was fixed at 3.

4(c) are quite high, revealing that elastic recovery is weakened substantially by increasing the initial plasticity index (ψ_0) of the rough surface.

The topography of the rough surface will be changed when two surfaces are subjected to contact deformations. The skewness (Sk) and kurtosis (Kt) parameters in the non-Gaussian probability density function vary with the mean separation of two contact surfaces. In the present study, these two parameters can be evaluated under the condition of either two contact surfaces under a constant load or the rough surface after elastic recovery. The results of the skewness and kurtosis parameters for the condition of two contact surfaces under various constant loads are shown in Figs. 5(a) and 5(b), respectively. The solutions of the skewness parameter varying with the mean separation were evaluated for four (Sk_0, Kt_0) sets. The skewness parameter generally has a negative value; its magnitude is lowered by decreasing the dimensionless contact load. The initial kurtosis in the figure was set to be a constant

value ($Kt_0=3$), but the initial skewness (Sk_0) varied in a range of -0.2 to -0.8 . The position of the right end of a $g(z^*)$ profile is governed by the negative magnitude of the initial skewness. In the case of $Sk_0=-0.2$, the right end point of $g(z^*)$ extends to a large, positive z^* value. Therefore, the skewness value evaluated at a small dimensionless load can still be obtained. At small contact loads, the skewness curve is asymptotic to a constant value. If the Sk_0 value is changed to -0.8 , the entire $g(z^*)$ profile is shifted leftward such that the z^* value corresponding to its right end point becomes small. Therefore, the Sk curve corresponding to $Sk_0=-0.8$ is present only in a small range of contact loads. By fixing the contact loads of two surfaces, the negative magnitude of Sk for the surface contacts under a constant load is lowered by increasing the initial skewness (Sk_0) of the rough surface.

As Fig. 5(b) shows, the behavior of kurtosis (Kt) due to the change in the dimensionless contact load (F_t^*) is exactly opposite

to that demonstrated in the skewness parameter. If the initial values of Sk_0 and Kt_0 are given, the Kt value evaluated for the surface contacts under a constant normal load is lowered by decreasing the contact load of two contact surface. In the figure, the initial kurtosis (Kt_0) was set to be 3. The kurtosis value evaluated at the same contact load is lowered by increasing the negative magnitude of the initial skewness (Sk_0).

The skewness and kurtosis results obtained after elastic recovery from a rough surface with a plasticity index $\psi_0=0.5$ are shown in Figs. 6(a-1) and 6(b-1), respectively. The initial conditions of the skewness and the kurtosis are exactly the same as those given in Fig. 6. In Fig. 6(a-1), the behavior exhibited in the skewness parameter is somewhat different from that demonstrated in Fig. 5(a). If the initial skewness (Sk_0) and kurtosis (Kt_0) are fixed, the skewness (Sk) after elastic recovery, varying with the contact load, shows the same behavior as that shown in Fig. 5(a). However, the effect of the initial skewness (Sk_0) magnitude on the skewness (Sk) after elastic recovery is exactly opposite to the effect on the skewness (Sk) created at the contact of a constant load if they are evaluated at the dimensionless contact load (F_t^*). This feature is related to the magnitude of the plasticity index ψ_0 . A relatively high possibility of elastic recovery generally occurs at the contact surface with a small ψ_0 . In addition, the degree of the elastic recovery corresponding to a surface with a small initial skewness (Sk_0) is always higher than that exhibited in a surface with a large initial skewness (Sk_0). The combined effect of a small plasticity index and a surface with a small initial skewness (Sk_0) makes the skewness (Sk) (negative magnitude) after the elastic recovery be overtaken by the combined effect of the same plasticity index and a large initial skewness. The involvement of the elastic recovery is obviously the main cause of the contrasting behavior of Figs. 5(a) and 6(a-1).

The kurtosis results obtained from the rough contact surface after the elastic recovery are shown in Fig. 6(b-1). The behavior demonstrated in this figure, as to the changes in the initial skewness (Sk_0) and the contact load, is similar to that shown in Fig. 5(b), except that the magnitude of Kt shown in Fig. 6(b-1) is much smaller than that shown in Fig. 5(b) if they are evaluated at the same (F_t^*). Significant differences in magnitude between Figs. 5(b) and 6(b-1) at large (F_t^*) values can be attributed to the surface having a small plasticity index ($\psi_0=0.5$) and thus a high elastic recovery. A strong elastic recovery makes the kurtosis values (Kt) corresponding to different Sk_0 values vary in a narrow range (from about 3.0 to 3.4).

The plasticity index ψ_0 is an influential factor to the skewness and kurtosis of surface asperities after elastic recovery. Figures 6(a-2) and 6(b-2) show the results of the skewness and kurtosis predicted by $\psi_0=2.0$, respectively. A large ψ_0 value means a rough surface with a characteristic that only a small portion of the entire probability density function is related to the elastic recovery. Therefore, the difference in the magnitude between the skewness during the contact of a constant load (see Fig. 5(a)) and the skewness obtained after the elastic recovery (see Fig. 6(a-2)) is rather limited. At a fixed dimensionless contact load (F_t^*), the skewness of a rough surface after the elastic recovery is lowered by increasing the initial skewness. This feature is similar to that demonstrated in the contacts under a constant normal load, but is exactly opposite to the behavior shown in Fig. 6(a-1) for $\psi_0=0.5$. Figure 6(b-2) shows the kurtosis results evaluated at various dimensionless contact loads. Due to its small effect on the elastic recovery, the behavior and magnitude exhibited in the kurtosis parameter are quite similar to those shown in the contact under a constant normal load.

Figures 7(a-1) and 7(b-1) show the contact loads obtained using the model, of constant-skewness and constant-kurtosis assumptions and the present model, respectively, evaluated at various mean separations. There are great differences between these two figures. The initial plasticity index (ψ_0) in each of these two figures

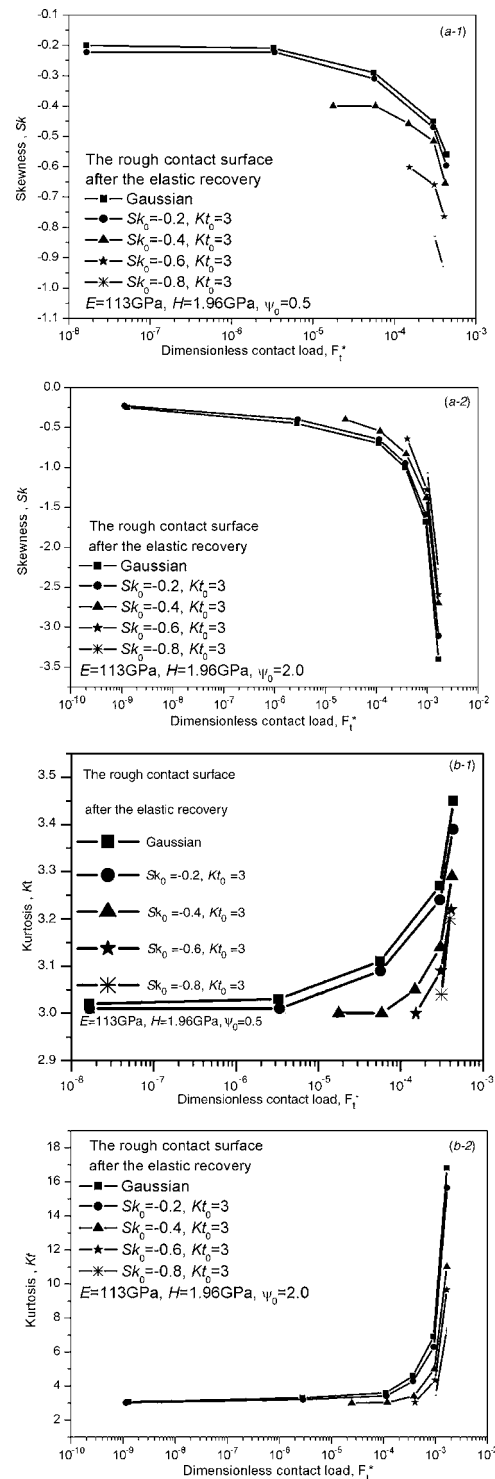


Fig. 6 Variations of (a) the skewness (Sk) and (b) the kurtosis (Kt) with the dimensionless contact load for the contact surfaces after completing the elastic recovery. These data were evaluated by changing the initial skewness (Sk_0) only; the initial kurtosis (Kt_0) was fixed at 3, and the initial plasticity indices used in the evaluation were 0.5 and 2.0, respectively.

is fixed at a value of 0.5. If constant skewness and constant kurtosis are assumed for the asperity heights (see Fig. 7(a-1)), significant differences in the contact load due to the difference in the initial skewness (Sk_0) are present in the region of large mean separations (d^*). As the mean surface separation decreases, the

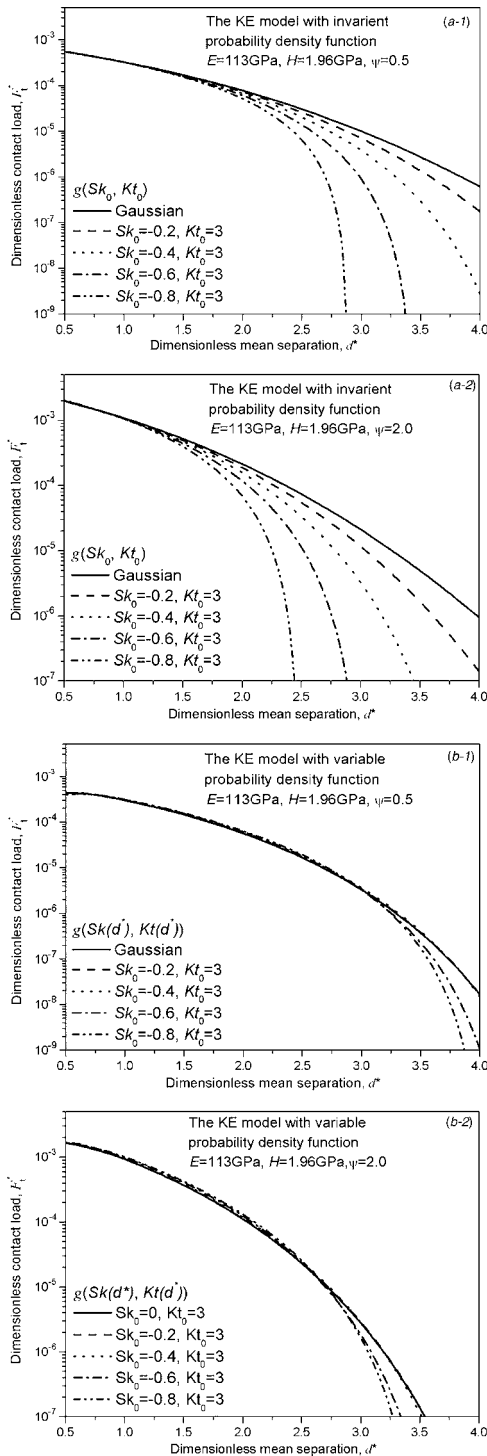


Fig. 7 Variations of the dimensionless contact load with the dimensionless mean surface separation. They are presented based on different skewness and kurtosis values before the roughness surface contacts. (a) Probability density functions $g(z^*)$ are invariant. (b) Probability density functions $g(z^*)$ are varied with dimensionless mean separation d^* . The initial plasticity indices used in the evaluation were 0.5 and 2.0, respectively.

differences among these five contact loads are also decreased. The differences become negligibly small when the mean separation (d^*) is lowered to a sufficiently small value. This behavior implies that the contact behavior arising at a sufficiently small mean surface separation is little dependent on the initial skewness and kur-

tosis of asperity heights. Nevertheless, the contact load predicted by a large initial (negative) skewness is always smaller than the contact load with a small initial (negative) skewness if they are evaluated at the same mean surface separation. As Fig. 7(b-1) shows, the magnitude sequence of the contact loads shown in large mean separation is opposite to the behavior exhibited in small mean separations. In the region of large mean separations, the load sequence shows that $(F_t^*)_{Sk_0=0} > (F_t^*)_{Sk_0=-0.2} > (F_t^*)_{Sk_0=-0.4} > (F_t^*)_{Sk_0=-0.6} > (F_t^*)_{Sk_0=-0.8}$. Conversely, the load sequence for mean separations varying in the range of about 1–3 shows that $(F_t^*)_{Sk_0=0} < (F_t^*)_{Sk_0=-0.2} < (F_t^*)_{Sk_0=-0.4} < (F_t^*)_{Sk_0=-0.6} < (F_t^*)_{Sk_0=-0.8}$. Nevertheless, the contact load evaluations based on these five initial skewnesses are almost asymptotic to the same constant value as the mean separation (d^*) is further reduced to a value smaller than 3. A constant contact load can be interpreted as the contact surfaces at a sufficiently small mean separation showing the same topography, making it independent of the initial skewness. The results in Fig. 7(a-1) show that the influence of the initial skewness on the contact load is significant only in the region of sufficiently large mean separations. The results in Fig. 7(b-1), however, show that the significance of the initial skewness on the contact load solution is enhanced as the mean separation is reduced to be sufficiently small. In the region of large mean separations, the differences in the contact load among these differences in the initial skewness are substantially narrowed when the present model is applied.

Figures 7(a-2) and 7(b-2) show the contact load solutions obtained under the same conditions, except for the plasticity index. The behavior demonstrated in Figs. 7(a-2) and 7(b-2) is basically similar to that exhibited in Figs. 7(a-1) and 7(b-1), respectively. However, the magnitudes of contact load shown at the same mean surface separation give the sequence $(F_t^*)_{\psi=0.5} < (F_t^*)_{\psi=2.0}$ if they are obtained in the subregion of small mean separations; however, the sequence $(F_t^*)_{\psi=0.5} > (F_t^*)_{\psi=2.0}$ is obtained in the subregion of large mean separations.

Figure 8(a) shows the contact load-area relationships evaluated at five different initial skewnesses; the initial plasticity index (ψ_0) applied in these evaluations is given a value of 0.5. The results in this figure were obtained by assuming constant skewness as well as constant kurtosis over the entire mean surface separation. Under the condition of having the same contact load, the contact areas with different values of the skewness of asperity heights show the sequence $(A_t^*)_{Sk_0=0} < (A_t^*)_{Sk_0=-0.2} < (A_t^*)_{Sk_0=-0.4} < (A_t^*)_{Sk_0=-0.6} < (A_t^*)_{Sk_0=-0.8}$. If the contact area results are evaluated using the present model, the behavior demonstrated in the contact area sequence due to the change in the initial skewness, as Fig. 8(b) shows, is the same as that shown in Fig. 8(a) if the contact load is fixed. However, the influence of different initial skewnesses on the contact load or the contact area exhibited in the present model is obviously smaller than that exhibited in the case of assuming constant skewness and constant kurtosis.

4 Conclusions

The probability density function ($g(z^*)$) of surface asperities strongly depends on the evaluations under the condition of either the surface contacts under a normal load or a surface obtained after elastic recovery. The initial skewness (Sk_0) and the initial kurtosis (Kt_0) are the governing factors as to the probability density function formed under a normal load as well as at a fixed mean separation, whereas the initial skewness and kurtosis and the plasticity index (ψ_0) are the governing factors of the $g(z^*)$ function defined for a surface after elastic recovery. The $g(z^*)$ profile is quite sharp in appearance and has a large value at its peak if it is obtained from the surface contacts under a constant normal load. The $g(z^*)$ profile obtained from the contact surface after elastic recovery is, however, dependent on the plasticity index (ψ_0) of the rough surface. The $g(z^*)$ profile defined for the contact

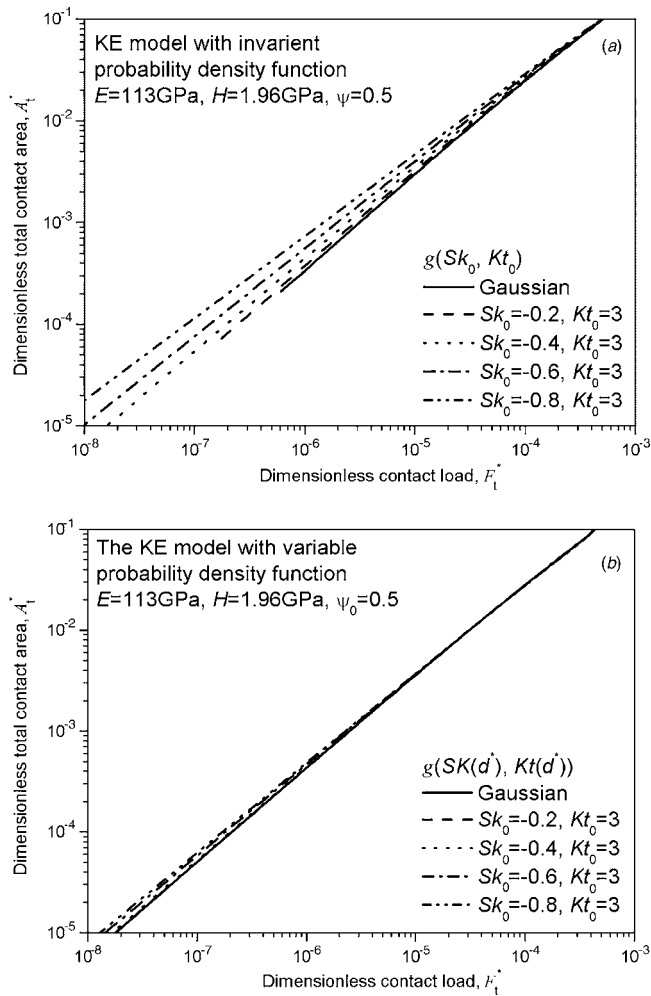


Fig. 8 Variations of the dimensionless total contact area with the dimensionless contact load. They are presented based on different skewness and kurtosis values before the roughness surface contacts. (a) Probability density functions $g(z^*)$ are invariant. (b) Probability density functions $g(z^*)$ are varied with dimensionless mean separation d^* .

surface after elastic recovery is similar to the $g(z^*)$ profile before experiencing contact deformations if the plasticity index is assumed to be small. However, the $g(z^*)$ profile for the contact surface after elastic recovery is similar to the $g(z^*)$ profile shown in the contacts under a normal constant load if a large plasticity index is assumed.

For the contact surface after elastic recovery, the influence of the mean separation of two contact surfaces on the $g(z^*)$ function is governed by the plasticity index (ψ_0) of the rough surface. The $g(z^*)$ profiles obtained at different mean separations are quite close if a small ψ_0 is assumed. However, the $g(z^*)$ profiles obtained at different mean separations vary significantly if a large ψ_0 is given. Big differences were found between the $g(z^*)$ function defined for the rough surface before any contact and the $g(z^*)$ function given for the contact surface after elastic recovery. These differences are enlarged by decreasing the mean separation.

The skewness (Sk) and the kurtosis (Kt) in the probability density function ($g(z^*)$) are expressed as a function of the initial skewness (Sk_0), the initial kurtosis (Kt_0), and the mean separation (d^*) of two contact surfaces if they are evaluated under a constant normal load. Apart from the above three factors, the plasticity index ψ_0 is needed if Sk and Kt are evaluated for the contact

surface after elastic recovery. If Sk_0 and Kt_0 are fixed in the evaluations, the skewness (Sk) is elevated by decreasing the contact load, and the kurtosis (Kt) is lowered by decreasing the contact load, no matter whether they are obtained at the condition of surface contacts under a constant normal load or the contact surface after the elastic recovery.

Comparison of the results predicted by the model of a single probability density function over the given interference and the present model reveals that both the contact load and the total contact area are overestimated if the model of a single probability density function is applied.

Nomenclature

- a = contact area of an asperity
- A = real area of contact
- A_n = apparent area
- d = separation based on asperity heights
- E = effective Young's modulus
- F = contact load
- g = probability density function of asperity heights
- h = separation based on surface heights
- H = hardness of the softer material
- K = maximum contact pressure factor
- Kt = kurtosis
- N = total number of asperities
- P = mean contact pressure
- R = radius of curvature of an asperity
- Sk = skewness
- y_s = distance between the mean of asperity heights and that of surface heights
- z = height of asperity measured from the mean of asperity heights
- η = area density of asperities
- δ = interference
- σ = standard deviation of surface height
- σ_s = standard deviation of asperity height
- ν = Poisson's ratio
- ψ = plasticity index

Subscripts

- 0 = initial value before asperity deformation occurs
- u = properties corresponding to the rough surfaces after elastic recovery
- d = properties corresponding to the rough surfaces on loading when the mean separation is d
- c = critical value at the point of initial yield
- e = elastic deformation
- ep = elastoplastic deformation
- p = fully plastic deformation
- t = total summation
- $*$ = dimensionless

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Extended Green's Solution for the Stresses in an Infinite Plate With Two Equal or Unequal Circular Holes

The distribution of stress in an isotropic and infinitely large plate perforated by circular holes has long attracted attention from both mathematical and engineering standpoints. Unfortunately, almost all existing solutions are only applicable to stress-free conditions at the boundary of the holes, which is not always the case in engineering applications. In an attempt to cover a wider range of applications, this paper presents the exact explicit solution for the stress distribution in an infinite plate containing two equal/unequal circular holes subjected to general in-plane stresses at infinity and internal pressures inside the holes, following the approach proposed by Green (1940, "General Bi-Harmonic Analysis for a Plate Containing Circular Holes," Proc. R. Soc. London, Ser. A, 176(964), pp. 121–139). The newly derived general solution has been verified not only with published solutions for special cases but also qualitatively with a comparable experimental testing program. In addition, some numerical examples are also provided to offer insight into the complexity of the interplay of parameters.

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Keywords: infinite plate, infinite plane, stress distribution, circular holes, elasticity, analytical solution

Introduction

The distribution of stress in an infinitely large plate perforated by circular holes has been studied by many investigators through different approaches. In 1921, Jeffery [1] pioneered the theory for plane stress and plane strain in bipolar coordinates, which was used by Ling [2] in 1948 to solve the problem of an infinite plate containing two equal circular holes under general far-field stresses. In 1980, Iwaki and Miyao [3] expanded Ling's solution to two unequal holes in an infinite plate under far-field tension and internal pressure or uniform shear on the walls of the holes. For more than two equal circular holes, Howland and Knight [4] solved a class of periodic configurations in 1939, taking advantage of the invariance of some coordinate transformations. One year later, Green [5] discovered a general method using coordinate transformation to calculate the stress distribution in an infinitely large plate containing any number of holes of any size under stresses applied at infinity. Although Green's solution is analytically powerful, it was not accompanied by many numerical examples and results because of the limits of the computational power of the day. The conformal mapping technique was also

used successfully by Haddon [6] to solve for two equal/unequal holes under uniaxial tension. The Schwarz alternating method of successive approximations has also been used successfully by Ting et al. [7], Ukadgaonker and Patil [8], and many others to solve for various configurations. A comprehensive treatise on the subject was compiled by Savin [9] in 1961. Unfortunately, except for the work of Iwaki and Miyao [3], all of the existing exact solutions are only applicable to stress-free conditions at the boundaries of the holes, which is not always the case in engineering problems. In an attempt to cover a wider range of applications, this paper seeks the closed-form solution to the problem of an infinite plate perforated by two equal/unequal circular holes subjected to general in-plane stresses at infinity and internal pressures inside the holes. Potential applications of the solution range from the classical structural integrity analysis of perforated plates, connections of cylindrical pipes to a large cylindrical pressure vessel, all the way to the stability analysis for multilateral junctions. Although only pressure is considered for the boundary conditions in this study, the method can be readily extended to other boundary conditions by minor changes as discussed in the Solution Method section.

Problem Description

A schematic of the problem is presented in Fig. 1. The two circular holes have centers O_1 and O_2 , and radii a_1 and a_2 , respec-

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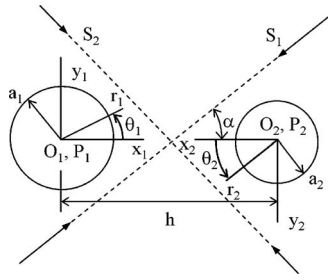


Fig. 1 Problem setup

tively. The distance from center to center between the holes is denoted as h . The pressures inside the holes are denoted by P_1 and P_2 . S_1 and S_2 are the applied major and minor principal stresses at infinity; α is the angle S_1 makes with the center-to-center line. A local right-handed Cartesian coordinate system is attached to each hole, with the origin coincides with the center of the hole, and the x -axis aligned along the center-to-center line with positive direction pointing toward the other hole, as shown in Fig. 1. A local polar coordinate system is also attached to each hole. Nondimensional polar coordinates and important nondimensional parameters are defined as follows:

$$\rho_1 = \frac{r_1}{h} \quad \rho_2 = \frac{r_2}{h} \quad (1)$$

$$\lambda_1 = \frac{a_1}{h} \quad \lambda_2 = \frac{a_2}{h} \quad (2)$$

The solution method is presented next using a notation similar to Green's [5].

Solution Method

If there were no holes in the plate, the stresses could be derived from an Airy stress function ϕ_0 . To account for the effects of the two holes, an additional set of Airy stress functions is added for each hole. These stress functions must produce zero stresses at infinity and single-valued stresses and displacements everywhere on the plate. The resultant stress function is expanded around each hole. Stresses are then derived and unknown coefficients are calculated using boundary conditions.

The given in-situ stresses are decomposed into components S_{xx} , S_{yy} , and S_{xy} . Using the compression positive convention, the Airy stress function ϕ_0 for Hole 1 is then represented as follows:

$$\phi_0 = h^2 \left[P_1 \lambda_1^2 \ln \rho_1 + P_2 \lambda_2^2 \ln \rho_2 + \frac{S_{xx}}{2} \rho_1^2 \sin^2 \theta_1 + \frac{S_{yy}}{2} \rho_1^2 \cos^2 \theta_1 + \frac{S_{xy}}{2} \rho_1^2 \sin 2\theta_1 \right] \quad (3)$$

The additional Airy stress functions must produce zero stresses at infinity and single-valued stresses and displacements. They are therefore of the form

$$\ln \rho$$

$$\theta$$

$$\rho^{-n} \cos n\theta \quad \text{for } n = 1, 2, \dots$$

$$\rho^{-n} \sin n\theta \quad \text{for } n = 1, 2, \dots$$

$$\rho^{-n} \cos(n+2)\theta \quad \text{for } n = 0, 1, 2, \dots$$

$$\rho^{-n} \sin(n+2)\theta \quad \text{for } n = 0, 1, 2, \dots$$

The additional set of stress functions for Hole 1 is a linear combination of the preceding functions.

$$\begin{aligned} \phi_1 = & {}^1A_0 \ln \rho_1 + {}^1B_0 \theta_1 + \sum_{n=1}^{\infty} [{}^1A_n \rho_1^{-n} \cos n\theta_1 + {}^1B_n \rho_1^{-n} \sin n\theta_1] \\ & + \sum_{n=0}^{\infty} [{}^1C_n \rho_1^{-n} \cos(n+2)\theta_1 + {}^1D_n \rho_1^{-n} \sin(n+2)\theta_1] \end{aligned} \quad (4)$$

The additional stress function for Hole 2 is obtained similarly,

$$\begin{aligned} \phi_2 = & {}^2A_0 \ln \rho_2 + {}^2B_0 \theta_2 + \sum_{n=1}^{\infty} [{}^2A_n \rho_2^{-n} \cos n\theta_2 + {}^2B_n \rho_2^{-n} \sin n\theta_2] \\ & + \sum_{n=0}^{\infty} [{}^2C_n \rho_2^{-n} \cos(n+2)\theta_2 + {}^2D_n \rho_2^{-n} \sin(n+2)\theta_2] \end{aligned} \quad (5)$$

where ${}^1A_n$, ${}^1B_n$, ${}^1C_n$, ${}^1D_n$, ${}^2A_n$, ${}^2B_n$, ${}^2C_n$, and ${}^2D_n$ ($n = 0, 1, 2, \dots$) are coefficients determined from boundary conditions.

The final Airy stress function for the full solution is the superposition of individual Airy stress functions,

$$\phi_f = \phi_0 + \phi_1 + \phi_2 \quad (6)$$

This function can be expressed in terms of ρ_1 and θ_1 using coordinate transformation formulas presented in Appendix A. The stresses around Hole 1 are then derived as follows:

$$\sigma_{rr} = \frac{1}{h^2} \left(\frac{1}{\rho_1} \frac{\partial \phi_f}{\partial \rho_1} + \frac{1}{\rho_1^2} \frac{\partial^2 \phi_f}{\partial \theta_1^2} \right) \quad (7)$$

$$\sigma_{\theta\theta} = \frac{1}{h^2} \frac{\partial^2 \phi_f}{\partial \rho_1^2} \quad (8)$$

$$\sigma_{r\theta} = -\frac{1}{h^2} \frac{\partial}{\partial \rho_1} \left(\frac{1}{\rho_1} \frac{\partial \phi_f}{\partial \theta_1} \right) \quad (9)$$

The boundary conditions at infinity are automatically satisfied. At the boundary of Hole 1, the radial stress must equal the applied pressure and the shear stress must vanish.

$$\sigma_{rr}|_{\rho_1=\lambda_1} = P_1 \quad (10)$$

$$\sigma_{r\theta}|_{\rho_1=\lambda_1} = 0 \quad (11)$$

Similarly, the final Airy stress function ϕ_f can be expressed in terms of ρ_2 and θ_2 . The stresses for Hole 2 are derived using equations similar to Eqs. (7)–(9). The boundary conditions at the boundary of Hole 2 are as follows:

$$\sigma_{rr}|_{\rho_2=\lambda_2} = P_2 \quad (12)$$

$$\sigma_{r\theta}|_{\rho_2=\lambda_2} = 0 \quad (13)$$

Equations (10)–(13) are solved simultaneously for the coefficients ${}^1A_n$, ${}^1B_n$, ${}^1C_n$, ${}^1D_n$, ${}^2A_n$, ${}^2B_n$, ${}^2C_n$, and ${}^2D_n$. Detailed derivation is presented in Appendix B. Although this solution covers only the boundary condition of internal pressures inside the holes, its method can be readily applied to other boundary conditions by using the appropriate function ϕ_0 and modifying the boundary equations (10)–(13).

Theoretical Verification

Equal Holes. The analytical solution for an infinite plate with two equal circular holes under general plane stresses has been derived by Ling [2] in 1948. Ling's solution can be regarded as a special case when both pressures are zero and when both circular holes are of equal size. Tables 1 and 2 show that the two solutions

Table 1 Ling's [2] tangential stress at the borehole wall. λ is defined as the ratio of the center-to-center distance and the diameter

Ling's λ	$S_H=S_h=1, \alpha=0 \text{ deg}$		$S_H=1, S_h=0, \alpha=0 \text{ deg}$ $\theta=\pm 90 \text{ deg}$	$S_H=1, S_h=0, \alpha=90 \text{ deg}$	
	$\theta=0 \text{ deg}$	$\theta=180 \text{ deg}$		$\theta=0 \text{ deg}$	$\theta=180 \text{ deg}$
1.5	2.255	2.887	2.623	3.151	3.264
2	2.158	2.411	2.703	3.066	3.020
3	2.080	2.155	2.825	3.020	2.992
5	2.033	2.049	2.927	3.004	2.997
8	2.014	2.018	2.970	3.001	2.999
Infinity	2.000	2.000	3.000	3.000	3.000

yield identical results for the tangential stress at the hole boundary. It is also evident from the last line of Table 2 that the newly derived solution converges to classical Kirsch's solution [10] for a single hole when the two holes are very far apart.

Ling [2] also provides tangential stress concentration factors when the two circular holes are tangential to each other. Unfortunately, those factors cannot be verified because the series in the newly derived solution do not converge when the two holes are tangential. However, this limitation does not hinder the application because tangential holes are physically impossible.

Bargui and Abousleiman [11] have also investigated Ling's solution for the tangential stress distribution at the boundary of the holes for different states of nonhydrostatic applied stress. Figure 2 shows the stress distribution for seven different stress anisotropies. The results are identical to those from Bargui and Abousleiman [11].

Unequal Holes. The analytical solution for an infinite plane with two unequal circular holes under uniaxial plane-stress loading has been derived by Haddon [6] using a conformal mapping technique. Haddon's solution applies when there is no pressure inside the holes and when one principal stress at infinity is zero. Table 3 shows that the two solutions yield identical results for the tangential stress at the hole boundary for 44/48 locations. Minor differences in the third decimal place at the other four locations could be because Haddon reportedly used an accuracy of 10^{-4} while the new results were calculated with an accuracy of 10^{-6} before rounding off for comparison.

The solution for two unequal holes under uniaxial tension at infinity, internal pressure, or uniform shear on the boundary of one hole has also been derived by Iwaki and Miyao [3] in 1980 using bipolar coordinates. Figure 3 shows the maximum tangential stress on the boundary of the holes obtained from the newly derived solution when there is only internal pressure inside Hole 2. The results are consistent with those reported by Iwaki and Miyao [3]. Iwaki and Miyao's solution is equivalent to the newly derived solution but cannot be extended to more than two circular holes in the plane. On the other hand, this solution method can be extended to cover any number of holes of any size, as shown originally by Green [5] in 1940.

Experimental Validation

To date, mechanical stability of a multilateral junction is still one of the most challenging problems in the petroleum industry. A multilateral well consists of a main or a mother wellbore with one or more deviating wellbore branches. These laterals or secondary wellbores are drilled to enhance the well productivity by increasing the drainage area and/or producing from isolated formations. Due to geomechanical effects, the second or even the main wellbore can be lost during drilling, completion, or production, leading to enormous losses and delays in production schedule. Due to the small angle between the two wellbores, the multilateral junction problem has been commonly modeled using two circular holes in an infinite plane, assuming that the plane-strain condition prevails [11,12]. Applied stresses include overburden stress, far-field horizontal principal stresses, and mud pressures inside the wellbores.

Papanastasiou et al. [13,14] have conducted a laboratory testing program on multilateral junctions. Figure 4 shows the breakout shapes they obtained.

Experiment 2 is expected to yield the same results as Experiment 1. Experiment 6 involves only a single borehole; therefore, it is not relevant to this study. Based on the data supplied by Papanastasiou et al. [13,14], the following configuration is used to verify Experiments 1 and 3–5:

$$a_1 = 18.5 \text{ mm}$$

$$a_2 = 15.5 \text{ mm}$$

$$h = 44 \text{ mm}$$

Figures 5–8 demonstrate the corresponding distribution of major principal stress simulating the aforementioned experiments. The experimental results are not accurate enough to make quantitative comparison. However, the stress distributions generated using the newly derived solution show excellent qualitative agreement with the reported breakout shapes.

Table 2 New solution shows identical results to Ling's solution

h ($a_1=a_2=1$)	$S_H=S_h=1, \alpha=0 \text{ deg}$		$S_H=1, S_h=0, \alpha=0 \text{ deg}$ $\theta=\pm 90 \text{ deg}$	$S_H=1, S_h=0, \alpha=90 \text{ deg}$	
	$\theta=0 \text{ deg}$	$\theta=180 \text{ deg}$		$\theta=0 \text{ deg}$	$\theta=180 \text{ deg}$
3	2.255	2.887	2.623	3.151	3.264
4	2.158	2.411	2.703	3.066	3.020
6	2.080	2.155	2.825	3.020	2.992
10	2.033	2.049	2.927	3.004	2.997
16	2.014	2.018	2.970	3.001	2.999
200	2.000	2.000	3.000	3.000	3.000

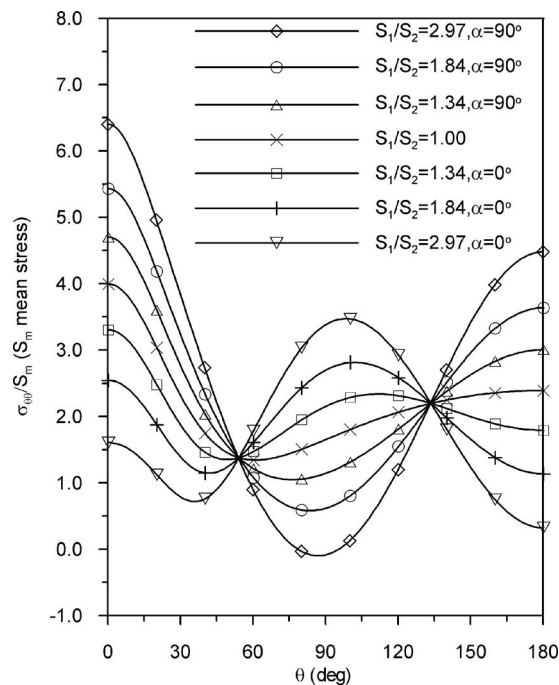


Fig. 2 Tangential stress at the boundary of either hole for different stress anisotropy (Ling's solution for $\lambda=1.20$), identical to the results by Bargui and Abousleiman [11]

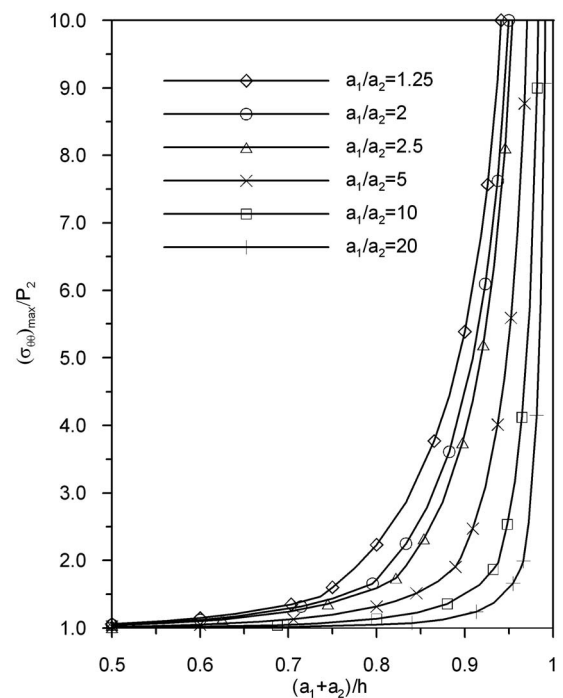


Fig. 3 Maximum tangential stress at the boundary of the holes when only Hole 2 is pressurized (identical to the results by Iwaki and Miyao [3])

Numerical Results

A simple example is investigated to demonstrate the new solution. The parameters are selected as follows: $S_1=1$, $S_2=0.8$, P_1

Table 3 Comparison with Haddon's solution [6]. The parameters are $h=4.5$, $a_1=2.5$, $a_2=1$, $S_{\max}=1$, $S_{\min}=0$, $P_1=P_2=0$, and $\alpha=45$ deg. Differing results are highlighted

θ (deg)	Tangential stress, Hole 1		Tangential stress, Hole 2	
	Haddon	Hoang and Abousleiman	Haddon	Hoang and Abousleiman
0	1.273	1.272	2.089	2.089
15	-0.694	-0.694	-0.067	-0.067
30	-0.711	-0.711	-1.563	-1.563
45	-0.710	-0.710	-1.932	-1.932
60	-0.464	-0.464	-1.335	-1.335
75	0.207	0.207	-0.137	-0.137
90	1.159	1.159	1.283	1.283
105	2.128	2.128	2.585	2.585
120	2.840	2.840	3.489	3.489
135	3.092	3.092	3.812	3.812
150	2.811	2.811	3.501	3.501
165	2.065	2.065	2.639	2.638
180	1.049	1.049	1.432	1.432
195	0.029	0.029	0.159	0.159
210	-0.728	-0.728	-0.902	-0.902
225	-1.030	-1.030	-1.531	-1.531
240	-0.808	-0.808	-1.618	-1.618
255	-0.138	-0.138	-1.170	-1.170
270	0.770	0.776	-0.282	-0.282
285	1.667	1.667	0.900	0.900
300	2.289	2.289	2.209	2.209
315	2.577	2.577	3.408	3.408
330	2.862	2.862	4.101	4.101
345	3.262	3.262	3.736	3.735

$=P_2=0.2$, and $a_1=1$. Only the distribution of tangential stress at the boundary of the holes is presented because this is the most important stress at the most critical locations.

Effects of Orientation of the Stress Field. Consider two holes of equal size, namely, $a_2=1$. Four orientations of the stress field are investigated: $\alpha=0$ deg, $\alpha=30$ deg, $\alpha=60$ deg, and $\alpha=90$ deg. The first orientation implies that the center-to-center line aligns with the maximum principal stress. The last case applies when the center-to-center line aligns with the minimum principal stress. The other two cases consider intermediate orientations.

Because of symmetry, the two holes have the same stress distribution. Figure 9 shows the tangential stress at the boundary of either hole for $\alpha=0$ deg for various separation distances of 0.2, 0.5, 1.0, and 2.0. The corresponding center-to-center distances are 2.2, 2.5, 3.0, and 4.0, respectively. The solution for a single hole is also presented as the base line for comparison. Interestingly, h

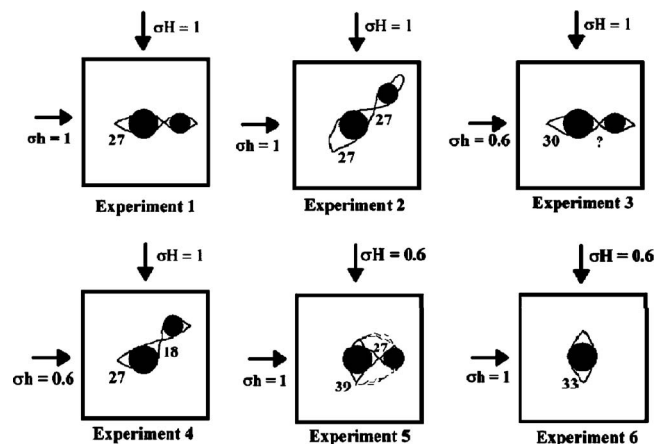


Fig. 4 Sketch of breakout shapes. Reproduced from Papanastasiou et al. [13,14]

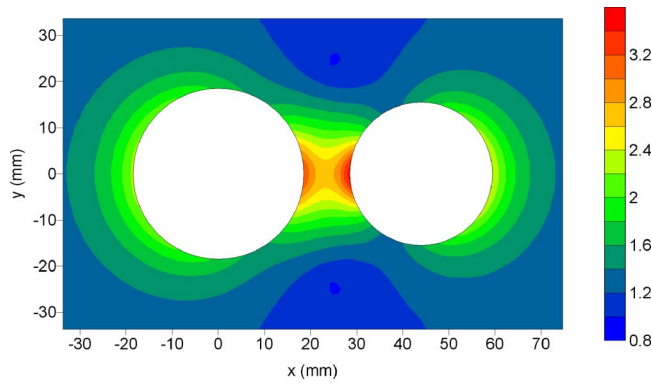


Fig. 5 Distribution of major principal stress reproducing the stress pattern in Fig. 4, Experiment 1: $S_H=1$, $S_h=1$, $\alpha=0$ deg, $P_1=P_2=0$

$=4.0$ and $h=3.0$ yield lower stress concentration factors compared to the single-hole solution. In other words, the maximum tangential stress at the hole boundary does not always increase with decreasing separation distance between the holes. As the separation distance decreases, the maximum initially at $\theta=90$ deg drifts away from the other hole (toward $\theta=180$ deg). High stress concentration also starts to develop around $\theta=0$ deg. For separation distances smaller than approximately 1, the area around $\theta=0$ deg becomes the critical region. For a separation distance of

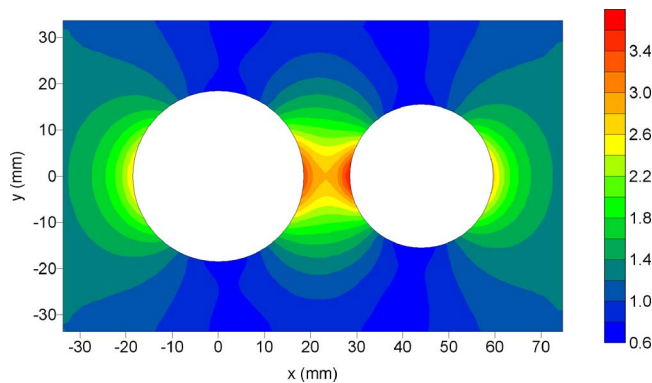


Fig. 6 Distribution of major principal stress reproducing the stress pattern in Fig. 4, Experiment 3: $S_H=1$, $S_h=0.6$, $\alpha=90$ deg, $P_1=P_2=0$

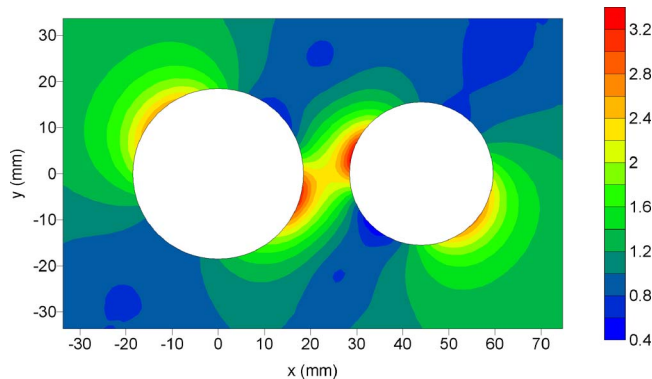


Fig. 7 Distribution of major principal stress reproducing the stress pattern in Fig. 4, Experiment 4: $S_H=1$, $S_h=0.6$, $\alpha=45$ deg, $P_1=P_2=0$

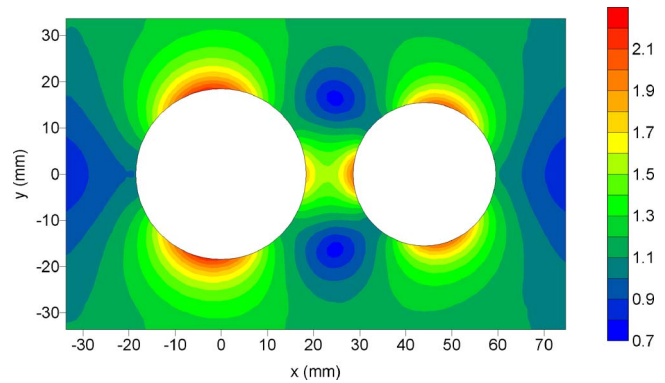


Fig. 8 Distribution of major principal stress reproducing the stress pattern in Fig. 4, Experiment 5: $S_H=1$, $S_h=0.6$, $\alpha=0$ deg, $P_1=P_2=0$

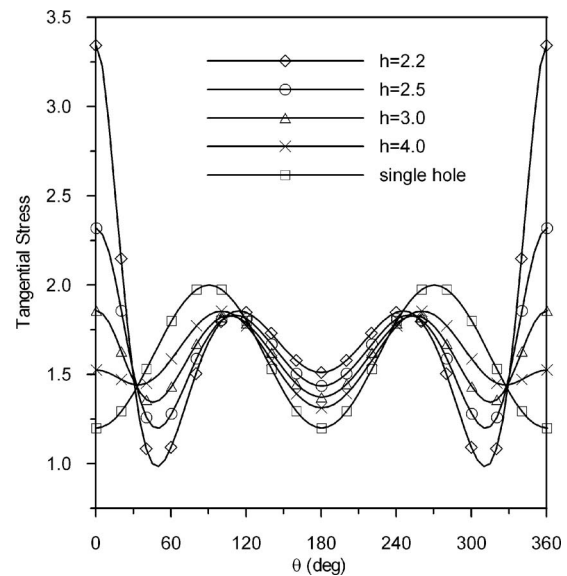


Fig. 9 Tangential stress at the boundary of either hole for $a_1=a_2=1$, $S_1=1$, $S_2=0.8$, $\alpha=0$ deg, $P_1=P_2=0.2$

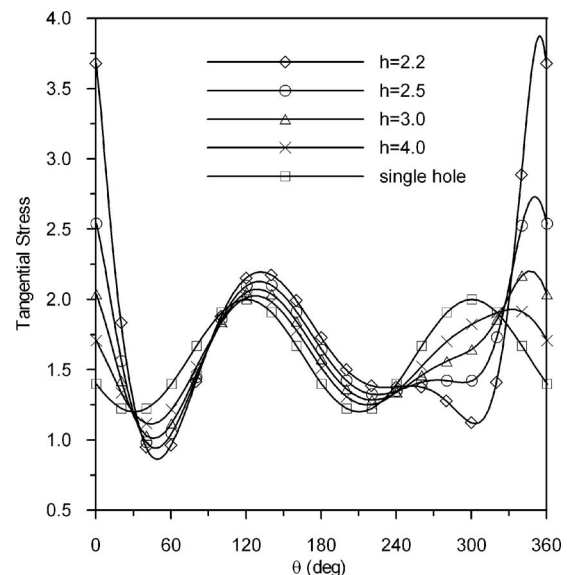


Fig. 10 Tangential stress at the boundary of either hole for $a_1=a_2=1$, $S_1=1$, $S_2=0.8$, $\alpha=30$ deg, $P_1=P_2=0.2$

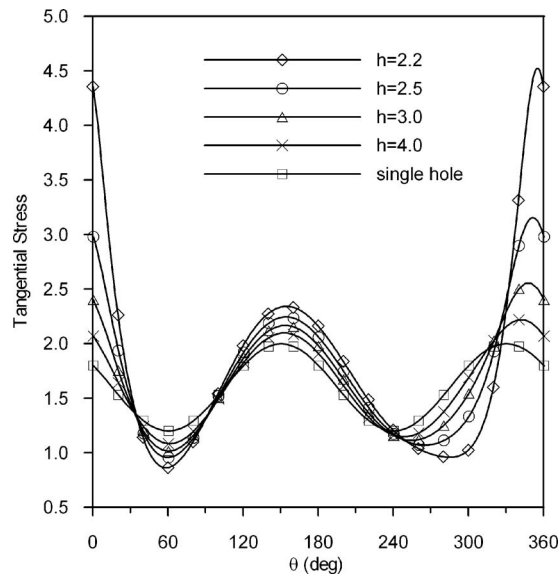


Fig. 11 Tangential stress at the boundary of either hole for $a_1=a_2=1$, $S_1=1$, $S_2=0.8$, $\alpha=60$ deg, $P_1=P_2=0.2$

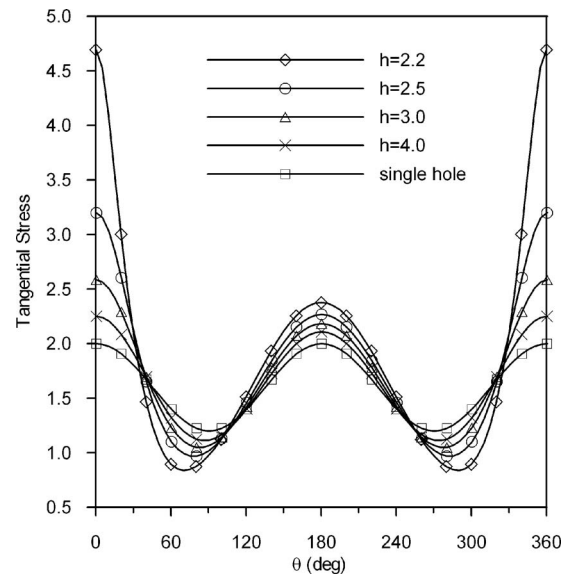


Fig. 12 Tangential stress at the boundary of either hole for $a_1=a_2=1$, $S_1=1$, $S_2=0.8$, $\alpha=90$ deg, $P_1=P_2=0.2$

0.5 ($h=2.5$), the maximum tangential stress at the boundary is 15.9% higher than the maximum value of 2 for the single-hole solution. For $h=2.2$, the maximum tangential stress increases by 67.2%.

Figure 10 demonstrates the tangential stress at the boundary of either hole for $\alpha=30$ deg for the same set of separation distances. The maximum stress initially at $\theta=90$ deg moves toward $\theta=180$ deg and increases in magnitude. At the same time, the maximum value initially at $\theta=270$ deg moves toward $\theta=0$ deg. The variation of stress distribution around $\theta=270$ deg is very complex. For any of the separation distances considered, the maximum tangential stress at the boundary of the holes for $\alpha=30$ deg is always higher than that for $\alpha=0$ deg. For instance, with $h=2.5$, the maximum value is 36.5% higher compared to the case of a single hole. For $h=2.2$, the increase in maximum tangential stress is 93.7%.

Figure 11 illustrates the stress distribution for $\alpha=60$ deg, with even higher stress concentration than for $\alpha=30$ deg. When $h=2.5$, the maximum tangential stress at the boundary increases 57.7% compared to that of a single hole. For a separation distance of 0.2, the maximum value reaches 126.2% higher than that of the single-hole solution.

The case of $\alpha=90$ deg is presented in Fig. 12. The maximum tangential stress at the hole boundary is at $\theta=0$ deg for all separation distances considered. For a separation distance of 0.5, the maximum value increases by 60.0% compared to the case of a single hole. For a separation distance of 0.2, the corresponding figure is 134.7%.

In short, the stress distribution varies significantly with the orientation of the applied stresses. Interpolating from the preceding cases, the plate is most stable when the center-to-center line aligns with the maximum principal stress, which has been observed by Bargui and Abousleiman [11].

Effects of Relative Ratio of Applied Stresses. Two lower values of minor principal stress are considered in this analysis, $S_2=0.7$ and $S_2=0.6$. The angle α is selected to be zero to minimize stress concentration based on the conclusions drawn from previous analysis. All other parameters are kept unchanged.

Figure 13 shows the tangential stress at the boundary of either hole for $S_2=0.7$ for the same separation distances. The trends observed in Fig. 9 for $S_2=0.8$ are repeated but with less pronounced effects for decreasing center-to-center distance. For h

$=2.5$, the maximum tangential stress is still lower than that for a single hole. For a separation distance of 0.2, the maximum value is only 30.1% higher compared to the case of a single hole.

Figure 14 depicts the distribution of tangential stress for $S_2=0.6$. Again, the same trends are observed but with even lower stress concentration. For $h=2.2$, the maximum tangential stress is still lower than that of the single-hole solution. In other words, the two holes help stabilize each other; two holes together are more stable than one single hole for all four separation distances considered.

Effects of Pressures Inside the Holes. Two lower pressures in the second hole are considered, $P_2=0.1$ and $P_2=0$. The angle α is still chosen to be zero. All other parameters are kept unchanged.

Because the two holes no longer have the same stress distribution, they are considered separately in this analysis. Figure 15 presents the tangential stress at the boundary of the first hole for

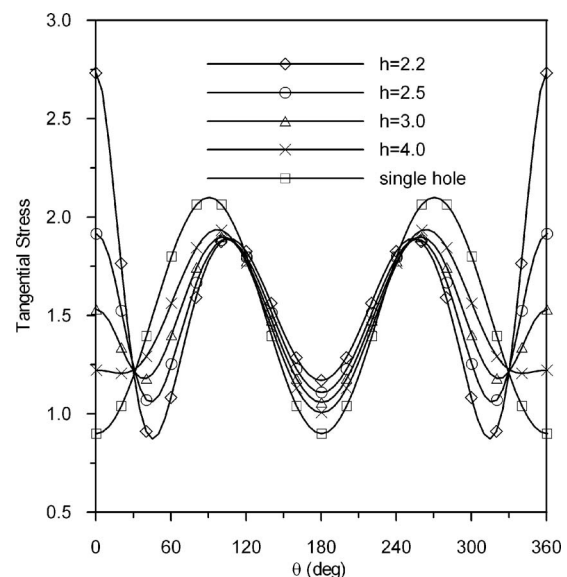


Fig. 13 Tangential stress at the boundary of either hole for $a_1=a_2=1$, $S_1=1$, $S_2=0.7$, $\alpha=0$ deg, $P_1=P_2=0.2$

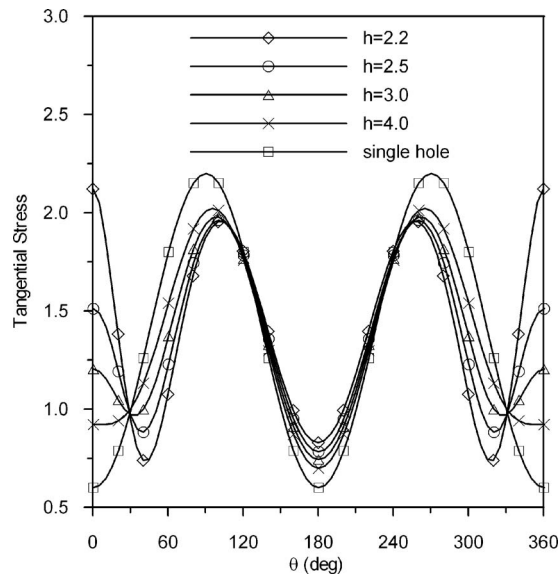


Fig. 14 Tangential stress at the boundary of either hole for $a_1=a_2=1$, $S_1=1$, $S_2=0.6$, $\alpha=0$ deg, $P_1=P_2=0.2$

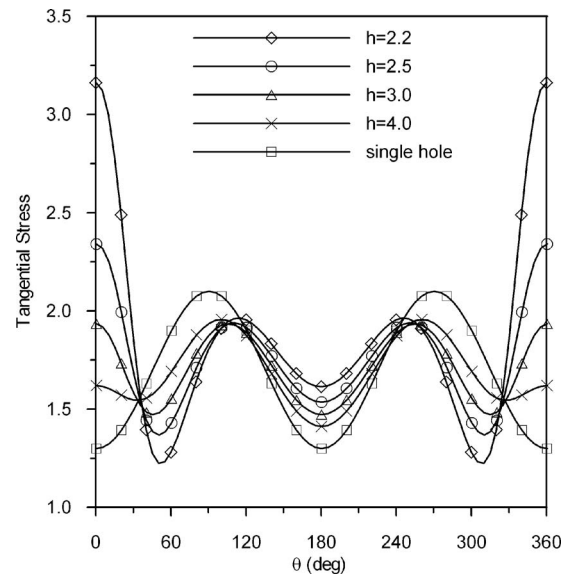


Fig. 16 Tangential stress at the boundary of Hole 2 for $a_1=a_2=1$, $S_1=1$, $S_2=0.8$, $\alpha=0$ deg, $P_1=0.2$, $P_2=0.1$

$P_2=0.1$. The trends observed in Fig. 9 for $P_2=0.2$ also show up but with more pronounced effects; for the same separation distance, the minima for $P_2=0.1$ are lower and the maxima are higher. The maximum stress at the boundary is 28.0% and 98.5% higher than the single hole for $h=2.5$ and $h=2.2$, respectively.

Figure 16 illustrates the distribution of tangential stress at the boundary of Hole 2 for $P_2=0.1$. The same trends as for $P_2=0.2$ are observed but with less pronounced effects. The maximum stress at the boundary is only 11.5% and 50.5% higher than the single hole for $h=2.5$ and $h=2.2$, respectively.

Similar conclusions can be drawn for the case of $P_2=0$. Figure 17 shows the stress distribution at the boundary of Hole 1. The increase of maximum stress relative to the case of a single hole shoots up to 40.2% and 129.9% for $h=2.5$ and $h=2.2$, respectively. Figure 18 displays the stress distribution for Hole 2. Except for the case of $h=2.2$, the same trends as for $P_2=0.1$ are observed but with even more definite effects. For $h=2.2$, the stress distri-

bution is complex near $\theta=0$ deg. There exist two maxima symmetrical about $\theta=0$ deg. $\theta=0$ deg itself is a local minimum.

Effects of the Size of Hole 2. All loading conditions are kept the same as the first analysis, namely, $S_1=1$, $S_2=0.8$, and $P_1=P_2=0.2$. α is again taken to be zero. The radius of Hole 1 is fixed at 1. Only the size of Hole 2 is varied. Two smaller values of a_2 are considered, $a_2=0.5$ and $a_2=0.25$.

Figure 19 shows the tangential stress at the boundary of Hole 1 for $a_2=0.5$. The same set of separation distances of 0.2, 0.5, 1.0, and 2.0 is used. Consequently, the new center-to-center distances are 1.7, 2.0, 2.5, and 3.5. The stress distribution for angles between 100 deg and 260 deg varies little with varying separation distance. The reason is that Hole 1 is twice as big as Hole 2. Therefore, regions far away from Hole 2 are not affected very

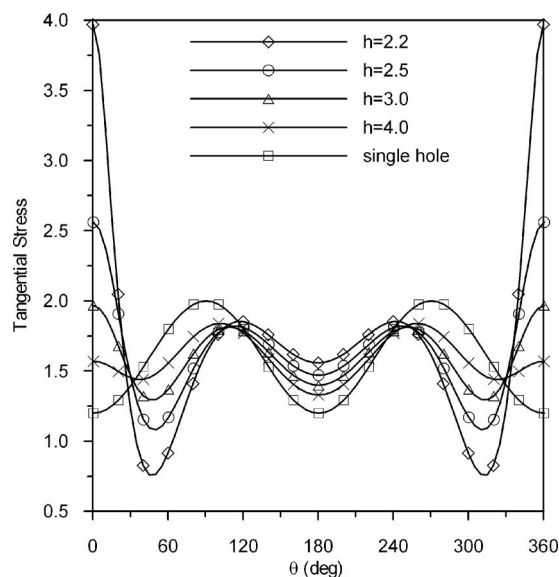


Fig. 15 Tangential stress at the boundary of Hole 1 for $a_1=a_2=1$, $S_1=1$, $S_2=0.8$, $\alpha=0$ deg, $P_1=0.2$, $P_2=0.1$

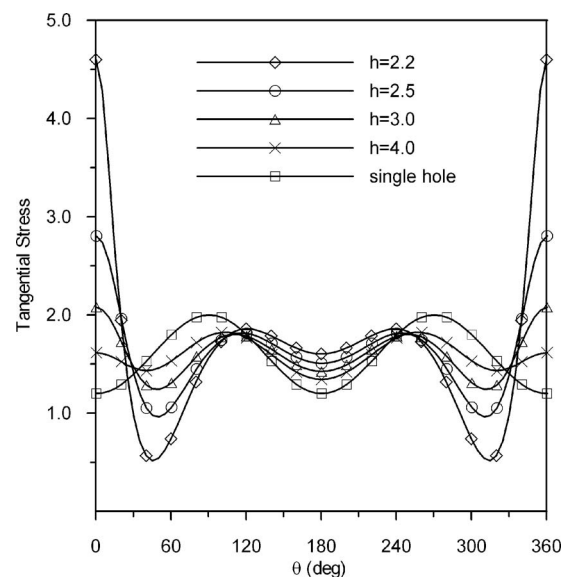


Fig. 17 Tangential stress at the boundary of Hole 1 for $a_1=a_2=1$, $S_1=1$, $S_2=0.8$, $\alpha=0$ deg, $P_1=0.2$, $P_2=0$

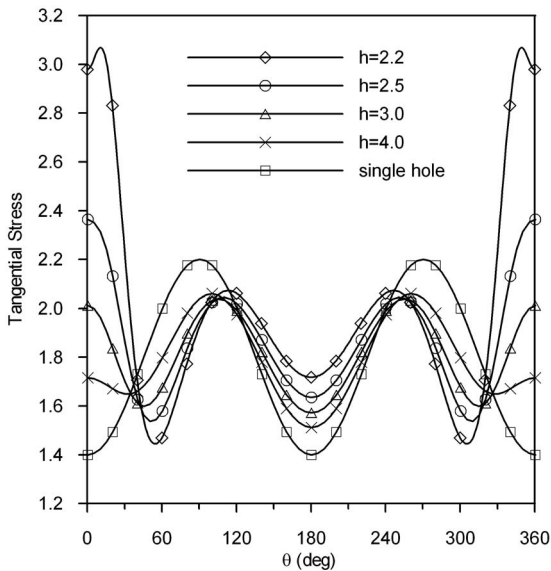


Fig. 18 Tangential stress at the boundary of Hole 2 for $a_1=a_2=1$, $S_1=1$, $S_2=0.8$, $\alpha=0$ deg, $P_1=0.2$, $P_2=0$

much. They are somewhat shielded from the effects caused by the second hole. For $h=1.7$, there are two maxima close to each other and symmetrical about $\theta=0$ deg while $\theta=0$ deg itself is a local minimum.

Figure 20 presents the tangential stress at the boundary of Hole 1 for $a_2=0.25$. The same trends are observed for this case. The shielding effect is much stronger because Hole 1 is now four times bigger than Hole 2. Most of the stress distribution stays essentially the same regardless of the separation distance. Only a small region directly facing Hole 2 has stress distribution strongly dependent on h . For a separation distance of 0.2 there are two maxima symmetrical about $\theta=0$ deg, similar to the case of $a_2=0.5$. However, they are much more distinguished for $a_2=0.25$.

Figures 21 and 22 show the stress distribution for Hole 2 when

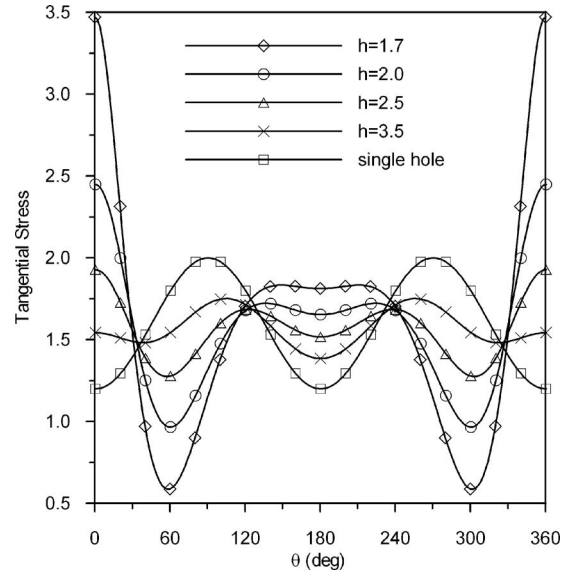


Fig. 21 Tangential stress at the boundary of Hole 2 for $a_1=1$, $a_2=0.5$, $S_1=1$, $S_2=0.8$, $\alpha=0$ deg, $P_1=P_2=0.2$

$a_2=0.5$ and $a_2=0.25$, respectively. They follow similar trends; as the separation distance decreases, two local maxima tend to develop at $\theta=0$ deg and $\theta=180$ deg.

Conclusions

An analytical solution for the stresses in an infinite plate subjected to nonhydrostatic state of far-field stress with two circular holes of any size has been derived and validated analytically as well as experimentally herein. The distribution of tangential stress at the boundary of the holes has also been investigated using the newly derived solution. The analysis shows that the maximum tangential stress at the hole boundary does not always increase with decreasing separation distance between the holes. The solution also suggests that the plate is most stable when the center-to-

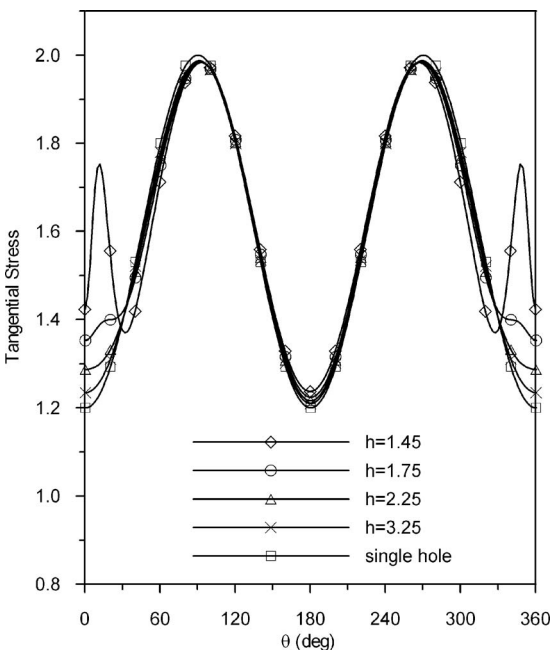


Fig. 20 Tangential stress at the boundary of Hole 1 for $a_1=1$, $a_2=0.25$, $S_1=1$, $S_2=0.8$, $\alpha=0$ deg, $P_1=P_2=0.2$

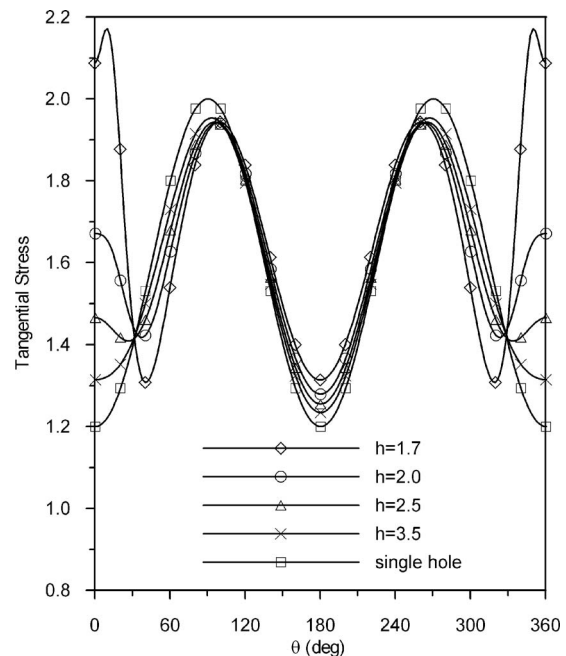


Fig. 19 Tangential stress at the boundary of Hole 1 for $a_1=1$, $a_2=0.5$, $S_1=1$, $S_2=0.8$, $\alpha=0$ deg, $P_1=P_2=0.2$

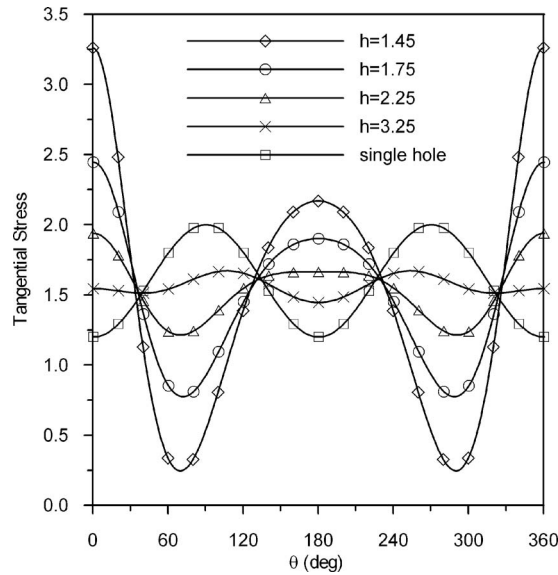


Fig. 22 Tangential stress at the boundary of Hole 2 for $a_1=1$, $a_2=0.25$, $S_1=1$, $S_2=0.8$, $\alpha=0$ deg, $P_1=P_2=0.2$

center line aligns with the maximum principal stress, which has been observed by earlier analyses [11]. Finally, the interplay between the various input parameters is very complex and could be very involved computationally, if conducted through numerical analysis.

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Nomenclature

Capital Letters

- O_1 = center of Hole 1
- O_2 = center of Hole 2
- P_1 = pressure in Hole 1
- P_2 = pressure in Hole 2
- S_1 = major principal stress at infinity
- S_2 = minor principal stress at infinity

Lowercase Letters

- a_1 = radius of Hole 1
- a_2 = radius of Hole 2
- h = center-to-center distance
- r_1 = radial coordinate from O_1
- r_2 = radial coordinate from O_2
- rX_n , $r=1,2$ = hole number
- $n=1,2,\dots$ = term number
- $X=A,B,C,D$ = term

Greek Symbols

- α = angle S_1 makes with the center-to-center line
- ρ_1 = nondimensional radial coordinate from O_1
- ρ_2 = nondimensional radial coordinate from O_2
- λ_1 = nondimensional radius of Hole 1
- λ_2 = nondimensional radius of Hole 2
- θ_1 = angular coordinate from O_1
- θ_2 = angular coordinate from O_2
- ϕ_f = final Airy stress function

- ϕ_0 = Airy stress function if there were no holes
- ϕ_1 = additional Airy stress function for Hole 1
- ϕ_2 = additional Airy stress function for Hole 2
- σ_{rr} = radial stress
- $\sigma_{\theta\theta}$ = tangential stress
- $\sigma_{r\theta}$ = shear stress

Appendix A: Coordinate Transformation

Formulas for transformation from coordinate system (ρ_2, θ_2) to coordinate system (ρ_1, θ_1) are as follows:

$$\ln \rho_2 = \sum_{n=1}^{\infty} -\frac{1}{n} \rho_1^n \cos n \theta_1 \quad (\text{A1})$$

$$\theta_2 = \sum_{n=1}^{\infty} -\frac{1}{n} \rho_1^n \sin n \theta_1 \quad (\text{A2})$$

$$\rho_2^{-n} \cos n \theta_2 = \sum_{k=1}^{\infty} \binom{n+k-1}{k} \rho_1^k \cos k \theta_1 \quad (\text{A3})$$

$$\rho_2^{-n} \sin n \theta_2 = \sum_{k=1}^{\infty} -\binom{n+k-1}{k} \rho_1^k \sin k \theta_1 \quad (\text{A4})$$

$$\rho_2^{-n} \cos(n+2) \theta_2 = \sum_{k=0}^{\infty} \left[\binom{n+k}{k} \rho_1^k \cos k \theta_1 - \binom{n+k+1}{k+1} \rho_1^{k+2} \cos k \theta_1 \right] \quad (\text{A5})$$

$$\rho_2^{-n} \sin(n+2) \theta_2 = \sum_{k=0}^{\infty} \left[-\binom{n+k}{k} \rho_1^k \sin k \theta_1 + \binom{n+k+1}{k+1} \rho_1^{k+2} \sin k \theta_1 \right] \quad (\text{A6})$$

The two sides of each of Eqs. (A1)–(A6) might differ by a constant. However, for our purpose, the difference can be ignored and they can be regarded as equal. The formulas for transformation from coordinate system (ρ_1, θ_1) to system (ρ_2, θ_2) are obtained from these equations by permutations.

Appendix B: Mathematical Solution

The solution outline has already been presented in the Solution Method section. Parts of it are repeated here for clarity. First, the stresses at infinity S_1 and S_2 are expanded into three components S_{xx} , S_{yy} , and S_{xy} . Taking two centers of pressure at O_1 and O_2 into account, the Airy stress function if there were no holes is as follows:

$$\phi_0 = h^2 \left[P_1 \lambda_1^2 \ln \rho_1 + P_2 \lambda_2^2 \ln \rho_2 + \frac{S_{xx}}{2} \rho_1^2 \sin^2 \theta_1 + \frac{S_{yy}}{2} \rho_1^2 \cos^2 \theta_1 + \frac{S_{xy}}{2} \rho_1^2 \sin 2 \theta_1 \right] \quad (\text{B1})$$

ϕ_0 takes a similar form when it is expanded around Hole 2. The additional Airy stress functions must produce zero stresses at infinity and single-valued stresses and displacements. They are therefore of the forms

$$\ln \rho$$

$$\theta$$

$$\rho^{-n} \cos n\theta \quad \text{for } n = 1, 2, \dots$$

$$\rho^{-n} \sin n\theta \quad \text{for } n = 1, 2, \dots$$

$$\rho^{-n} \cos(n+2)\theta \quad \text{for } n = 0, 1, 2, \dots$$

$$\rho^{-n} \sin(n+2)\theta \quad \text{for } n = 0, 1, 2, \dots$$

The additional set of stress functions for Hole 1 is a linear combination of the preceding functions.

$$\begin{aligned} \phi_1 = & {}^1A_0 \ln \rho_1 + {}^1B_0 \theta_1 + \sum_{n=1}^{\infty} [{}^1A_n \rho_1^{-n} \cos n\theta_1 + {}^1B_n \rho_1^{-n} \sin n\theta_1] \\ & + \sum_{n=0}^{\infty} [{}^1C_n \rho_1^{-n} \cos(n+2)\theta_1 + {}^1D_n \rho_1^{-n} \sin(n+2)\theta_1] \end{aligned} \quad (\text{B2})$$

The additional stress function for Hole 2 is obtained similarly,

$$\begin{aligned} \phi_2 = & {}^2A_0 \ln \rho_2 + {}^2B_0 \theta_2 + \sum_{n=1}^{\infty} [{}^2A_n \rho_2^{-n} \cos n\theta_2 + {}^2B_n \rho_2^{-n} \sin n\theta_2] \\ & + \sum_{n=0}^{\infty} [{}^2C_n \rho_2^{-n} \cos(n+2)\theta_2 + {}^2D_n \rho_2^{-n} \sin(n+2)\theta_2] \end{aligned} \quad (\text{B3})$$

where ${}^1A_n$, ${}^1B_n$, ${}^1C_n$, ${}^1D_n$, ${}^2A_n$, ${}^2B_n$, ${}^2C_n$, and ${}^2D_n$ ($n = 0, 1, 2, \dots$) are coefficients determined from boundary conditions.

The final Airy stress function for the full solution is the superposition of the individual Airy stress functions.

$$\phi_f = \phi_0 + \phi_1 + \phi_2 \quad (\text{B4})$$

This function can be expressed in terms of ρ_1 and θ_1 using coordinate transformation formulas presented in Appendix A. The stresses around Hole 1 are then derived as follows:

$$\sigma_{rr} = \frac{1}{h^2} \left(\frac{1}{\rho_1} \frac{\partial \phi_f}{\partial \rho_1} + \frac{1}{\rho_1^2} \frac{\partial^2 \phi_f}{\partial \theta_1^2} \right) \quad (\text{B5})$$

$$\sigma_{\theta\theta} = \frac{1}{h^2} \frac{\partial^2 \phi_f}{\partial \rho_1^2} \quad (\text{B6})$$

$$\sigma_{r\theta} = -\frac{1}{h^2} \frac{\partial}{\partial \rho_1} \left(\frac{1}{\rho_1} \frac{\partial \phi_f}{\partial \theta_1} \right) \quad (\text{B7})$$

The boundary conditions at infinity are automatically satisfied. At the boundary of Hole 1, the radial stress must equal the applied pressure and the shear stress must vanish.

$$\sigma_{rr}|_{\rho_1=\lambda_1} = P_1 \quad (\text{B8})$$

$$\sigma_{r\theta}|_{\rho_1=\lambda_1} = 0 \quad (\text{B9})$$

Similarly, the final Airy stress function ϕ_f can be expressed in terms of ρ_2 and θ_2 . The stresses for Hole 2 are derived using equations similar to Eqs. (B5)–(B7). The boundary conditions at the boundary of Hole 2 are as follows:

$$\sigma_{rr}|_{\rho_2=\lambda_2} = P_2 \quad (\text{B10})$$

$$\sigma_{r\theta}|_{\rho_2=\lambda_2} = 0 \quad (\text{B11})$$

The stresses corresponding to ϕ_0 can be easily derived. At the boundary of Hole 1, they take the forms

$$\sigma_{rr} = \sum_{k=1}^{\infty} {}^1U_k \sin k\theta_1 + \sum_{k=0}^{\infty} {}^1S_k \cos k\theta_1 \quad (\text{B12})$$

$$\sigma_{r\theta} = \sum_{k=1}^{\infty} {}^1T_k \sin k\theta_1 + \sum_{k=0}^{\infty} {}^1V_k \cos k\theta_1 \quad (\text{B13})$$

For ϕ_0 given in Eq. (B1), the coefficients are as follows:

$${}^1U_2 = {}^1V_2 = S_{xy}$$

$${}^1S_0 = P_1 + (S_{xx} + S_{yy})/2$$

$${}^1S_2 = -{}^1T_2 = P_2 \lambda^2 + (S_{xx} - S_{yy})/2$$

$${}^1S_k = -{}^1T_k = P_2 \lambda^k (k-1) \quad (\text{B14})$$

All other coefficients are zero. Similarly, stresses at the boundary of Hole 2 corresponding to ϕ_0 are as follows:

$$\sigma_{rr} = \sum_{k=1}^{\infty} {}^2U_k \sin k\theta_2 + \sum_{k=0}^{\infty} {}^2S_k \cos k\theta_2 \quad (\text{B15})$$

$$\sigma_{r\theta} = \sum_{k=1}^{\infty} {}^2T_k \sin k\theta_2 + \sum_{k=0}^{\infty} {}^2V_k \cos k\theta_2 \quad (\text{B16})$$

Equations (B8)–(B11) are solved simultaneously to obtain

$$\begin{aligned} {}^1A_n = \sum_{p=0}^{\infty} {}^1A_n^{(p)} \quad {}^1B_n = \sum_{p=0}^{\infty} {}^1B_n^{(p)} \quad {}^1C_n = \sum_{p=0}^{\infty} {}^1C_n^{(p)} \quad {}^1D_n = \sum_{p=0}^{\infty} {}^1D_n^{(p)} \\ {}^2A_n = \sum_{p=0}^{\infty} {}^2A_n^{(p)} \quad {}^2B_n = \sum_{p=0}^{\infty} {}^2B_n^{(p)} \quad {}^2C_n = \sum_{p=0}^{\infty} {}^2C_n^{(p)} \quad {}^2D_n = \sum_{p=0}^{\infty} {}^2D_n^{(p)} \end{aligned} \quad (\text{B17})$$

The first terms of the series are as follows:

$${}^1A_0^{(0)} = h^2 \lambda_1^2 (P_1 - {}^1S_0)$$

$${}^1A_n^{(0)} = \frac{h^2 [(n+2) {}^1T_n - n {}^1S_n] \lambda_1^{n+2}}{2n(n+1)} \quad (n \geq 1)$$

$${}^1B_0^{(0)} = 0$$

$${}^1B_n^{(0)} = -\frac{h^2 [(n+2) {}^1V_n + n {}^1U_n] \lambda_1^{n+2}}{2n(n+1)} \quad (n \geq 1)$$

$${}^1C_n^{(0)} = -\frac{h^2 ({}^1S_{n+2} - {}^1T_{n+2}) \lambda_1^{n+2}}{2(n+1)} \quad (n \geq 0)$$

$${}^1D_n^{(0)} = \frac{h^2 ({}^1U_{n+2} + {}^1V_{n+2}) \lambda_1^{n+2}}{2(n+1)} \quad (n \geq 0)$$

$${}^2A_0^{(0)} = h^2 \lambda_2^2 (P_2 - {}^2S_0)$$

$${}^2A_n^{(0)} = \frac{h^2 [(n+2) {}^2T_n - n {}^2S_n] \lambda_2^{n+2}}{2n(n+1)} \quad (n \geq 1)$$

$${}^2B_0^{(0)} = 0$$

$${}^2B_n^{(0)} = -\frac{h^2 [(n+2) {}^2V_n + n {}^2U_n] \lambda_2^{n+2}}{2n(n+1)} \quad (n \geq 1)$$

$${}^2C_n^{(0)} = -\frac{h^2 ({}^2S_{n+2} - {}^2T_{n+2}) \lambda_2^{n+2}}{2(n+1)} \quad (n \geq 0)$$

$${}^2D_n^{(0)} = \frac{h^2 ({}^2U_{n+2} + {}^2V_{n+2}) \lambda_2^{n+2}}{2(n+1)} \quad (n \geq 0) \quad (\text{B18})$$

Other terms obey the following recursive relationships:

$$\begin{aligned}
{}^1A_n^{(p+1)} &= \sum_{k=0}^{\infty} {}^2A_k^{(p)k} \alpha_n^{21} + {}^2C_k^{(p)k} \gamma_n^{21} \quad (n \geq 0) \\
{}^1B_n^{(p+1)} &= \sum_{k=0}^{\infty} {}^2B_k^{(p)k} \eta_n^{21} + {}^2D_k^{(p)k} \tau_n^{21} \quad (n \geq 1) \\
{}^1C_n^{(p+1)} &= \sum_{k=0}^{\infty} {}^2A_k^{(p)k} \kappa_{n+2}^{21} + {}^2C_k^{(p)k} \nu_{n+2}^{21} \quad (n \geq 0) \\
{}^1D_n^{(p+1)} &= \sum_{k=0}^{\infty} {}^2B_k^{(p)k} \theta_{n+2}^{21} + {}^2D_k^{(p)k} \psi_{n+2}^{21} \quad (n \geq 0) \\
{}^2A_n^{(p+1)} &= \sum_{k=0}^{\infty} {}^1A_k^{(p)k} \alpha_n^{12} + {}^1C_k^{(p)k} \gamma_n^{12} \quad (n \geq 0) \\
{}^2B_n^{(p+1)} &= \sum_{k=0}^{\infty} {}^1B_k^{(p)k} \eta_n^{12} + {}^1D_k^{(p)k} \tau_n^{12} \quad (n \geq 1) \\
{}^2C_n^{(p+1)} &= \sum_{k=0}^{\infty} {}^1A_k^{(p)k} \kappa_{n+2}^{12} + {}^1C_k^{(p)k} \nu_{n+2}^{12} \quad (n \geq 0) \\
{}^2D_n^{(p+1)} &= \sum_{k=0}^{\infty} {}^1B_k^{(p)k} \theta_{n+2}^{12} + {}^1D_k^{(p)k} \psi_{n+2}^{12} \quad (n \geq 0) \quad (B19)
\end{aligned}$$

with new parameters

$$\begin{aligned}
{}^k\gamma_0^{21} &= 2(k+1)\lambda_1^2 \\
{}^k\gamma_0^{12} &= 2(k+1)\lambda_2^2 \\
{}^0\alpha_n^{21} &= -\frac{n-1}{n}\lambda_1^{2n} \quad (n \geq 2) \\
{}^0\alpha_n^{12} &= -\frac{n-1}{n}\lambda_2^{2n} \quad (n \geq 2) \\
{}^k\alpha_n^{21} &= (n-1)\binom{k+n-1}{n}\lambda_1^{2n} \quad (n \geq 1, k \geq 1)
\end{aligned}$$

$${}^k\alpha_n^{12} = (n-1)\binom{k+n-1}{n}\lambda_2^{2n} \quad (n \geq 1, k \geq 1)$$

$${}^k\gamma_n^{21} = (n-1)\binom{k+n}{n}\lambda_1^{2n} - n\binom{k+n+1}{n+1}\lambda_1^{2n+2} \quad (n \geq 1, k \geq 0)$$

$${}^k\gamma_n^{12} = (n-1)\binom{k+n}{n}\lambda_2^{2n} - n\binom{k+n+1}{n+1}\lambda_2^{2n+2} \quad (n \geq 1, k \geq 0)$$

$${}^0\kappa_n^{21} = \lambda_1^{2n-2} \quad (n \geq 2)$$

$${}^0\kappa_n^{12} = \lambda_2^{2n-2} \quad (n \geq 2)$$

$${}^k\kappa_n^{21} = -n\binom{k+n-1}{n}\lambda_1^{2n-2} \quad (n \geq 2, k \geq 1)$$

$${}^k\kappa_n^{12} = -n\binom{k+n-1}{n}\lambda_2^{2n-2} \quad (n \geq 2, k \geq 1)$$

$${}^k\nu_n^{21} = -n\binom{k+n}{n}\lambda_1^{2n-2} + (n+1)\binom{k+n+1}{n+1}\lambda_1^{2n} \quad (n \geq 2, k \geq 0)$$

$${}^k\nu_n^{12} = -n\binom{k+n}{n}\lambda_2^{2n-2} + (n+1)\binom{k+n+1}{n+1}\lambda_2^{2n} \quad (n \geq 2, k \geq 0)$$

$${}^0\eta_n^{21} = {}^0\alpha_n^{21} \quad {}^0\eta_n^{12} = {}^0\alpha_n^{12} \quad (n \geq 1)$$

$${}^k\eta_n^{21} = -{}^k\alpha_n^{21} \quad {}^k\eta_n^{12} = -{}^k\alpha_n^{12} \quad (n \geq 1, k \geq 1)$$

$${}^k\tau_n^{21} = -{}^k\gamma_n^{21} \quad {}^k\tau_n^{12} = -{}^k\gamma_n^{12} \quad (n \geq 1, k \geq 0)$$

$${}^0\theta_n^{21} = {}^0\kappa_n^{21} \quad {}^0\theta_n^{12} = {}^0\kappa_n^{12} \quad (n \geq 1)$$

$${}^k\theta_n^{21} = {}^k\kappa_n^{21} \quad {}^k\theta_n^{12} = {}^k\kappa_n^{12} \quad (n \geq 1, k \geq 1)$$

$${}^k\psi_n^{21} = -{}^k\nu_n^{21} \quad {}^k\psi_n^{12} = -{}^k\nu_n^{12} \quad (n \geq 1, k \geq 0) \quad (B20)$$

All other parameters are zero.

Substitute these coefficients back into the final Airy stress function in Eq. (B4). Stresses around Hole 1 are again derived from Eqs. (B5)–(B7). The radial stress around Hole 1 takes the form

$$\begin{aligned}
\sigma_{rr} = & S_{rr} + \frac{P_1\lambda_1^2}{\rho_1^2} + P_2\lambda_2^2 \sum_{k=2}^{\infty} (k-1)\rho_1^{k-2} \cos k\theta_1 + \frac{1}{h^2} \left\{ \frac{{}^1A_0}{\rho_1^2} + {}^2A_0 \sum_{k=2}^{\infty} (k-1)\rho_1^{k-2} \cos k\theta_1 - \sum_{n=1}^{\infty} n(n+1){}^1A_n\rho_1^{-n-2} \cos n\theta_1 \right. \\
& - \sum_{n=1}^{\infty} {}^2A_n \sum_{k=2}^{\infty} \binom{k+n-1}{k} k(k-1)\rho_1^{k-2} \cos k\theta_1 - \sum_{n=1}^{\infty} n(n+1){}^1B_n\rho_1^{-n-2} \sin n\theta_1 + \sum_{n=1}^{\infty} {}^2B_n \sum_{k=2}^{\infty} \binom{k+n-1}{k} k(k-1)\rho_1^{k-2} \sin k\theta_1 \\
& - \sum_{n=0}^{\infty} (n+1)(n+4){}^1C_n\rho_1^{-n-2} \cos(n+2)\theta_1 + \sum_{n=0}^{\infty} {}^2C_n \left\{ -2(n+1) - 2\binom{n+2}{2} \rho_1 \cos \theta_1 + \sum_{k=2}^{\infty} \left[-\binom{n+k}{k} k(k-1)\rho_1^{k-2} \cos k\theta_1 \right. \right. \\
& \left. \left. + \binom{n+k+1}{k+1} (k+1)(k-2)\rho_1^k \cos k\theta_1 \right] \right\} - \sum_{n=0}^{\infty} (n+1)(n+4){}^1D_n\rho_1^{-n-2} \sin(n+2)\theta_1 + \sum_{n=0}^{\infty} {}^2D_n \left\{ 2\binom{n+2}{2} \rho_1 \sin \theta_1 \right. \\
& \left. \left. + \sum_{k=2}^{\infty} \left[\binom{n+k}{k} k(k-1)\rho_1^{k-2} \sin k\theta_1 - \binom{n+k+1}{k+1} (k+1)(k-2)\rho_1^k \sin k\theta_1 \right] \right\} \right\} \quad (B21)
\end{aligned}$$

The shear stress around Hole 1 takes the form

$$\begin{aligned} \sigma_{r\theta} = & S_{r\theta} + P_2 \lambda_2^2 \sum_{k=2}^{\infty} (1-k) \rho_1^{k-2} \sin k \theta_1 + \frac{1}{h^2} \left\{ -^2 A_0 \sum_{k=2}^{\infty} (k-1) \rho_1^{k-2} \sin k \theta_1 - \sum_{n=1}^{\infty} n(n+1) ^1 A_n \rho_1^{-n-2} \sin n \theta_1 + \sum_{n=1}^{\infty} ^2 A_n \sum_{k=2}^{\infty} \binom{k+n-1}{k} k(k-1) \rho_1^{k-2} \sin k \theta_1 \right. \\ & - 1) \rho_1^{k-2} \sin k \theta_1 + \sum_{n=1}^{\infty} n(n+1) ^1 B_n \rho_1^{-n-2} \cos n \theta_1 + \sum_{n=1}^{\infty} ^2 B_n \sum_{k=2}^{\infty} \binom{k+n-1}{k} k(k-1) \rho_1^{k-2} \cos k \theta_1 - \sum_{n=0}^{\infty} (n+1)(n+2) ^1 C_n \rho_1^{-n-2} \sin(n \\ & + 2) \theta_1 + \sum_{n=0}^{\infty} ^2 C_n \left\{ -2 \binom{n+2}{2} \rho_1 \sin \theta_1 + \sum_{k=2}^{\infty} \left[\binom{n+k}{k} k(k-1) \rho_1^{k-2} \sin k \theta_1 - \binom{n+k+1}{k+1} k(k+1) \rho_1^k \sin k \theta_1 \right] \right\} + \sum_{n=0}^{\infty} (n+1)(n \\ & + 2) ^1 D_n \rho_1^{-n-2} \cos(n+2) \theta_1 + \sum_{n=0}^{\infty} ^2 D_n \left\{ -2 \binom{n+2}{2} \rho_1 \cos \theta_1 + \sum_{k=2}^{\infty} \left[\binom{n+k}{k} k(k-1) \rho_1^{k-2} \cos k \theta_1 - \binom{n+k+1}{k+1} k(k \right. \right. \\ & \left. \left. + 1) \rho_1^k \cos k \theta_1 \right] \right\} \left. \right\} \end{aligned} \quad (B22)$$

The tangential stress around Hole 1 is as follows:

$$\begin{aligned} \sigma_{\theta\theta} = & S_{\theta\theta} - \frac{P_1 \lambda_1^2}{\rho_1^2} - P_2 \lambda_2^2 \sum_{k=2}^{\infty} (k-1) \rho_1^{k-2} \cos k \theta_1 + \frac{1}{h^2} \left\{ -\frac{^1 A_0}{\rho_1^2} - ^2 A_0 \sum_{k=2}^{\infty} (k-1) \rho_1^{k-2} \cos k \theta_1 + \sum_{n=1}^{\infty} n(n+1) ^1 A_n \rho_1^{-n-2} \cos n \theta_1 \right. \\ & + \sum_{n=1}^{\infty} ^2 A_n \sum_{k=2}^{\infty} \binom{k+n-1}{k} k(k-1) \rho_1^{k-2} \cos k \theta_1 + \sum_{n=1}^{\infty} n(n+1) ^1 B_n \rho_1^{-n-2} \sin n \theta_1 - \sum_{n=1}^{\infty} ^2 B_n \sum_{k=2}^{\infty} \binom{k+n-1}{k} k(k-1) \rho_1^{k-2} \sin k \theta_1 \\ & + \sum_{n=0}^{\infty} n(n+1) ^1 C_n \rho_1^{-n-2} \cos(n+2) \theta_1 + \sum_{n=0}^{\infty} ^2 C_n \left\{ -2(n+1) - 6 \binom{n+2}{2} \rho_1 \cos \theta_1 + \sum_{k=2}^{\infty} \left[\binom{n+k}{k} \right. \right. \\ & \left. \left. \times k(k-1) \rho_1^{k-2} \cos k \theta_1 - \binom{n+k+1}{k+1} (k+1)(k+2) \rho_1^k \cos k \theta_1 \right] \right\} + \sum_{n=0}^{\infty} n(n+1) ^1 D_n \rho_1^{-n-2} \sin(n+2) \theta_1 \\ & \left. + \sum_{n=0}^{\infty} ^2 D_n \left\{ 6 \binom{n+2}{2} \rho_1 \sin \theta_1 + \sum_{k=2}^{\infty} \left[-\binom{n+k}{k} k(k-1) \rho_1^{k-2} \sin k \theta_1 + \binom{n+k+1}{k+1} (k+1)(k+2) \rho_1^k \sin k \theta_1 \right] \right\} \right\} \end{aligned} \quad (B23)$$

with applied polar stress components

$$S_{rr} = \frac{S_{xx} + S_{yy}}{2} + \frac{S_{xx} - S_{yy}}{2} \cos 2\theta + S_{xy} \sin 2\theta \quad (B24)$$

$$S_{\theta\theta} = \frac{S_{xx} + S_{yy}}{2} + \frac{S_{yy} - S_{xx}}{2} \cos 2\theta - S_{xy} \sin 2\theta \quad (B25)$$

$$S_{r\theta} = \frac{S_{yy} - S_{xx}}{2} \sin 2\theta + S_{xy} \cos 2\theta \quad (B26)$$

while S_{xx} , S_{yy} , and S_{xy} are Cartesian stress components of S_1 and S_2 .

The final Airy stress function ϕ_f can also be expressed in the (ρ_2, θ_2) coordinate system. The stresses around Hole 2 are then easily derived. The radial stress around Hole 2 takes the form

$$\begin{aligned} \sigma_{rr} = & S_{rr} + \frac{P_2 \lambda_2^2}{\rho_2^2} + P_1 \lambda_1^2 \sum_{k=2}^{\infty} (k-1) \rho_2^{k-2} \cos k \theta_2 + \frac{1}{h^2} \left\{ \frac{^2 A_0}{\rho_2^2} + ^1 A_0 \sum_{k=2}^{\infty} (k-1) \rho_2^{k-2} \cos k \theta_2 - \sum_{n=1}^{\infty} n(n+1) ^2 A_n \rho_2^{-n-2} \cos n \theta_2 \right. \\ & - \sum_{n=1}^{\infty} ^1 A_n \sum_{k=2}^{\infty} \binom{k+n-1}{k} k(k-1) \rho_2^{k-2} \cos k \theta_2 - \sum_{n=1}^{\infty} n(n+1) ^2 B_n \rho_2^{-n-2} \sin n \theta_2 + \sum_{n=1}^{\infty} ^1 B_n \sum_{k=2}^{\infty} \binom{k+n-1}{k} k(k-1) \rho_2^{k-2} \sin k \theta_2 \\ & - \sum_{n=0}^{\infty} (n+1)(n+4) ^2 C_n \rho_2^{-n-2} \cos(n+2) \theta_2 + \sum_{n=0}^{\infty} ^1 C_n \left\{ -2(n+1) - 2 \binom{n+2}{2} \rho_2 \cos \theta_2 + \sum_{k=2}^{\infty} \left[-\binom{n+k}{k} k(k-1) \rho_2^{k-2} \cos k \theta_2 \right. \right. \\ & \left. \left. + \binom{n+k+1}{k+1} (k+1)(k-2) \rho_2^k \cos k \theta_2 \right] \right\} - \sum_{n=0}^{\infty} (n+1)(n+4) ^2 D_n \rho_2^{-n-2} \sin(n+2) \theta_2 + \sum_{n=0}^{\infty} ^1 D_n \left\{ 2 \binom{n+2}{2} \rho_2 \sin \theta_2 \right. \end{aligned}$$

$$+ \sum_{k=2}^{\infty} \left[\binom{n+k}{k} k(k-1) \rho_2^{k-2} \sin k\theta_2 - \binom{n+k+1}{k+1} (k+1)(k-2) \rho_2^k \sin k\theta_2 \right] \Bigg\} \quad (\text{B27})$$

The shear stress around Hole 2 is as follows:

$$\begin{aligned} \sigma_{r\theta} = & S_{r\theta} + P_1 \lambda_1^2 \sum_{k=2}^{\infty} (1-k) \rho_2^{k-2} \sin k\theta_2 + \frac{1}{h^2} \left\{ - {}^1A_0 \sum_{k=2}^{\infty} (k-1) \rho_2^{k-2} \sin k\theta_2 - \sum_{n=1}^{\infty} n(n+1)^2 A_n \rho_2^{-n-2} \sin n\theta_2 + \sum_{n=1}^{\infty} {}^1A_n \right. \\ & \times \sum_{k=2}^{\infty} \binom{k+n-1}{k} k(k-1) \rho_2^{k-2} \sin k\theta_2 + \sum_{n=1}^{\infty} n(n+1)^2 B_n \rho_2^{-n-2} \cos n\theta_2 + \sum_{n=1}^{\infty} {}^1B_n \sum_{k=2}^{\infty} \binom{k+n-1}{k} k(k-1) \rho_2^{k-2} \cos k\theta_2 - \sum_{n=0}^{\infty} (n+1)(n \\ & + 2)^2 C_n \rho_2^{-n-2} \sin(n+2)\theta_2 + \sum_{n=0}^{\infty} {}^1C_n \left\{ - 2 \binom{n+2}{2} \rho_2 \sin \theta_2 + \sum_{k=2}^{\infty} \left[\binom{n+k}{k} k(k-1) \rho_2^{k-2} \sin k\theta_2 - \binom{n+k+1}{k+1} k(k+1) \rho_2^k \sin k\theta_2 \right] \right\} \\ & + \sum_{n=0}^{\infty} (n+1)(n+2)^2 D_n \rho_2^{-n-2} \cos(n+2)\theta_2 + \sum_{n=0}^{\infty} {}^1D_n \left\{ - 2 \binom{n+2}{2} \rho_2 \cos \theta_2 + \sum_{k=2}^{\infty} \left[\binom{n+k}{k} k(k-1) \rho_2^{k-2} \cos k\theta_2 \right. \right. \\ & \left. \left. - \binom{n+k+1}{k+1} k(k+1) \rho_2^k \cos k\theta_2 \right] \right\} \Bigg\} \quad (\text{B28}) \end{aligned}$$

Finally, the tangential stress around Hole 2 is

$$\begin{aligned} \sigma_{\theta\theta} = & S_{\theta\theta} - \frac{P_2 \lambda_2^2}{\rho_2^2} - P_1 \lambda_1^2 \sum_{k=2}^{\infty} (k-1) \rho_2^{k-2} \cos k\theta_2 + \frac{1}{h^2} \left\{ - \frac{{}^2A_0}{\rho_2^2} - {}^1A_0 \sum_{k=2}^{\infty} (k-1) \rho_2^{k-2} \cos k\theta_2 + \sum_{n=1}^{\infty} n(n+1)^2 A_n \rho_2^{-n-2} \cos n\theta_2 \right. \\ & + \sum_{n=1}^{\infty} {}^1A_n \sum_{k=2}^{\infty} \binom{k+n-1}{k} k(k-1) \rho_2^{k-2} \cos k\theta_2 + \sum_{n=1}^{\infty} n(n+1)^2 B_n \rho_2^{-n-2} \sin n\theta_2 - \sum_{n=1}^{\infty} {}^1B_n \sum_{k=2}^{\infty} \binom{k+n-1}{k} k(k-1) \rho_2^{k-2} \sin k\theta_2 \\ & + \sum_{n=0}^{\infty} n(n+1)^2 C_n \rho_2^{-n-2} \cos(n+2)\theta_2 + \sum_{n=0}^{\infty} {}^1C_n \left\{ - 2(n+1) - 6 \binom{n+2}{2} \rho_2 \cos \theta_2 + \sum_{k=2}^{\infty} \left[\binom{n+k}{k} k(k-1) \rho_2^{k-2} \cos k\theta_2 - \binom{n+k+1}{k+1} \right. \right. \\ & \left. \left. \times (k+1)(k+2) \rho_2^k \cos k\theta_2 \right] \right\} + \sum_{n=0}^{\infty} n(n+1)^2 D_n \rho_2^{-n-2} \sin(n+2)\theta_2 + \sum_{n=0}^{\infty} {}^1D_n \left\{ 6 \binom{n+2}{2} \rho_2 \sin \theta_2 + \sum_{k=2}^{\infty} \left[- \binom{n+k}{k} k(k-1) \rho_2^{k-2} \right. \right. \\ & \left. \left. \times \sin k\theta_2 + \binom{n+k+1}{k+1} (k+1)(k+2) \rho_2^k \sin k\theta_2 \right] \right\} \Bigg\} \quad (\text{B29}) \end{aligned}$$

Although only the solution for the boundary condition of internal pressures inside the holes is presented here, the method can be readily applied to other boundary conditions by using the appropriate function ϕ_0 and modifying the boundary equations (B8)–(B11).

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Three-Dimensional Constitutive Creep/Relaxation Model of Carbon Cathode Materials

In order to adequately simulate the behavior of a Hall–Héroult electrolysis cell, a finite element model must take into account the properties of each material forming the cell structure and those contained in it. However, there is some lack of full knowledge of the mechanical behavior of these materials, e.g., the long term viscoelastic (creep/relaxation) behavior of the carbon cathode. In this present paper, a three-dimensional viscoelastic model is devised and proposed, being ready to be implemented in a finite element code. This 3D viscoelastic model was developed from the thermodynamics of irreversible processes, where the selection of the model's internal variables was based on a phenomenological approach. The model has been developed at a particular reference state; therefore, the model parameters are represented by constant constitutive tensors. The model's particular parameters were identified for three different types of cathode carbon, i.e., semigraphitic, graphitic, and graphitized. [DOI: 10.1115/1.2840044]

Keywords: graphite, cathodes, rheology, mechanical properties, viscoelasticity

1 Introduction

1.1 Generalities. Aluminum is produced industrially by the so-called Hall–Héroult process [1]. This long established process consists mainly by the electrolysis of refined alumina, previously dissolved in a mixture of molten cryolite at a temperature of around 980°C [2]. The electrolysis cell is schematized in Fig. 1. During the process, multiphysical phenomena are occurring in the electrolysis cell, these being of a thermal, mechanical, chemical, electrical, etc., character. Knowledge of the thermochemicoelectromechanical behavior of the cell itself, and of the liquid phases it contains, is the key to the optimization of the whole process. However, the prevailing high temperatures, molten phases, and an overall corrosive environment make the obtaining of in situ process measurements very difficult, and indeed, sometimes impossible, to achieve.

Numerical modeling is therefore an invaluable tool to gain insights into the complex phenomena taking place during such transitory steps as in smelting pot “startup.” Adequate modeling of the mechanical responses of the cell lining is critical for the detection and minimization of the risk of cathode block cracking or the development of gaps in the lining, from which liquid phases could leak out. The baking of the “ramming paste,” the quasibrittle nature of the carbon blocks, the stress relaxation/creep phenomenon, and the contact interfaces are examples of the key elements needed to be considered within the compass of a finite element model of the Hall–Héroult electrolysis cell.

The cell lining of the Hall–Héroult electrolysis cell (Fig. 1) constitutes one of the cell's major components. The lining itself contains a range of various components, each having those specific functions and properties to enable it to play a precise role in the thermoelectromechanical behavior of the cell [2]. The lining materials can be grouped into two material categories, i.e., refrac-

tory types and carbon materials. In previous studies of cell design, an integrated approach has been adopted for the finite element simulation of cell preheating, where all of the important phenomena are coupled, and some physical phenomena, and their interactions, have been addressed, e.g., D'Amours [3] (carbon materials), D'Amours et al. [4] (carbon materials), Goulet [5] (contact interfaces), Richard [6] (refractory), Richard et al. [7] (refractory), and Désilets et al. [8] (cell preheating simulation).

1.2 Strain in Carbon Materials. In the Hall–Héroult electrolysis cell, the carbon materials, as with many other materials, are affected by the high prevailing temperatures, the Joule effect, chemical contamination, the baking of lining materials (ramming paste), etc. [2]. All of these phenomena induce stresses and strains in the cell materials. Thus, the three-dimensional total strain of a material, the carbon cathode/“baked” ramming paste in this case, can be expressed as

$$[\varepsilon] = [\varepsilon^e] + [\varepsilon^p] + [\varepsilon^{ch}] + [\varepsilon^T] + [\varepsilon^a] + [\varepsilon^v] + \sum [\varepsilon^{\text{other}}] \quad (1)$$

where all strains are second-order tensors. The elastic strain $[\varepsilon^e]$ is generally a well known parameter since it is related to Young's modulus and Poisson's ratio, classical parameters that are obtained from the mechanical characterization of materials [2,9–13]. The plastic strain $[\varepsilon^p]$ of carbon materials has also been investigated [3,4], by defining a pseudoelastoplastic constitutive law, based on a failure envelop. This model also takes into account the hardening and softening of the carbon material. The chemical strain $[\varepsilon^{ch}]$ may be associated with chemical contamination or reaction. In the Hall–Héroult cell, this strain is related to the carbon cathode swelling [14], and an analytical model, based on Fick's law, has also been proposed by Zolochovsky et al. [15]. The thermal strain $[\varepsilon^T]$ is related to the thermal expansion and is also a well known material property [2]. The time-dependant viscoelastoplasticity of the material is taken into account through the viscoelastic strains $[\varepsilon^a]$ and viscoplastic strains $[\varepsilon^v]$, which correspond to the anelastic (dissipative reversible) and viscous (dissipative irreversible) mechanisms, respectively. The viscoelastic strain is also associated with the creep/relaxation phenomenon.

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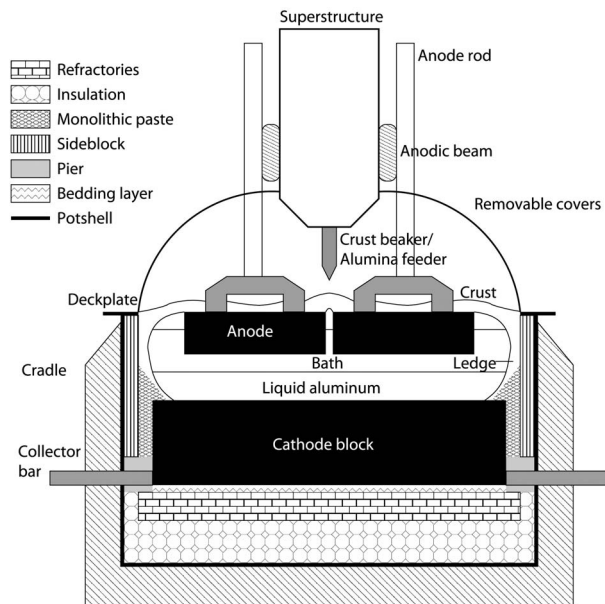


Fig. 1 Diagram of the Hall-Héroult electrolysis cell [6]

The last term of Eq. (1) represents any other strains that should be taken into consideration for a specific case (e.g., strain due to the magnetic forces).

1.3 Mechanical Aspects of the Cathode Block. The carbon materials employed in the cell lining include the ramming paste and the cathode. The ramming paste behavior has been addressed by D'Amours et al. [4] by developing a three-dimensional chemoelastoplastic model, which is a function of the hydrostatic pressure and the ramming paste baking. In fact, before pot startup, the ramming paste is green and its baking will begin during the preheating phase [2]. Once the paste is fully baked, its behavior is similar to that of a cathode block. The other carbon material, i.e., the carbon cathode, is one of the most important parts of the cell because it is mainly this component that determines the overall life expectancy of the cell [2]. In fact, high temperatures and sodium diffusion [15–20] all lead to the cathode block expansion.

The cell mechanical behavior is a strain driven problem. In fact, the materials of the cell lining are enclosed in the pot shell [2], thereby restraining the expansion of these materials. Therefore, it is very important that the cathode behavior during the operation of the Hall-Héroult cell be well understood. The model developed by D'Amours et al. [4] can assess the hardening, softening, and plastic strain of carbon materials. However, the mechanical behavior of the cathode under a state of sustained stress, i.e., on the long-term viscoelastic (creep/relaxation) behavior, is still lacking in some relevant knowledge.

Up to five years ago, no measurement was available on the creep behavior of carbon materials situated in the electrolysis cell (Fig. 1). The creep of these materials was first demonstrated by Hop [19]. The experimental results obtained from Hop's experiments were purely axial ones and were obtained for a short time period of a few hours only. The creep behavior has since been described by means of a 1D logarithmic equation. However, this cannot be used in the context of a three-dimensional, finite element analysis.

Creep and relaxation models are generally based on a rheological approach, involving unidirectional considerations. However, Fafard et al. [21] have extended this approach for the three-dimensional case where the scalar parameters of the rheological model have become fourth-order tensors. The so developed model is compatible with the finite element method and thus can be used in a finite element analysis. By taking into account the effect of

external parameters, such as temperature, sodium migration, and bath penetration in the lining (cathode swelling), the reduced time method can be used [21,22]. However, in the present case, using the reduced time method leads to a system of equations with non-constant coefficients and thus cannot be solved analytically. The solution will then be obtained by considering all model parameters to be constant. The dependencies of external parameters will be taken into account in the numerical simulation by substituting the constant parameters of each tensor by an algebraic expression function of temperature, chemical contamination, etc., evaluated at each time step.

Three-dimensional models of the viscoelastic behavior of the carbon cathode, adapted for a finite element analysis, are non-existent. However, concrete is a material macroscopically similar to the carbon cathode blocks and various authors have proposed models for this comparative situation [21,23,24]. Constitutive laws proposed by these authors are developed using the rheology and thermodynamics of irreversible processes. Due to the similarity of both materials (carbon cathode and concrete), this methodology has been applied to aid in the development of the three-dimensional, viscoelastic constitutive law applicable to the carbon cathode materials.

However, concrete models are generally developed in order to predict the longitudinal strains existing under uniaxial stresses and are extended to 3D cases by analogy with classical Hooke's law in considering a "creep" Poisson's ratio [23,24]. For many carbon materials, e.g., POCO graphite [25–27], experimental tests are generally performed at their industrial operational temperature (around 1000°C); so, the measurement of radial strains involves major technical difficulties. Moreover, in order to achieve the same stress levels, at both ambient and high temperatures, experiments at high temperatures must be conducted on small samples (~10 mm diameter). This is due to the degradation of the mechanical properties of some apparatus metallic parts at the elevated temperature, e.g., the extension/compression rod entering the test furnace [19]. As Poisson's ratio for carbon materials is generally small (~0.15), measuring the radial strains developed in those small samples is thus irrelevant.

Therefore, a parameter's identification generally relies on the axial strains only, as was the case of Fafard et al. [21] with concrete. However, in the case of the three-dimensional model, as developed here, and the similar one by Fafard et al. [21], the use of axial information only is not sufficient. This point will be expanded upon in the paper.

The model's parameters were identified for three different types of carbon cathode: semigraphitic (SG), graphitic (GQ), and graphitized (GZ). Experimental viscoelastic data for the identification comes from Picard [28]. The later works presents the experimental setup and experimental results. The main experimental objectives were as follows:

- to measure the axial and radial strains under uniaxial, sustained stresses of virgin samples at the ambient temperature (reference state)
- to confirm the presence of the creep/relaxation phenomena in the carbon cathode
- to compare the creep behavior of three types of carbon cathode (SG, GQ, and GZ)
- to propose creep mechanisms
- to provide data for the parameter's identification of the three-dimensional viscoelastic model for the three types of carbon cathode examined (SG, GQ, and GZ)

In this paper, the main objective is that of presenting a consistent three-dimensional viscoelastic model, compatible with classical solid mechanic equations, and usable in a finite element code for the carbon cathode material used in a Hall-Héroult cell. The proposed constitutive model is suitable for contaminated carbon materials at different temperatures. However, this paper presents the results of parameter identification of the three-dimensional

Table 1 Description of the carbon cathode blocks

	Semigraphitic	Graphitic	Graphitized
Filler	Anthracitic (20% graphite)	Graphitic	Petroleum coke
Binder	Pitch	Pitch	Pitch
HTT ^a (°C)	1200	1200	2400–3000

^aHeat treatment temperature.

model based on the viscoelastic experimental data of Picard [28], at the so-called “reference state,” i.e., at room temperature and for virgin material (noncontaminated). In a consistent viscoelastic model, it is important to start from parameters at a known reference state and then allow evolution of those parameters, for example, as a function of temperature and sodium contamination. This approach makes it possible, for example, to analyze the thermochemomechanical behavior of the electrolysis cell in a transient state during startup.

Tables 1 and 2 now present brief details of the selected carbon materials and some of their physical properties [2,17,28].

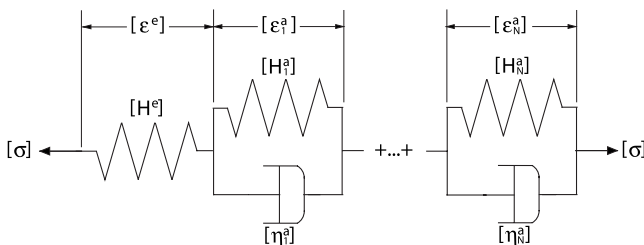
2 Three-Dimensional Viscoelastic Model

2.1 Proposed Viscoelastic Model. To identify the parameters of a model, the thermodynamic approach is used and thus internal state variables (ISVs) need to be defined [21]. Instead of choosing generic ISV [21], the choice of these variables is based on the use of a phenomenological approach. To represent the viscoelastic behavior of the carbon materials, a Kelvin–Voigt rheological model has been chosen, based on experimental observations, as shown in Fig. 2, where the parameters of the model are fourth-order tensors, while the strain and the stress are second-order tensors. As presented further in this paper, the model of Fig. 2 represents adequately the viscoelastic behavior of the studied material. One may refer to Simo et al. [29] for other viscoelastic models and their finite element implementation.

Assuming the conditions of a virgin material, having no permanent strain due to the earlier damage, constant temperature, and without other environmental phenomena that can cause additive strains, the total strain $[\varepsilon]$ (second-order tensor), as defined in Eq. (1), is reduced to

Table 2 Selected properties of carbon cathode blocks

	Semigraphitic	Graphitic	Graphitized
Young's modulus (MPa)	7695	4658	3913
Poisson's ratio	0.14	0.12	0.11
Crushing strength (MPa)	34.7	22.1	20.1
Real density (g/cm ³)	2.05–2.18	2.10–2.19	2.19–2.24
Apparent porosity (%)	15.4–16.4	17.1–17.5	16.9–17.3
Electrical resistivity ($\mu\Omega$ m)	12–18	16–24	8–14

**Fig. 2 Diagram of the three-dimensional viscoelastic rheological model**

$$[\varepsilon] = [\varepsilon^e] + \sum_{\alpha=1}^N [\varepsilon_\alpha^a] \quad (2)$$

where $[\varepsilon^e]$ is the instantaneous elastic strain, $[\varepsilon_\alpha^a]$ is the anelastic strain associated with the Kelvin–Voigt element α , and N is the number of Kelvin–Voigt elements. Thus, the phenomenological ISVs, related to the viscoelastic rheological model, are $[\varepsilon^e]$, associated with the elastic strain, and $[\varepsilon_\alpha^a]$, associated with the anelastic strain of the α viscoelastic element.

In the present paper, anelasticity means a reversible time-dependant, or reversible dissipative, mechanism. Therefore, due to this reversibility behavior, the anelastic strain (time dependent) is fully recovered at the end of the recovery period ($t_{\text{recovery}} \rightarrow \infty$). The zone known as the “recovery period” represents the period of time during which the strains evolve after the removal of all external stress applied to the solid.

The 3D rheological model proposed (Fig. 2), being constituted of one elastic element (non dissipative reversible mechanism) and N anelastic elements (dissipative reversible mechanisms), the total strain will be entirely recovered if the external stress is removed. It is important to note that the last remark is valid only for the viscoelastic behavior of the material, as represented by Eq. (2). In the case where material damage or permanent strains have occurred, residual strains will be observed at the end of the recovery period.

2.2 Thermodynamic Framework Applied to Solid Mechanics. The fundamental thermodynamic equations applied to solid mechanic are well known and are thus introduced without any preliminary considerations. The methodology used here is similar to that of Fafard et al. [21]. Using the Voigt notation, the Helmholtz free energy ψ is defined by

$$\rho\psi = \frac{1}{2} \langle \varepsilon^e \rangle [H^e] \{ \varepsilon^e \} + \frac{1}{2} \sum_{\alpha=1}^N \langle \varepsilon_\alpha^a \rangle [H_\alpha^a] \{ \varepsilon_\alpha^a \} \quad (3)$$

where $[H^e]$ is the Voigt form of the fourth-order elasticity tensor and $[H_\alpha^a]$ is the fourth-order tensor related to the spring of the α Kelvin–Voigt element. The parameters defining those tensors must be identified using an appropriate test setup. In order to take into account the dissipative mechanism of the model, a dissipative potential ϕ , per unit of volume, can be chosen as a quadratic form of its arguments:

$$\rho\phi = \frac{1}{2} \sum_{\alpha=1}^N \langle \dot{\varepsilon}_\alpha^a \rangle [\eta_\alpha^a] \{ \dot{\varepsilon}_\alpha^a \} \quad (4)$$

where the fourth-order tensor $[\eta_\alpha^a]$ is related to the dashpot of the α Kelvin–Voigt element. The parameters of this tensor must also be identified. Based on the thermodynamics of irreversible processes (TIP) and on the choice of the ISVs, the Clausius–Duhem inequality can be written as

$$\varphi_T + \varphi_M \geq 0 \quad (5)$$

where φ_T represents the thermal terms and φ_M the mechanical terms. For the viscoelastic problem, and assuming that the thermal part φ_T is satisfied, Eq. (5) can be rewritten, with the mechanical terms only, as

$$\varphi_m = [\sigma] : [\dot{\varepsilon}] - \dot{\psi} \geq 0 \quad (6)$$

$$\left([\sigma] - \rho \frac{\partial \psi}{\partial [\varepsilon^e]} \right) : [\dot{\varepsilon}^e] + \sum_{\alpha=1}^N \left([\sigma] - \rho \frac{\partial \psi}{\partial [\varepsilon_\alpha^a]} \right) : [\dot{\varepsilon}_\alpha^a] \geq 0 \quad (7)$$

where $[\sigma]$ is the stress tensor. The first term in the Eq. (7) is associated with the elastic strain [30] and, based on the fact that this strain is related to a reversible mechanism, one can write

$$[\sigma] = \rho \frac{\partial \psi}{\partial [\varepsilon^e]} \quad (8)$$

The Clausius–Duhem inequality (7) can then be rewritten as

$$\sum_{\alpha=1}^N \left([\sigma] - \rho \frac{\partial \psi}{\partial [\varepsilon_{\alpha}^a]} \right) : [\dot{\varepsilon}_{\alpha}^a] \geq 0 \quad (9)$$

To satisfy the former equation, the complementary evolution law proposed in the following equation is postulated:

$$[\sigma] - \rho \frac{\partial \psi}{\partial [\varepsilon_{\alpha}^a]} = \frac{\partial \phi}{\partial [\varepsilon_{\alpha}^a]} \quad (10)$$

Assuming that $[\eta_{\alpha}^a]$ are positive semidefinite, the use of Eqs. (4) and (10) ensures that Eq. (9) is satisfied and thus the Clausius–Duhem equation is also satisfied. Using the definitions of ψ and ϕ in Eq. (10) leads to, in Voigt notation,

$$[\eta_{\alpha}^a] \{\varepsilon_{\alpha}^a\} + [H_{\alpha}^a] \{\varepsilon_{\alpha}^a\} = \{\sigma\} \quad (11)$$

for a given α . Due to the choice of the ISV, the fourth-order tensors that need to be identified ($[\eta_{\alpha}^a]$ and $[H_{\alpha}^a]$) correspond to the tensors in the rheological model, presented earlier (Fig. 2).

In Eq. (11), the fourth-order tensors could be expressed as a function of different parameters, such as the temperature. The latter equation can be solved analytically or through numerical integration. Numerical integration of Eq. (11) can be done even if $[\eta_{\alpha}^a]$ and $[H_{\alpha}^a]$ are time-dependent tensors. In the present case, the time dependency is being taken into account implicitly through the parameters $T(t)$ and $\xi(t)$, representing the temperature and chemical contamination, respectively. For simplification, the shortened notation T and ξ will be used from now on.

Analytically, temperature and chemical contamination dependencies are generally taken into account through a scalar function in order to define an equivalent time, a method often used in this kind of problem [21,22]. The fourth-order tensors are then defined as follows:

$$[\eta_{\alpha}^a(T, \xi)] = f_{\alpha}(T, \xi) [\overline{\eta}_{\alpha}^a], \quad [H_{\alpha}^a(T, \xi)] = g_{\alpha}(T, \xi) [\overline{H}_{\alpha}^a], \quad [H^e(T, \xi)] = h(T, \xi) [\overline{H}^e] \quad (12)$$

The tensors with overbars express the quantity at the reference state, while the scalars multiplying the tensors express the dependencies of temperature (T) and chemical contamination (ξ) in the present case. However, the equivalent time method (see, for example, Refs. [21,22]) cannot be used to solve Eq. (11) analytically since this leads to a differential equation system with nonconstant coefficients.

In order to obtain an analytical solution, Eq. (11) must be a nonhomogeneous differential equation with constant coefficients. Thus, Eq. (11) will be solved at the reference state, i.e., by considering all coefficients independent of T and ξ and thus of time. This assumption implies that there is no evolution of the parameters. By assuming that the mechanical strains/stresses have no influence on the thermal and chemical variables, the mechanical problem can be uncoupled from the thermal and the chemical ones. Moreover, stating that the mechanical problem must respect the second principle of the thermodynamics, as shown by Eq. (6), is more restrictive and also ensures that the thermochemomechanical problem respects the second principle.

The dependencies of temperature T and chemical contamination ξ , or any other parameter, will then be taken into account in the finite element model. In fact, in a finite element model, the components of all fourth-order tensors will be defined as algebraic equations, which will be a function of T and ξ . The identification of these algebraic functions will be done experimentally. However, this paper focuses on the identification of the fourth-order tensor parameters at the reference state only.

2.3 Topology of the Tensors. Assuming that the carbon cathode is an isotropic material, all of the tensors in the rheological model have the form of the elastic one. In fact, since the tensors are related to the same material, it is logical to consider that they have the same topology [31], as defined hereafter:

$$[A(B, \beta)] = \frac{B}{(1-2\beta)(1+\beta)} \begin{bmatrix} 1-\beta & \beta & \beta & 0 & 0 & 0 \\ \beta & 1-\beta & \beta & 0 & 0 & 0 \\ \beta & \beta & 1-\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\beta)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\beta)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\beta)/2 \end{bmatrix} \quad (13)$$

Hence, each tensor of the three-dimensional model is defined as function of two positive independent variables, one being similar to Young's modulus (B) and the other one similar to Poisson's ratio (β). These variables must be identified in a laboratory, as explained later in this paper. In the case of the elastic tensor $[H^e]$, the variables are Young's modulus (E) and the Poisson's ratio (ν). The parameters of the tensors, as defined in Eqs. (3) and (4), which also correspond to the tensors of the rheological model, based on a phenomenological approach, are summarized below:

$$\begin{aligned} [H^e] &= [H^e(E, \nu)] \\ [H_{\alpha}^a] &= [H_{\alpha}^a(E_{H_{\alpha}^a}, \mu_{\alpha})] \\ [\eta_{\alpha}^a] &= [\eta_{\alpha}^a(E_{\eta_{\alpha}^a}, \varsigma_{\alpha})] \end{aligned} \quad (14)$$

2.4 Analytical Solution

2.4.1 Generalities. In finite element analysis, strain history is very important. In fact, generally, the finite element method is based on strain evolution, solved with the use of the classical interpolation method. From the strain increment, the incremental stress can be estimated. Plasticity, damage, hardening, etc., are all phenomena that can affect the evolution of strains and are taken into account through a numerical integration scheme [3,4,6,32]. It is thus important to solve the local system, based on strain prediction, to do a numerical implementation of the three-dimensional viscoelastic model.

Before implementing the model, all fourth-order tensors of Eq. (11) need to be identified. This latter equation can be developed for the relaxation case or the creep case, assuming the material has a linear viscoelasticity. Therefore, performing creep or relaxation tests leads to the identification of the same tensor parameters.

However, creep tests are easier to carry out since maintaining a constant strain demands a more complex apparatus. Moreover, it is easier to solve Eq. (11) for the creep case.

2.4.2 Creep Case. Starting from Eq. (11), one can rewrite it as, in Voigt notation,

$$\{\dot{\varepsilon}_\alpha^a\} + [\eta_\alpha^a]^{-1} [H_\alpha^a] \{\varepsilon_\alpha^a\} = [\eta_\alpha^a]^{-1} \{\sigma\} \quad (15)$$

for a given α . In the creep situation, $\{\sigma\}$ is known and constant. In order to solve Eq. (15), the diagonalization technique [21] (modal projection) is used. First, one needs to determine the eigenvalues and eigenvectors of the following system of equations:

$$([\eta_\alpha^a]^{-1} [H_\alpha^a] - \lambda_\alpha [I]) \{x_\alpha\} = 0 \quad (16)$$

where λ_α and $\{x_\alpha\}$ are the eigenvalues and eigenvectors, respectively, for the system. Hence, it can be easily demonstrated that

$$[X_\alpha]^{-1} [\eta_\alpha^a]^{-1} [H_\alpha^a] [X_\alpha] = [D_\alpha] \quad (17)$$

with $[X_\alpha]$ containing all the eigenvectors, placed column by column, while $[D_\alpha]$ is a diagonal matrix containing all of the corresponding eigenvalues. The modal projection [21] is then defined as

$$\{\varepsilon_\alpha^a\} = [X_\alpha] \{z_\alpha^a\} \quad (18)$$

and Eq. (15) can be rewritten as

$$[X_\alpha]^{-1} \{\dot{\varepsilon}_\alpha^a\} + [X_\alpha]^{-1} [\eta_\alpha^a]^{-1} [H_\alpha^a] \{\varepsilon_\alpha^a\} = [X_\alpha]^{-1} [\eta_\alpha^a]^{-1} \{\sigma\} \quad (19)$$

$$\{z_\alpha^a\} + [D_\alpha] \{z_\alpha^a\} = [X_\alpha]^{-1} [\eta_\alpha^a]^{-1} \{\sigma\} = \{F_\alpha\} \quad (20)$$

Equation (20) is now a fully uncoupled system (six equations), where the i th equation is

$$\dot{z}_{\alpha_i}^a + D_{\alpha_{ii}} z_{\alpha_i}^a = F_{\alpha_i} \quad (21)$$

Moreover, in the creep case and by always considering the tensors at a reference state, F_{α_i} is a constant. Equation (18) can then be rewritten, using standard analytical solution of Eq. (21) and by considering that the anelastic strain at the initial condition are $z_{\alpha_i}^a = \varepsilon_{\alpha_i}^a = 0$ (at $t=0$ only elastic strain can undergoes for the proposed model), as

$$\{\varepsilon_\alpha^a\} = [X_\alpha] \left\{ \begin{array}{c} F_{\alpha_1} \int_0^\tau e^{-\lambda_{\alpha_1}(t-\tau)} d\tau \\ F_{\alpha_2} \int_0^\tau e^{-\lambda_{\alpha_2}(t-\tau)} d\tau \\ \vdots \\ F_{\alpha_6} \int_0^\tau e^{-\lambda_{\alpha_6}(t-\tau)} d\tau \end{array} \right\} \quad (22)$$

From Eq. (8), the total strain can be expressed as

$$\{\varepsilon\} = [H^e]^{-1} \{\sigma\} + [X] \sum_{\alpha=1}^N \left\{ \begin{array}{c} F_{\alpha_1} \int_0^\tau e^{-\lambda_{\alpha_1}(t-\tau)} d\tau \\ F_{\alpha_2} \int_0^\tau e^{-\lambda_{\alpha_2}(t-\tau)} d\tau \\ \vdots \\ F_{\alpha_6} \int_0^\tau e^{-\lambda_{\alpha_6}(t-\tau)} d\tau \end{array} \right\} \quad (23)$$

where $[X_\alpha] = [X]$. In fact, all tensors have the same topology, as stated earlier, thus the eigenvectors of Eq. (16) are constant and identical for any value of α . The matrix $[X]$ is presented in Eq. (24). The column i of the matrix of eigenvectors $[X]$ is associated with the eigenvalues λ_{α_i}

$$[X] = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (24)$$

$$\lambda_{\alpha_1} = \frac{E_{H_\alpha^a}(1-2s_\alpha)}{E_{\eta_\alpha^a}(1-2\mu_\alpha)}, \quad \lambda_{\alpha_{2,3,4,5,6}} = \frac{E_{H_\alpha^a}(1+s_\alpha)}{E_{\eta_\alpha^a}(1+\mu_\alpha)} \quad (25)$$

Finally, the vector F_α can be expressed as

$$\begin{aligned} \{F_\alpha\} &= [X_\alpha]^{-1} [\eta_\alpha^a]^{-1} \{\sigma\} \\ &= - \frac{(2s_\alpha-1)\sigma_1 + (2s_\alpha-1)\sigma_2 + (2s_\alpha-1)\sigma_3}{3E_{\eta_\alpha^a}} \\ &\quad - \frac{(s_\alpha+1)\sigma_1 - 2(s_\alpha+1)\sigma_2 + (s_\alpha+1)\sigma_3}{3E_{\eta_\alpha^a}} \\ &\quad - \frac{(s_\alpha+1)\sigma_1 + (s_\alpha+1)\sigma_2 - 2(s_\alpha+1)\sigma_3}{3E_{\eta_\alpha^a}} \\ &\quad \times \frac{2(1+s_\alpha)\tau_{12}}{E_{\eta_\alpha^a}} \frac{2(1+s_\alpha)\tau_{13}}{E_{\eta_\alpha^a}} \frac{2(1+s_\alpha)\tau_{23}}{E_{\eta_\alpha^a}} \end{aligned} \quad (26)$$

For identification purposes, Eq. (23) is solved by assuming a uniaxial creep test (uniaxial compression under sustained stress, with free lateral stress) on a cylindrical carbon sample. The applied constant stress and the integrated strain equations are then

$$\langle \sigma \rangle = \langle -\sigma_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \rangle \quad (27)$$

$$\varepsilon_1 = -\sigma_1 \left(\frac{1}{E_{H^e}} + \frac{1}{9} \sum_{\alpha=1}^N K_\alpha^{a-1} \frac{1-e^{-\lambda_{\alpha_1} t}}{\lambda_{\alpha_1}} + \frac{1}{3} \sum_{\alpha=1}^N G_\alpha^{a-1} \frac{1-e^{-\lambda_{\alpha_2} t}}{\lambda_{\alpha_2}} \right) \quad (28)$$

$$\varepsilon_r = -\sigma_1 \left(-\frac{\nu}{E_{H^e}} + \frac{1}{9} \sum_{\alpha=1}^N K_\alpha^{a-1} \frac{1-e^{-\lambda_{\alpha_1} t}}{\lambda_{\alpha_1}} - \frac{1}{6} \sum_{\alpha=1}^N G_\alpha^{a-1} \frac{1-e^{-\lambda_{\alpha_2} t}}{\lambda_{\alpha_2}} \right) \quad (29)$$

with

$$K_\alpha^{a-1} = \frac{3(1-2s_\alpha)}{E_{\eta_\alpha^a}}, \quad G_\alpha^{a-1} = \frac{2(1+s_\alpha)}{E_{\eta_\alpha^a}} \quad (30)$$

where σ_1 is the constant axial stress, ε_1 the corresponding strain, and ε_r the hoop strain. All other strain components are null. The first part of Eqs. (28) and (29) represents the elastic strain. To take into account the creep strain $\{\varepsilon^{\text{creep}}\}$ only, these equations can be rewritten as

$$\varepsilon_1^{\text{creep}} = -\sigma_1 \left(\frac{1}{9} \sum_{\alpha=1}^N K_\alpha^{a-1} \frac{1-e^{-\lambda_{\alpha_1} t}}{\lambda_{\alpha_1}} + \frac{1}{3} \sum_{\alpha=1}^N G_\alpha^{a-1} \frac{1-e^{-\lambda_{\alpha_2} t}}{\lambda_{\alpha_2}} \right) \quad (31)$$

$$\varepsilon_r^{\text{creep}} = -\sigma_1 \left(\frac{1}{9} \sum_{\alpha=1}^N K_\alpha^{a-1} \frac{1-e^{-\lambda_{\alpha_1} t}}{\lambda_{\alpha_1}} - \frac{1}{6} \sum_{\alpha=1}^N G_\alpha^{a-1} \frac{1-e^{-\lambda_{\alpha_2} t}}{\lambda_{\alpha_2}} \right) \quad (32)$$

The coefficients K_α^{a-1} and G_α^{a-1} are related to the hydrostatic and deviatoric creep mechanisms, respectively. Moreover, the coefficients λ_{α_i} are positive and $1/\lambda_{\alpha_i}$ are related to a relaxation time. The elastic parameters E and ν of the tensor $[H^e]$ must be identi-

fied through standard compression tests, while the anelastic parameters E_{H^a} , μ_α , E_{H^e} , and s_α must be identified through experimental creep tests.

At $t=0$, Eqs. (28) and (29) can be simplified to express the instantaneous elastic strain:

$$\varepsilon_1 = -\sigma_1 \frac{1}{E_{H^e}}, \quad \varepsilon_r = \sigma_1 \frac{\nu}{E_{H^e}} \quad \text{with} \quad \frac{\varepsilon_r}{\varepsilon_1} = -\nu \quad (33)$$

At $t \rightarrow \infty$, the steady-state solution is found,

$$\varepsilon_1 = -\sigma_1 \left(\frac{1}{E_{H^e}} + \frac{1}{9} \sum_{\alpha=1}^N K_\alpha^{-1} + \frac{1}{3} \sum_{\alpha=1}^N G_\alpha^{-1} \right) \quad (34)$$

$$\varepsilon_r = -\sigma_1 \left(-\frac{\nu}{E_{H^e}} + \frac{1}{9} \sum_{\alpha=1}^N K_\alpha^{-1} - \frac{1}{6} \sum_{\alpha=1}^N G_\alpha^{-1} \right) \quad (35)$$

where

$$K_\alpha^{-1} = \frac{3(1-2\mu_\alpha)}{E_{H^a}}, \quad G_\alpha^{-1} = \frac{2(1+\mu_\alpha)}{E_{H^a}} \quad (36)$$

It can be observed that when $t \rightarrow \infty$, the steady-state solution depends only on the parameters representing the spring elements in Fig. 2, designated by $[H^e]$ and $[H^a]$. In fact, the solution at $t \rightarrow \infty$ leads to $\{\dot{\varepsilon}\}=0$, thus the steady-state solution is independent of time and those parameters related to time function, such as $[\eta_\alpha^a]$. The same result would be observed if the rheological model was 1D rather than 3D.

At $t \rightarrow \infty$, the three-dimensional viscoelastic model leads to “creep” Poisson’s ratio ν^{creep} defined by Eq. (37). The creep strains are defined by Eqs. (31) and (32).

$$\nu^{\text{creep}} = -\frac{\varepsilon_r^{\text{creep}}}{\varepsilon_1^{\text{creep}}} \quad (37)$$

Performing the ratio $\varepsilon_r^{\text{creep}}$ over $\varepsilon_1^{\text{creep}}$ at $t \rightarrow \infty$ leads to

$$\nu^{\text{creep}} = \frac{\sum_{i=1}^N (\mu_i / E_{H_j^a})}{\sum_{i=1}^N (1/E_{H_j^a})} \quad (38)$$

3 Parameter Identification

3.1 Generalities. In the three-dimensional viscoelastic model developed in this paper, there are several parameters that need to be identified. The first are the two coefficients of the fourth-order elastic tensor $[H^e]$, i.e., Young’s modulus (E) and Poisson’s ratio (ν), which have been easily determined by means of standard compression tests [28] and can be found in Table 2. Then, there are the two coefficients for each fourth-order tensor $[H_\alpha^a]$ and $[\eta_\alpha^a]$, as shown by Eq. (14). The last parameter, the number of Kelvin–Voigt elements N , is fixed. Therefore, only the creep strain needs to be taken into account since the elastic strain parameters are already known at this point [28]. Therefore, Eqs. (31) and (32) will be used for the identification process.

All parameters, except the elastic ones (E and ν), have been obtained through use of the least squares method to minimize the error of an objective function, through the use of a genetic algorithm. The genetic algorithm (GA) only requires range values for each parameter. However, a GA does not converge to a unique solution. Since $E_{H_\alpha^a}$ and $E_{\eta_\alpha^a}$ are similar to Young’s modulus and μ_α and s_α to Poisson’s ratio, the intervals have been chosen as

$$0 < \{\mu_\alpha, s_\alpha\} < 0.5 \quad (39)$$

$$0 < \{E_{H_\alpha^a}, E_{\eta_\alpha^a}\} < 10^7 \quad (40)$$

where the upper limit, 10^7 , of both $E_{H_\alpha^a}$ and $E_{\eta_\alpha^a}$ has been determined after many trials. The units of $E_{H_\alpha^a}$ are MPa, while those of $E_{\eta_\alpha^a}$ are MPa h. To take into account both the axial and the radial strains, the objective function to be minimized has the following form:

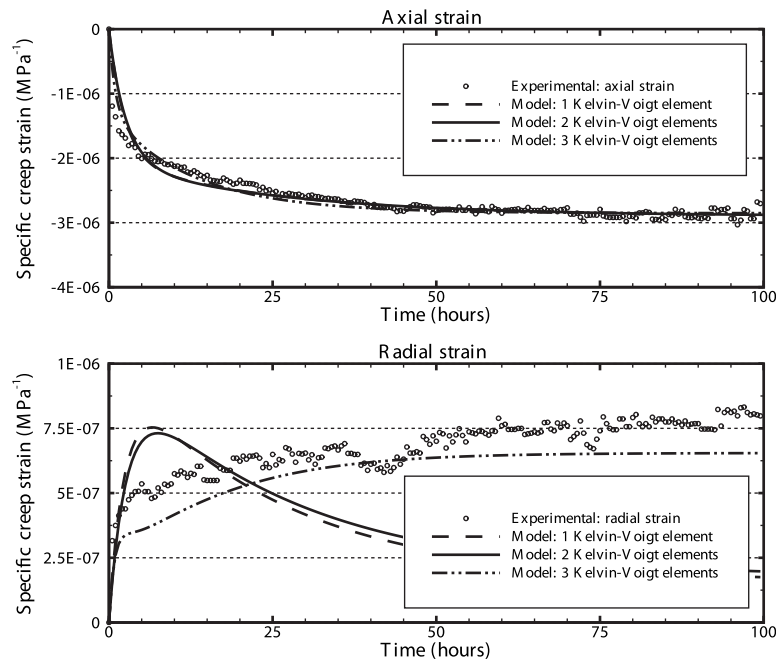


Fig. 3 Parameter’s identification of the SG carbon material with axial strains only

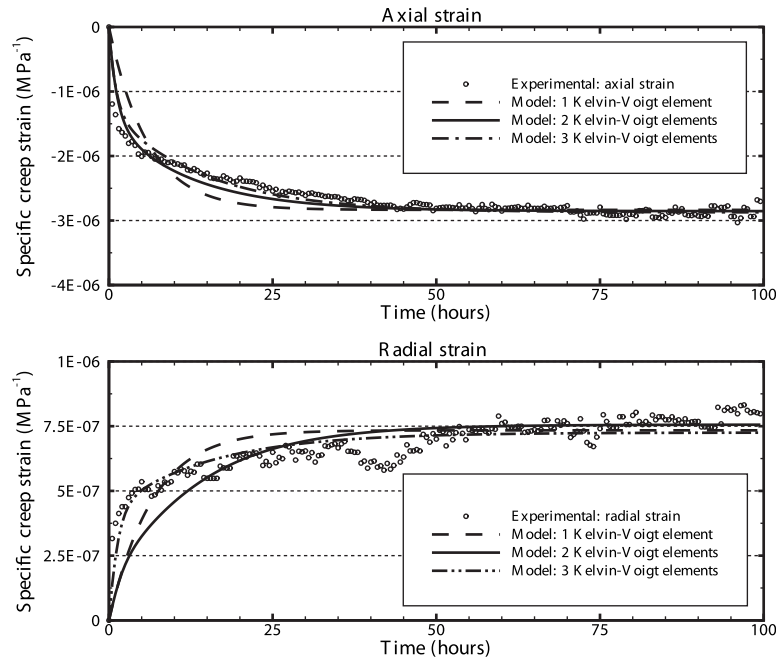


Fig. 4 Parameter's identification of the SG carbon material

$$\begin{aligned} & \frac{w_{\text{axial}}}{w_{\text{radial}} + w_{\text{axial}}} \sum_{i=1}^n (\varepsilon_{m_{\text{axial}_i}} - \varepsilon_{c_{\text{axial}_i}})^2 \\ & + \frac{w_{\text{radial}}}{w_{\text{radial}} + w_{\text{axial}}} \sum_{i=1}^n (\varepsilon_{m_{\text{radial}_i}} - \varepsilon_{c_{\text{radial}_i}})^2 \end{aligned} \quad (41)$$

where w_{axial} and w_{radial} are the weights, respectively, applied to the axial and radial data; c means that the data are calculated via Eqs. (31) and (32); m signals that the data are the measured one; n is the number of data used in the minimization method. The optimization method used was the Genetic Algorithm Optimization Toolbox (GAOT) for MATLAB.

Experimental data were obtained by conducting creep tests at three different stress levels (20%, 30%, and 40% of the crushing strength σ_u) on the three carbon cathode materials (SG, GQ, GZ). The crushing strength of each material is presented in Table 2. The details of these experiments are presented in Picard [28]. However, it is important to note that the optimization is based on the mean values for the specific creep. The specific creep (SC) strain is defined by

$$\{\varepsilon^{\text{SC}}\} = \frac{\{\varepsilon\} - \{\varepsilon^e\}}{|\sigma_1|} \quad (42)$$

where $|\sigma_1|$ is the absolute value of the applied constant stress in the axial direction. Thus, Eqs. (31) and (32) need to be normalized by $|\sigma_1|$, before using the GA.

3.2 Preliminary Results. Radial strains are rarely measured in creep tests. Thus, parameter identification is generally based only on the axial strains [21]. However, in the case of this three-dimensional model, the use of axial information only is not sufficient. In fact, using $w_{\text{radial}}=0$ in the objective function (41) leads to the results shown in Fig. 3. Clearly, using only the axial strain information could lead to a very good correlation between the model and the experiment in the axial direction. However, in the radial direction, the model prediction is not physically admissible. In fact, it is impossible to observe an increase, followed by a decrease, of the radial strain under constant sustained stress.

Hence, one must use both radial and axial strains in order to obtain relevant, three-dimensional viscoelastic parameters, even if the measured radial strains are often of poor quality [28].

To put emphasis on the axial strains without totally neglecting the radial ones, the “weight method” is used, as shown by the Eq. (41). Even if there is no optimum ratio, the weights used were generally as follows: $w_{\text{radial}}=1$ and $w_{\text{axial}}=5$, which is similar to Poisson's ratio.

3.3 Parameters Identification. Using both axial and radial datasets leads to a very good prediction in the two directions (Figs. 4–6). Thus, the radial strain can be seen as a supplementary constraint to the optimization problem, i.e., to converge toward a solution with a physical meaning. Also, as shown in Figs. 4–6, the determination of the parameters was performed using three different configurations, i.e. using one, two, and three Kelvin–Voigt elements ($N=1,2,3$). Using more than three Kelvin–Voigt elements does not improve the model prediction. Moreover, the acceptance of a solution obtained with the GA is based on both the error given by the objective, Eq. (41), and the visual aspect of that solution judged to be relevant. Also, it must be mentioned that the radial strains of the GZ material have been estimated by assuming $\varepsilon_{\text{radial}} = -\nu \varepsilon_{\text{axial}}$, where ν is Poisson's ratio, since the measured radial strains were unusable for the parameter's identification [28].

Since that the predictions made with $N=2$ and $N=3$ are very similar for the long-term behavior, a model with $N=2$ has been chosen and the parameters are presented in Table 3. This choice is based on finite element considerations; the complexity of the stiffness matrix increases with the number of anelastic elements [6]. However, in the case of the GZ material, only one anelastic element ($N=1$) could fairly represent the viscoelastic behavior, while in the case of the GQ and SG materials, two anelastic elements ($N=2$) are a minimum. Both GQ and SG materials have a high quantity of anthracite or amorphous coke content, while the structure of the GZ material is almost 100% GQ [2,33]. Thus, one can assume that the creep of a highly uniform, oriented lattice (GZ) arises from one dominating mechanism ($N=1$), while the creep mechanisms of a more globally disoriented lattice (SG, GQ) are more complex ($N>1$). In general, the anthracitic part is related to

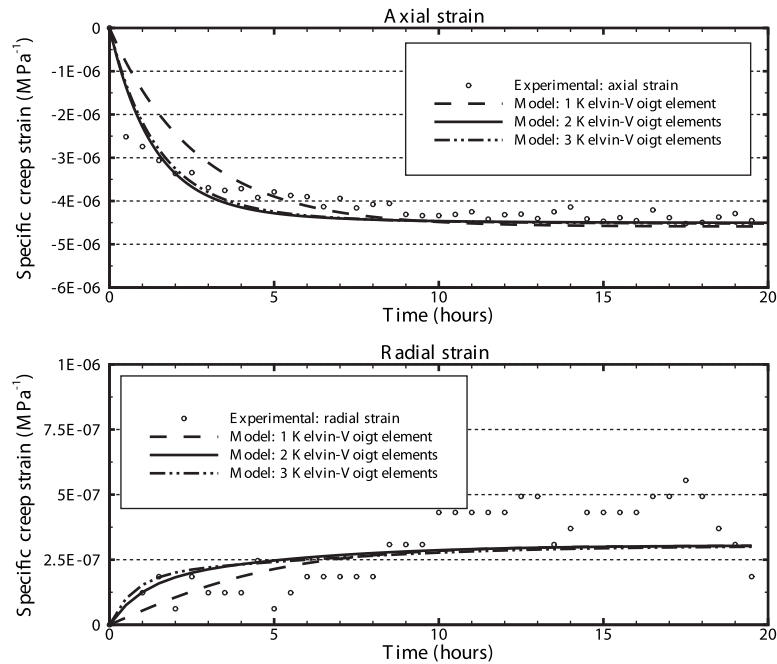


Fig. 5 Parameter's identification of the GQ carbon material

the binder. Hence, one can assume that the binder structure is responsible for the supplementary anelastic elements needed to adequately represent the viscoelastic behavior of the GQ and SG materials.

Figures 7–9 present the hydrostatic and deviatoric creep mechanisms, related to each Kelvin–Voigt element (anelastic mechanism) of the three-dimensional model and Table 4 presents the values of the coefficients K_{α}^a and G_{α}^a , related to the hydrostatic and deviatoric creep mechanisms, respectively, and those of the coefficients $\lambda_{\alpha_1}^{-1}$, related to a relaxation time. The bold lines in Figs. 7–9 are the results related to the first or second mechanism, while

the nonbold lines represent the total strains associated with different mechanisms in the axial or radial direction. It can be observed that the long-term creep strain of the SG material (Fig. 7) is driven by the second deviatoric mechanism and that both Kelvin–Voigt elements contribute to model the creep behavior. In the GQ case (Fig. 8), the first anelastic element seems to be responsible for the long-term behavior, while the second element is responsible for the short time behavior and it is the dominant one. Thus, this shows up the rapidity of the creep mechanisms in the GQ material. The results of the GZ material (Fig. 9) clearly show that only one Kelvin–Voigt element has been used since the second element

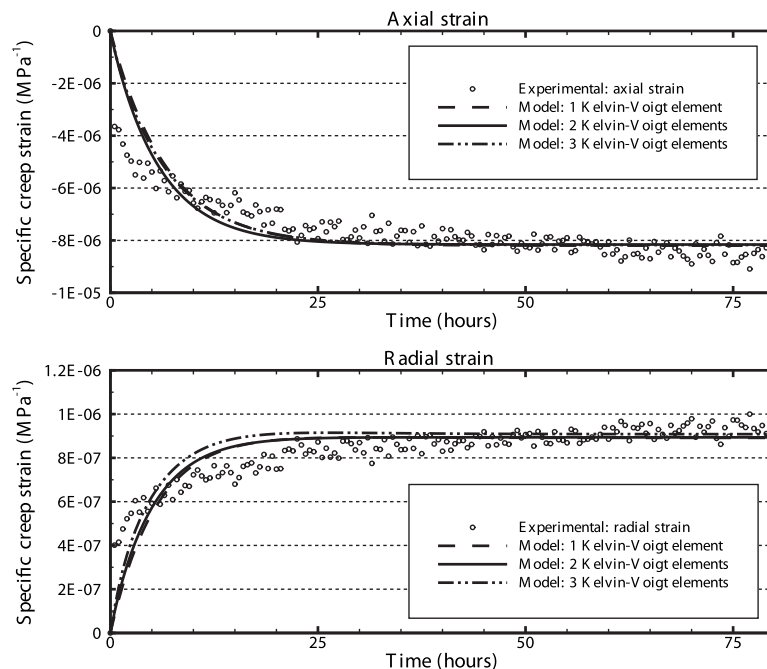


Fig. 6 Parameter's identification of the GZ carbon material

Table 3 Parameters of the rheological model with two Kelvin–Voigt elements ($N=2$)

	$E_{\eta_1^a}$ (MPa h)	s_1	$E_{H_1^a}$ (MPa)	μ_1	$E_{\eta_2^a}$ (MPa h)	s_2	$E_{H_2^a}$ (MPa)	μ_2
SG	840,306	0.07	712,153	0.13	7,562,665	0.20	689,502	0.39
GQ	6,193,999	0.11	1,760,191	0.27	322,398	0.05	253,990	0.04
GZ	731,232	0.12	124,487	0.11	5,198,490	0.18	8,031,829	0.23

merely contribute to the creep behavior. This is in accordance with the observations stated above and based on Fig. 6. All these observations on the three materials are relevant in both axial and radial directions. It is important to note that since all anelastic elements (Kelvin–Voigt elements) are placed in series, it is possible to modify their order without altering the results.

Also, based on the results presented in Table 4, the higher relaxation times $\lambda_{\alpha_1}^{-1}$ associated with each material can be ordered as follows:

$$GQ < GZ < SG \quad (43)$$

This classification is similar to the one of $t_{e_{total}}^c$ (time required to reach a steady state) presented in Picard [28].

Moreover, in Table 5, Poisson's ratio ν is compared with creep Poisson's ratio ν^{creep} at $t \rightarrow \infty$, defined in Eq. (38), for the three

materials. In the case of the GZ material, both Poisson's ratios are identical since the radial strains were estimated using $\varepsilon_{radial} = -\nu \varepsilon_{axial}$.

Creep Poisson's ratio of the other two materials (SG and GQ) may, however, indicate an evolution of that ratio over time. Performing creep Poisson's ratio $\nu^{creep}(t)$ for the (SG) material leads to the result shown in Fig. 10. A similar result was obtained for the GQ material. This result (Fig. 10) clearly shows the evolution of creep Poisson's ratio and points to the necessity for taking into account both the axial and the radial strains for the identification process, not just the axial one alone. Also, this result means that the assumption made to estimate the radial strain of the GZ material, i.e., $\varepsilon_{radial} = -\nu \varepsilon_{axial}$, is inadequate since ν is not constant throughout the creep. However, as shown by Fig. 3, it is impera-

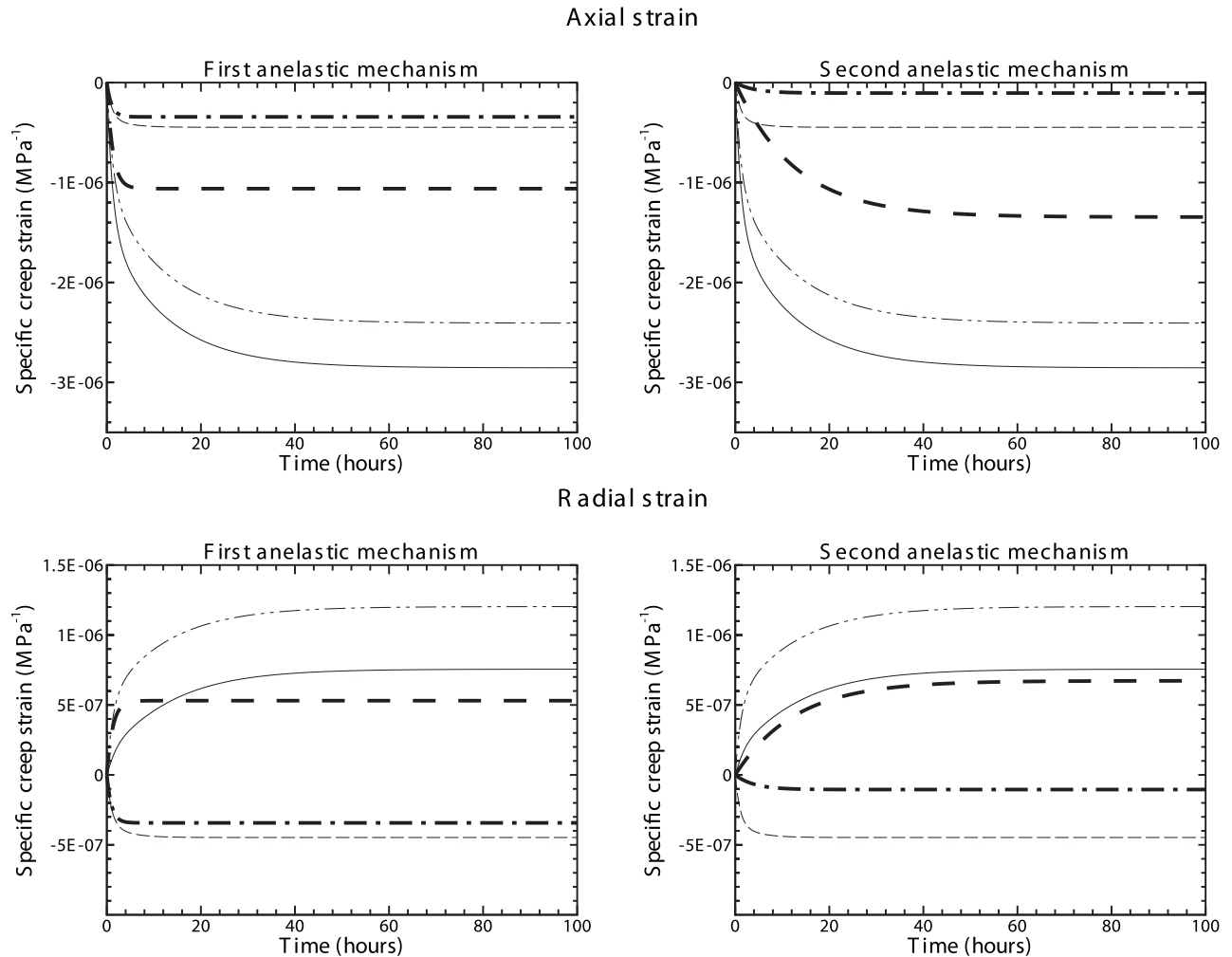


Fig. 7 Hydrostatic and deviatoric creep mechanisms for the SG material. (—) hydrostatic creep strain of the anelastic mechanism α ; (---) deviatoric creep strain of the anelastic mechanism α ; (- - -) total hydrostatic creep strain; (— · —) total deviatoric creep strain; (—) total creep strain.

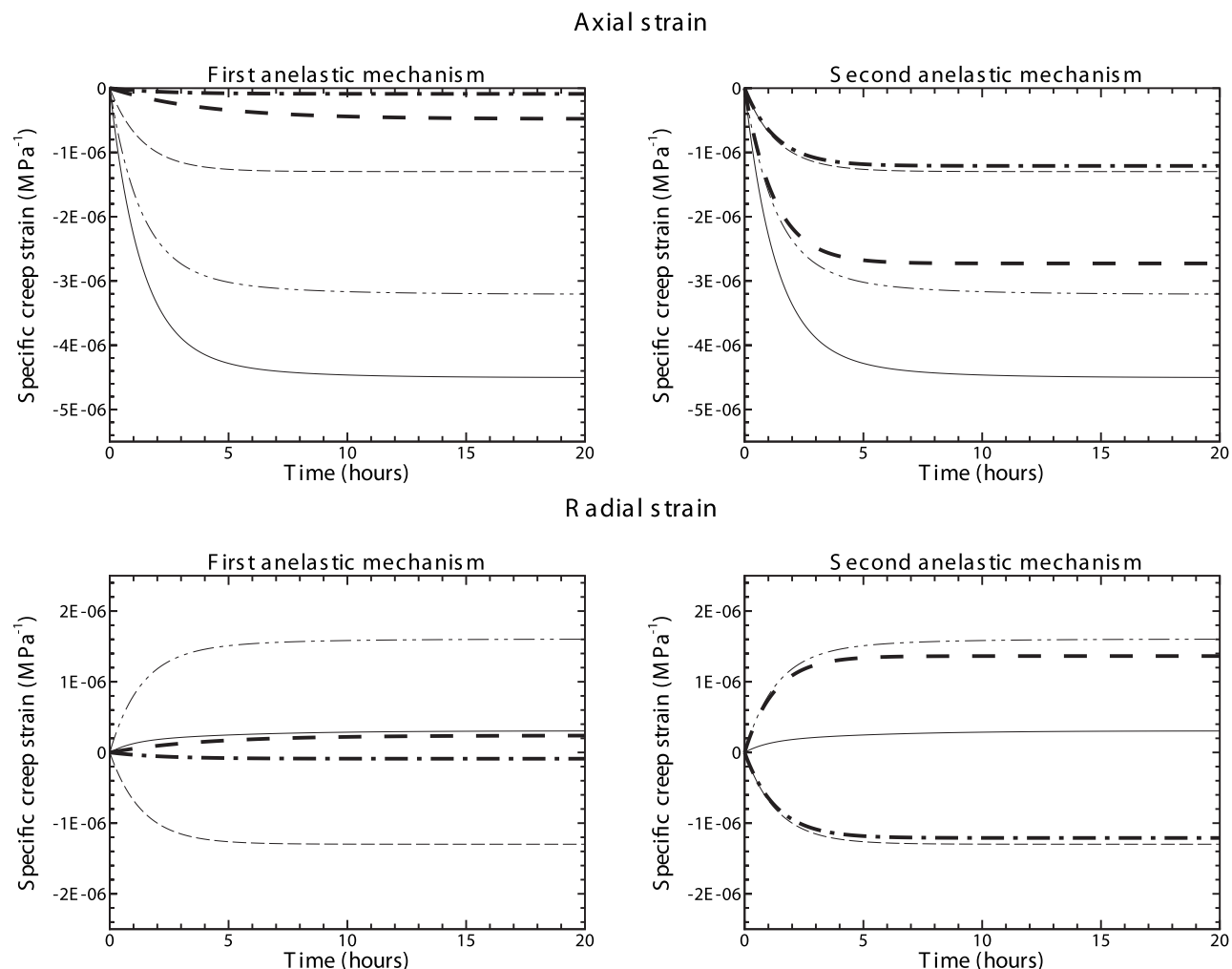


Fig. 8 Hydrostatic and deviatoric creep mechanisms for the GQ material. (—) hydrostatic creep strain of the anelastic mechanism α ; (---) deviatoric creep strain of the anelastic mechanism α ; (- - -) total hydrostatic creep strain; (— · —) total deviatoric creep strain; (—) total creep strain.

tive to use a radial dataset for the parameters' identification. Thus, it is better to use an estimate of the radial strain than not to use one at all.

4 Conclusions

The main contribution of this paper is the development of a three-dimensional, viscoelastic constitutive model, based on the work of Fafard et al. [21], and using ISVs, based on a phenomenological approach, related to a one-dimensional rheological model, being extended to the three-dimensional case. The 3D model so developed can be used in a finite element code.

The long-term creep tests were carried out at a reference state, i.e., with virgin material at ambient temperature, to identify, with the help of a GA, the constant parameters of the fourth-order tensors of the three-dimensional viscoelastic model. The dependencies of the temperature and sodium/bath penetration will be taken into account through the numerical integration performed in the finite element analysis.

Moreover, while developing a three-dimensional model, it is imperative to take into account the radial strain of the samples in the optimization method to, in turn, identify correctly the parameters of the constitutive tensors. In fact, if the identification is performed only with the axial dataset, the parameters found cannot be used in Eq. (29) to predict the radial strain.

Finally, the number N of anelastic elements, or Kelvin–Voigt elements, necessary to represent the long-term viscoelastic behavior (creep/relaxation) of carbon materials seems to be related to the heterogeneity of the latter. A more homogenous structure requires less anelastic elements. This could be linked to the deformation mechanisms proposed by Picard [28] and based on works performed on similar materials [33–38].

5 Discussion

Future works will consist mainly of two program phases that will be developed in parallel. The first will focus on long-term creep tests at high temperatures in order to determine its influence on the parameters of the proposed model. The creep tests should also be extended to the operational condition of a Hall–Héroult electrolysis cell, i.e., at 950°C under electrolysis. The second phase will focus on the numerical modeling by adding the viscoelastic behavior of the carbon cathode to the elastoplastic behavior already implemented by D'Amours et al. [4] (see also D'Amours [3]). It will be undertaken through the use of the finite element toolbox, FESH++, which stands for Finite Element Shell in C++ [8].

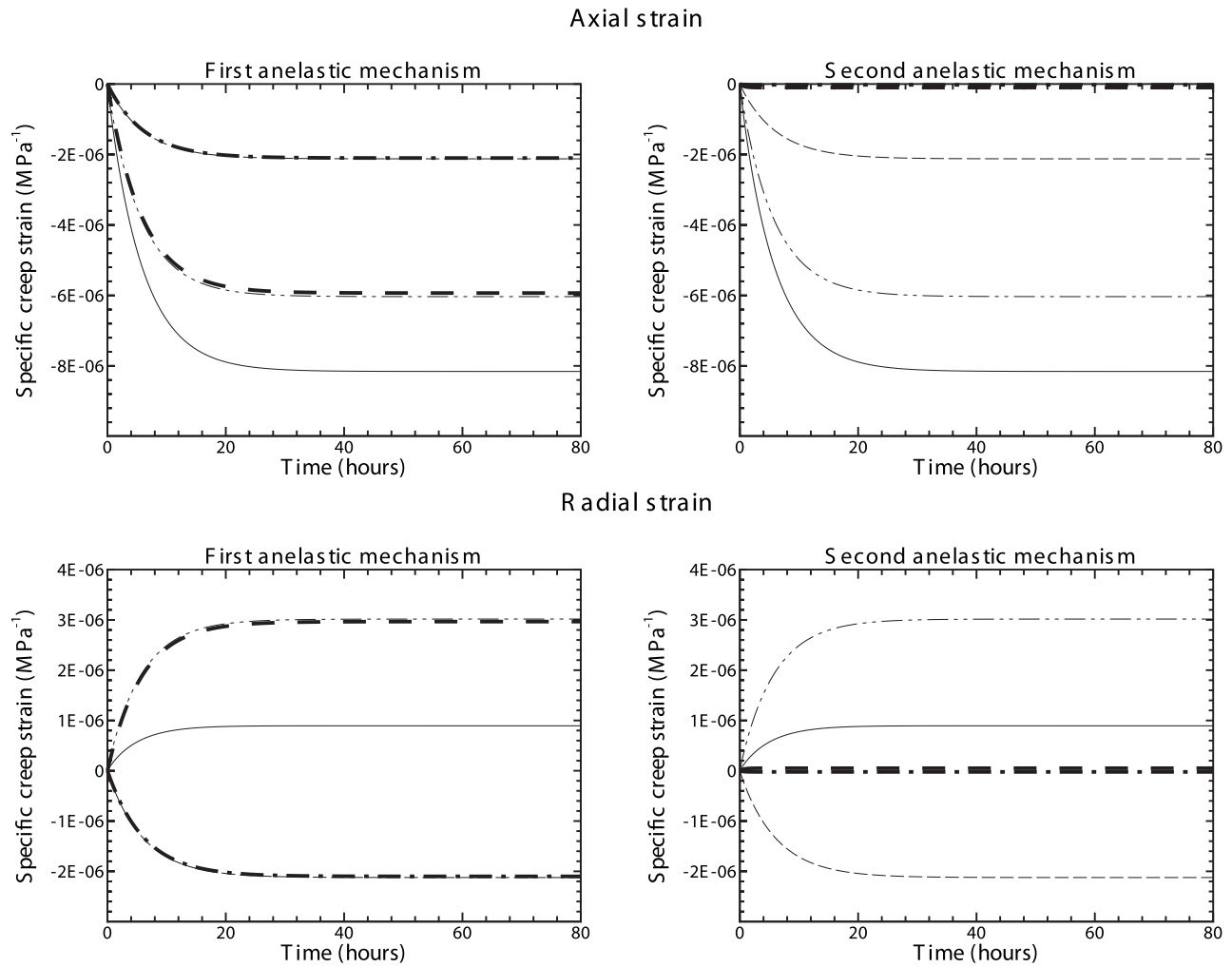


Fig. 9 Hydrostatic and deviatoric creep mechanisms for the graphitized material. (—) hydrostatic creep strain of the anelastic mechanism α ; (---) deviatoric creep strain of the anelastic mechanism α ; (- - -) total hydrostatic creep strain; (— — —) total deviatoric creep strain; (—) total creep strain.

Table 4 Related hydrostatic creep, deviatoric creep, and relaxation time coefficients

First anelastic mechanism				
	K_1^α (MPa h)	λ_{11}^{-1} (h)	G_1^a (MPa h)	λ_{12}^{-1} (h)
SG	326,133	1.00643	392,458	1.24955
GQ	2,685,990	2.14236	2,775,940	3.99316
GZ	324,197	6.12911	325,258	5.78861
Second anelastic mechanism				
	K_2^a (MPa h)	λ_{21}^{-1} (h)	G_2^a (MPa h)	λ_{22}^{-1} (h)
SG	4,226,420	3.97864	3,146,470	12.7028
GQ	120,544	1.31235	152,905	1.25115
GZ	2,729,040	0.55790	2,198,060	0.67122

Table 5 Creep Poisson's ratio

	ν	ν^{creep}
SG	0.14	0.26
GQ	0.12	0.07
GZ	0.11	0.11

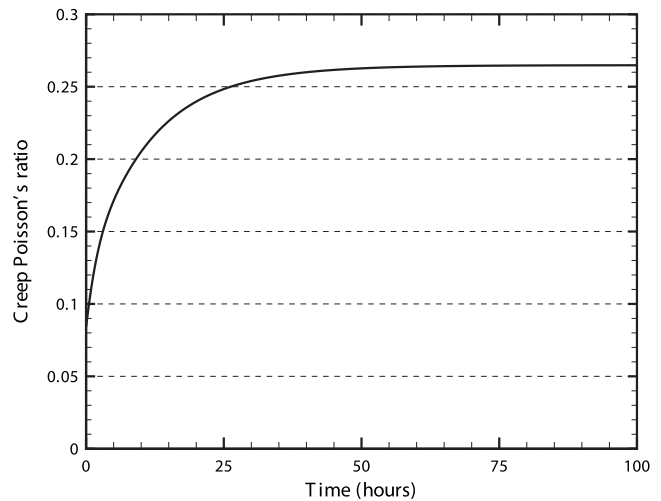


Fig. 10 Creep Poisson's ratio of the SG material, based on Eq. (37)

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Nomenclature

E	= Young's modulus
$E_{H\alpha}^a, \mu_\alpha$	= constitutive parameters associated with the tensor $[H_\alpha^a]$
$E_{\eta_\alpha}^a, \zeta_\alpha$	= constitutive parameters associated with the tensor $[\eta_\alpha^a]$
G_α^{-1}	= coefficients related to the deviatoric creep mechanism
$[H^e]$	= fourth-order elasticity tensor
$[H_\alpha^a]$	= fourth-order tensor related to the spring of the α Kelvin–Voigt element
K_α^{-1}	= coefficients related to the hydrostatic mechanism
N	= number of Kelvin–Voigt element
T	= temperature
$[X_\alpha]$	= matrix of eigenvectors
t	= time
w	= weight
$\{x_\alpha\}$	= eigenvectors
$\{z_\alpha^a\}$	= modal coordinates
$[\varepsilon]$	= total strain
$[\varepsilon^e]$	= elastic strain
$[\varepsilon^p]$	= plastic strain
$[\varepsilon^{ch}]$	= chemical strain
$[\varepsilon^T]$	= thermal strain
$[\varepsilon^a]$	= anelastic strain
$[\varepsilon_\alpha^a]$	= anelastic strain
$[\varepsilon^v]$	= viscous strain
$\{\varepsilon^{SC}\}$	= SC strain
ε_1	= axial strain
ε_r	= hoop strain
ε_r^{creep}	= radial creep strain
ε_1^{creep}	= axial creep strain
λ_α	= eigenvalues
$[\eta_\alpha^a]$	= fourth-order tensor related to the dashpot of the α Kelvin–Voigt element
ν	= Poisson's ratio
ν^{creep}	= "creep Poisson's ratio"
$[\sigma]$	= stress tensor
σ_1	= axial stress
σ_u	= crushing strength
ϕ	= dissipative potential
ψ	= Helmholtz free energy
ξ	= chemical contamination index
α	= Kelvin–Voigt element

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Nonisothermal Transient Flow in Natural Gas Pipeline

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The fully implicit finite-difference method is used to solve the continuity, momentum, and energy equations for flow within a gas pipeline. This methodology (1) incorporates the convective inertia term in the conservation of momentum equation, (2) treats the compressibility factor as a function of temperature and pressure, and (3) considers the friction factor as a function of the Reynolds number and pipe roughness. The fully implicit method representation of the equations offers the advantage of guaranteed stability for a large time step, which is very useful for gas pipeline industry. The results show that the effect of treating the gas in a nonisothermal manner is extremely necessary for pipeline flow calculation accuracies, especially for rapid transient process. It also indicates that the convective inertia term plays an important role in the gas flow analysis and cannot be neglected from the calculation. [DOI: 10.1115/1.2840046]

Keywords: fully implicit method, nonisothermal, compressibility factor, natural gas pipeline, Joule–Thompson effect, convective inertia term

Introduction

Mathematical models are used to design, optimize, and operate increasingly complex natural gas pipeline systems. Researchers continue to develop unsteady mathematical models that focus on the unsteady nature of these systems. Many related design problems, however, could be solved using steady-state modeling.

Several researchers attempted different numerical methods to solve the compressible unsteady one-dimensional flow. Thorley and Tiley [1] provided an excellent literature review of solution methods for transient pipeline analysis. Some methods reviewed included the method of characteristics, the explicit and implicit finite-difference methods, and finite element methods.

Wylie et al. [2], and Issa and Spalding [3] used the method of characteristics. An advantage of the method of characteristics is that it can handle discontinuities in the simulation [1]. Although the main disadvantage is that it is comparatively slow, the time steps are restricted by the stability criterion such that the time step needs to be small enough to satisfy the Courant condition.

Rachford and Dupont [4] used a Galerkin finite element method by considering two-dimensional elements in space-time to simulate isothermal transient gas flow. This method has not been commonly used for gas transient flow modeling because computing time and the storage requirement are high. On the other hand, this method can handle some boundary conditions better than finite-difference methods. The element size, shape, and distribution are relatively flexible, so that nonuniform internal distribution of nodal points is possible.

There are several explicit finite-difference methods such as first-order and second-order approximations. In general, a first-order approximation is not sufficiently accurate for modeling gas transients in a pipeline, and so attention is focused on the second-order methods [1]. The main disadvantage of the second-order approximation is that these methods require a large amount of computer time and hence are not suitable for the analysis of large systems or for the evaluation of unsteady flows over long periods of time. Poloni et al. [5] used the Lax–Wendroff method, which used the explicit method.

Wylie et al. [6] presented an implicit finite-difference method that uses central differences and compared this method with the

method of characteristics. They showed that the implicit method is very accurate for large time steps and so in the implicit procedure, the maximum practical time increment is limited by the frequency of the variables imposed at the boundary conditions, rather than the stability criterion limit of the method of characteristics.

Heath and Blunt [7] used the Crank–Nicolson method to solve the conservation of mass and conservation of momentum equations for slow transients in isothermal gas flow. The main disadvantage of this method is that it does not always give a stable solution according to the Neumann stability analysis of large time steps for nonlinear problems.

Kiuchi [8] described a fully implicit finite-difference method to solve isothermal unsteady compressible flow. After neglecting the convective inertia term in the momentum equation, the von Neumann stability analysis on the finite-difference equations of a pipe showed that the equations were unconditionally stable. This method was compared with other methods such as the method of characteristics, the Lax–Wendroff method, the Guys method, and the Crank–Nicolson method. This comparison showed that the fully implicit method was very accurate for a small number of sections and a large time step. Furthermore, the short computational time makes this model very appealing for gas pipeline applications.

Zhou and Adewumi [9] presented a new method for solving one-dimensional, isothermal transient natural gas flow in a horizontal pipeline without neglecting any terms in the conservation of momentum equation. In simulating transient flow of single-phase natural gas in pipelines, most previous investigators had neglected the convective term in the momentum equation, which resulted in a loss of accuracy in the simulation results.

Osiadacz and Chaczykowski [10] compared isothermal and nonisothermal transient models for gas pipelines using constant compressibility and friction factors, while neglecting the convective inertia term. They showed that there exists a significant difference in the pressure profile along the pipeline between isothermal and nonisothermal conditions and this difference increases when the density of the gas increases.

The objective of the current study is to simulate nonisothermal, one-dimensional compressible flow through a gas pipeline with more detail by considering (1) the compressibility factor as a function of pressure and temperature, (2) the friction factor as a function of the Reynolds number and pipe roughness, and (3) the convective inertia term in the momentum equation. The continuity, momentum, and energy equations are written in terms of the

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mass flow rate $\dot{m}$. This is a matter of convenience since the primary interest, in this case, is the mass flow rate as a function of time and location. This is accomplished by replacing the velocity with the mass flow rate. The method of solution is the fully implicit finite-difference method, which is very suitable for gas pipeline simulation because of its large step size and low computation time [1,8]. The algorithm used to solve the nonlinear finite-difference equations of a pipe is based on the Newton–Raphson Method.

Governing Equations

The nonisothermal flow of natural gas in pipelines is governed by the time-dependent continuity, momentum, and energy equations, and an equation of state for homogeneous, geometrically one-dimensional single-phase flow. By solving these equations, the behavior of gas parameters can be obtained along the pipeline.

Issa and Spalding [3], Deen and Reintsema [11], and Thorley and Tiley [1] developed the basic equations for one-dimensional, unsteady, compressible flow that include the effects of wall friction and heat transfer.

In continuity equation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \quad (1)$$

In momentum equation,

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial x} + \frac{\partial P}{\partial x} = -\frac{w}{A} - \rho g \sin \theta \quad (2)$$

where

$$w = \frac{f \rho v |v|}{8} \pi D$$

Ouyang and Aziz [12] discussed the friction factor (f) calculation in detail and applicable ranges of Reynolds number and pipe roughness for different correlations. In this study, the Colebrook friction factor equation is used as follows:

$$f^{0.5} = -2 \log \left(\frac{\varepsilon}{3.7D} + \frac{2.51}{\text{Re} f^{0.5}} \right) \quad (3)$$

This equation is widely used in natural gas industry and is based on experimental data. It combines both partially and fully turbulent flow regimes and is most suitable for cases where the pipeline is operating in transition zone [13].

In conservation of energy,

$$\rho \frac{\partial h}{\partial t} + \rho v \frac{\partial h}{\partial x} - \frac{\partial P}{\partial t} - v \frac{\partial P}{\partial x} = \frac{\Omega + wv}{A} \quad (4)$$

Ω is the heat flow into the pipe per unit length of pipe per unit time as follows:

$$\Omega = -\frac{4UA(T - T_G)}{D} \quad (5)$$

In equation of state,

$$\frac{P}{\rho} = ZRT \quad (6)$$

where Z is the modified form of the formulation of Dranchuck et al. [14] as

$$Z = 1 + \left(A_1 + \frac{A_2}{Tr} + \frac{A_3}{Tr^3} \right) \rho r + \left(A_4 + \frac{A_5}{Tr} \right) \rho r^2 + \frac{A_6}{Tr^3} \rho r^3 \quad (7)$$

A_1 – A_6 is given in Table 1. To obtain h in terms of P , Z , and T , Zemansky [15] described the thermodynamic identity

Table 1 Coefficients used in compressibility factor equation

A_1	A_2	A_3	A_4	A_5	A_6
0.31506	−1.0467	−0.5783	0.5353	−0.6123	0.6895

$$dh = Cp dT + \left\{ \frac{T}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_P + 1 \right\} \frac{dP}{\rho} \quad (8)$$

The resulting set of equations is

$$\left(\frac{\partial P}{\partial t} \right) + v \left(\frac{\partial P}{\partial x} \right) + \rho V_w^2 \left(\frac{\partial v}{\partial x} \right) = \frac{V_w^2}{C_p T} \left[1 + \frac{T}{Z} \left(\frac{\partial Z}{\partial T} \right)_P \right] \frac{\Omega + wv}{A} \quad (9)$$

$$\left(\frac{\partial v}{\partial t} \right) + v \left(\frac{\partial v}{\partial x} \right) + \frac{1}{\rho} \left(\frac{\partial P}{\partial x} \right) = -\frac{w}{A\rho} - g \sin \theta \quad (10)$$

$$\begin{aligned} \left(\frac{\partial T}{\partial t} \right) + v \left(\frac{\partial T}{\partial x} \right) + \frac{V_w^2}{C_p} \left[1 + \frac{T}{Z} \left(\frac{\partial Z}{\partial T} \right)_P \right] \left(\frac{\partial v}{\partial x} \right) \\ = \frac{V_w^2}{C_p P} \left[1 - \frac{P}{Z} \left(\frac{\partial Z}{\partial P} \right)_T \right] \frac{\Omega + wv}{A} \end{aligned} \quad (11)$$

The parameter V_w is

$$V_w = \sqrt{\frac{ZRT}{\left\{ 1 - \frac{P}{Z} \left(\frac{\partial Z}{\partial P} \right)_T - \frac{P}{\rho C_p T} \left[1 + \frac{T}{Z} \left(\frac{\partial Z}{\partial T} \right)_P \right]^2 \right\}}} \quad (12)$$

The continuity, momentum, and energy equations can then be written in terms of the mass flow rate $\dot{m}$. This is a matter of convenience since the primary interest, in this case, is the mass flow rate as a function of time and location. This is accomplished by replacing the velocity with the mass flow rate.

$$v = \frac{\dot{m}}{\rho A} = \frac{\dot{m} ZRT}{PA} \quad (13)$$

Therefore [16,17],

$$\begin{aligned} \frac{\partial P}{\partial t} + \frac{\dot{m} ZRT}{PA} \left[1 - \frac{V_w^2}{ZRT} \left(1 - \frac{P}{Z} \left(\frac{\partial Z}{\partial P} \right)_T \right) \right] \frac{\partial P}{\partial x} \\ + \dot{m} V_w^2 \left(\frac{1}{\dot{m}} \frac{\partial \dot{m}}{\partial x} + \frac{1}{T} \left(1 + \frac{T}{Z} \left(\frac{\partial Z}{\partial T} \right)_P \right) \frac{\partial T}{\partial x} \right) \\ = \frac{V_w^2}{C_p T} \left\{ 1 + \frac{T}{Z} \left(\frac{\partial Z}{\partial T} \right)_P \right\} \left(\frac{\Omega}{A} + \frac{\dot{m} ZRT}{PA^2} w \right) \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{ZRT}{AP^2} \left\{ P \left(\frac{\partial \dot{m}}{\partial t} + \frac{\dot{m} ZRT}{PA} \frac{\partial \dot{m}}{\partial x} \right) - \dot{m} \left(1 - \frac{P}{Z} \left(\frac{\partial Z}{\partial P} \right)_T \right) \left(\frac{\partial P}{\partial t} + \frac{\dot{m} ZRT}{PA} \frac{\partial P}{\partial x} \right) \right. \\ \left. + \frac{P \dot{m}}{T} \left(1 + \frac{T}{Z} \left(\frac{\partial Z}{\partial T} \right)_P \right) \left(\frac{\partial T}{\partial t} + \frac{\dot{m} ZRT}{PA} \frac{\partial T}{\partial x} \right) \right\} + \frac{ZRT}{P} \frac{\partial P}{\partial x} \\ = -\frac{w ZRT}{PA} - g \sin \theta \end{aligned} \quad (15)$$

$$\begin{aligned} \frac{\partial T}{\partial t} + \frac{V_w^2}{C_p} \left(1 + \frac{T}{Z} \left(\frac{\partial Z}{\partial T} \right)_P \right) \frac{ZRT}{AP^2} \left[P \frac{\partial \dot{m}}{\partial x} - \dot{m} \left(1 - \frac{P}{Z} \left(\frac{\partial Z}{\partial P} \right)_T \right) \frac{\partial P}{\partial x} \right] \\ + \frac{\dot{m} ZRT}{PA} \left(1 + \frac{V_w^2}{C_p T} \left(1 + \frac{T}{Z} \left(\frac{\partial Z}{\partial T} \right)_P \right) \right) \frac{\partial T}{\partial x} \\ = \frac{V_w^2}{C_p P} \left\{ 1 - \frac{P}{Z} \left(\frac{\partial Z}{\partial P} \right)_T \right\} \left(\frac{\Omega}{A} + \frac{\dot{m} ZRT}{PA^2} w \right) \end{aligned} \quad (16)$$

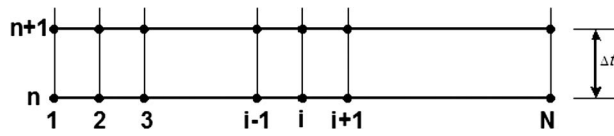


Fig. 1 Mesh of the solution

Numerical Formulation Using the Fully Implicit Method

The fully implicit method consists of transforming Eqs. (14)–(16) from partial differential equations to algebraic equations by using finite-difference approximations for the partial derivatives. Figure 1 shows a mesh used in this calculation. The pipe has (N) nodes and (n) time levels.

The partial derivatives with respect to time are approximated by [8]

$$\frac{\partial Y}{\partial t} = \frac{(Y_{i+1}^{n+1} + Y_i^{n+1} - Y_{i+1}^n - Y_i^n)}{2\Delta t} \quad O(\Delta t)^2 \quad (17)$$

The geometric partial derivatives are transformed by [8]

$$\frac{\partial Y}{\partial x} = \frac{Y_{i+1}^{n+1} - Y_i^{n+1}}{\Delta x} \quad O(\Delta x) \quad (18)$$

Finally, the individual terms at an interface between the nodes are approximated by [8]

$$Y = \frac{Y_{i+1}^{n+1} + Y_i^{n+1}}{2} \quad (19)$$

The parameter Y represents P , $\dot{m}$, and T in Eqs. (17)–(19). Substituting Eqs. (17)–(19) into Eqs. (14)–(16) results in three sets of equations for each node and without considering node N , there will be $(3N-3)$ equations for a pipe. The number of unknown values at time level $n+1$, which consists of pressure, temperature, and mass flow rate at each node, is $3N$. Three equations will come from boundary condition, and then there are $3N$ unknowns and $3N$ equations. These equations are nonlinear and the Newton–Raphson method is applied to solve these equations for the compressible, nonisothermal transient flows through a pipe.

Solution Comparison With the Kiuchi Result

Kiuchi [8] applied the fully implicit finite-difference method to the isothermal formulation of the conservation equations:

$$\frac{\partial P}{\partial t} + \frac{V_w^2}{A} \frac{\partial \dot{m}}{\partial x} = 0 \quad (20)$$

$$\frac{1}{A} \frac{\partial \dot{m}}{\partial t} + \frac{\partial}{\partial x} \left(\frac{\dot{m}^2 V_w^2}{PA^2} \right) + \frac{\partial P}{\partial x} + \left(\frac{f V_w^2}{2DA^2} \right) \frac{\dot{m} |\dot{m}|}{P} + \frac{Pg}{V_w^2} \sin \theta = 0 \quad (21)$$

The wave speed V_w in his formulation was $\sqrt{ZRT}$. The first and second terms in the momentum equation (Eq. (21)) are the convective terms. In the Kiuchi study, the second term (convective inertia) was neglected due to the assumption of low flow velocity with respect to wave speed. In order to compare the model described in this study with the results presented by Kiuchi, the following adjustments were made.

- The convective inertia term was temporarily set to zero.
- The flow field was treated as isothermal.
- The friction factor was assumed to be constant with a value of 0.008.

The comparison uses the system described by Kiuchi, as illustrated in Fig. 2. This system is characterized by a simple straight

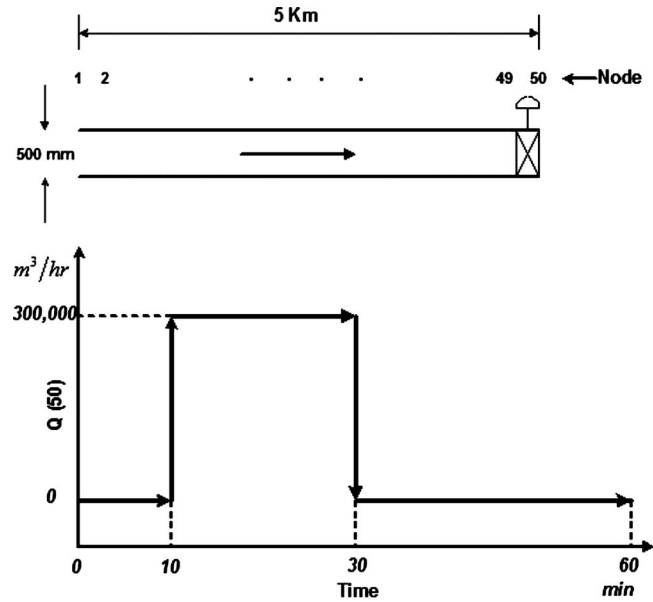


Fig. 2 Pipe information and boundary condition for flow through the valve

pipe segment of 5 km in length with a 500 mm internal diameter that holds a gas of molecular weight 18.0 at a pressure of 5 MPa.

At time $t=10$ min, the outlet valve opens and the gas out flow increases from zero to 300,000 scmh (scmh denotes standard cubic meters per hour), while the inlet pressure is maintained at 5 MPa. After maintaining this condition for 20 min, the outlet valve closes. Solutions are performed using different grid densities (5, 10, 20, 30, 40, 50, 60, ...) to ensure a grid-independent solution. A grid density of 50 is found to be sufficient for this particular problem.

Kiuchi compared his method with Crank–Nicolson, method of characteristics, Lax–Wendroff method, and Guys method. The Crank–Nicolson method gave an unstable solution in the case of a large time step. Lax–Wendroff method and the method of characteristics use the explicit method and gave a correct answer when pipes are divided into sufficiently small sections for both rapid and slow transient phenomena but it took significant computation time. Kiuchi also showed that Guys method, which uses the implicit method, has good stability for a small time step and has greatly damped oscillation.

Figures 3–5 compare the results from the current study to those from the Kiuchi model, Figs. 3(b), 4(b), and 5(b), and show the variation of flow rates at Node 1 with respect to time for different time steps. As shown in the figures, the results of the present work are identical to the Kiuchi solutions. As illustrated in Fig. 6 for time step $\Delta t=0.01$ min, the flow rate at the inlet end of the pipe

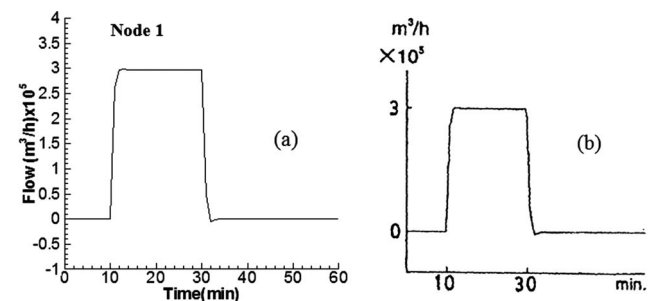


Fig. 3 Comparison of present work (a) with Kiuchi model (b) for $\Delta t=1.0$ min

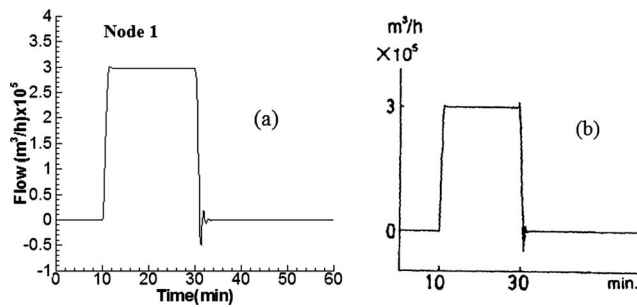


Fig. 4 Comparison of present work (a) with Kiuchi model (b) for $\Delta t=0.1$ min

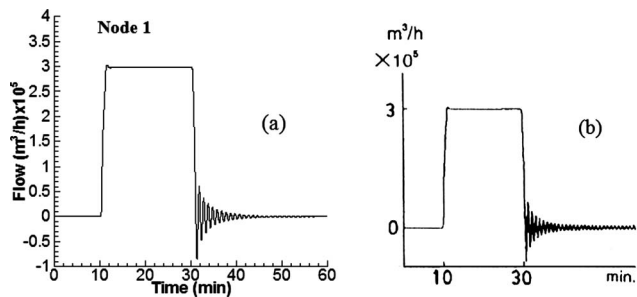


Fig. 5 Comparison of present work (a) with Kiuchi model (b) for $\Delta t=0.01$ min

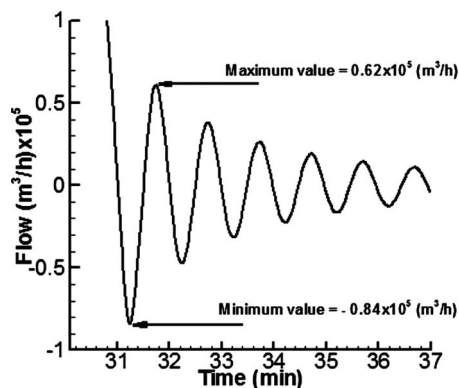


Fig. 6 Oscillation on flow rate at closing valve process for $\Delta t=0.01$ min

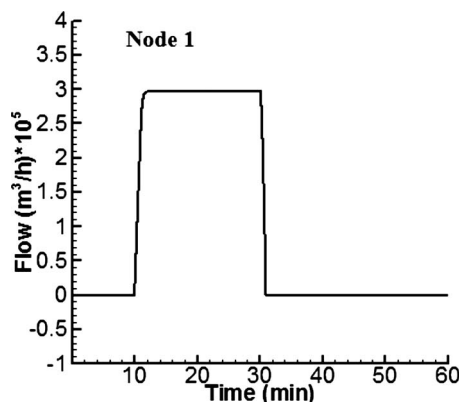


Fig. 7 Variation of inlet flow rate without considering temporal inertia term in momentum equation $\Delta t=0.01$ min

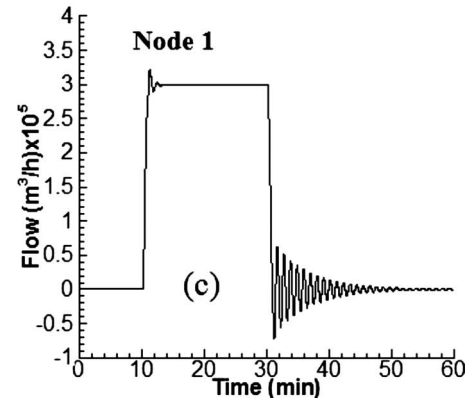
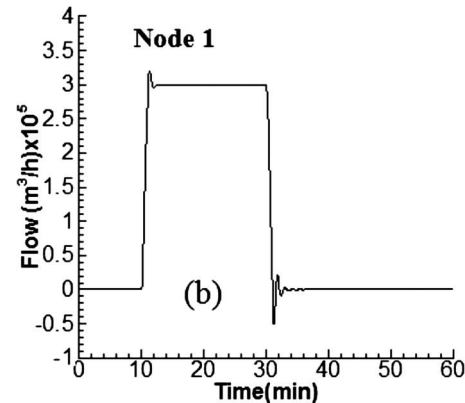
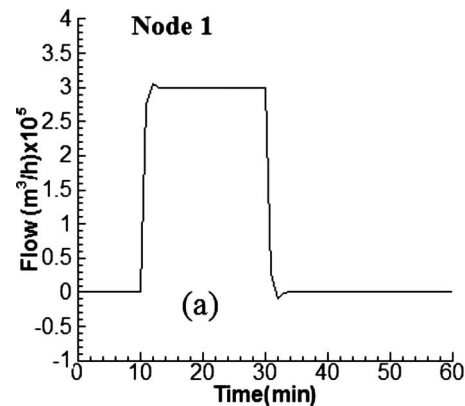


Fig. 8 Solution for isothermal model by considering convective inertia term for (a) $\Delta t=1.0$ min, (b) $\Delta t=0.1$ min, and (c) $\Delta t=0.01$ min

decreases to zero after closing the valve at time 31 min and thereafter, it continues to oscillate around the final value ("0"). The oscillation gradually damps to zero in about 60 min. The oscillation in the flow rate is also found to be symmetrical about the steady-state value. The maximum value during the oscillation is $0.62 \times 10^5 \text{ m}^3/\text{h}$, while the minimum value is about $-0.84 \times 10^5 \text{ m}^3/\text{h}$.

As the time step decreases, the ability to capture the physically existing flow oscillation became more apparent. Eventually, this oscillation damps out due to conservation of mass and momentum.

Now, if the temporal inertia term ($\partial m / \partial t$) in the momentum equation is neglected, the oscillation will disappear, as shown in Fig. 7. With a decreasing time step, the impact of the temporal inertia term increases and the oscillation appears in the results. Comparing Figs. 3 and 7, as the time step increases, the results reach when the temporal inertia term is neglected.

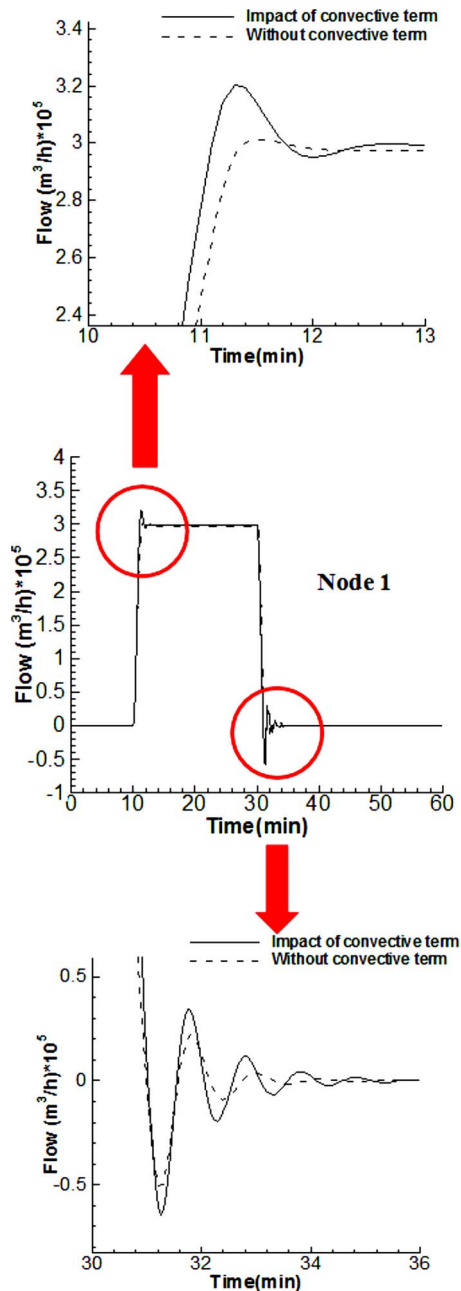


Fig. 9 Comparison of the result for impact of convective inertia term and without convective inertia for $\Delta t=0.1$ min

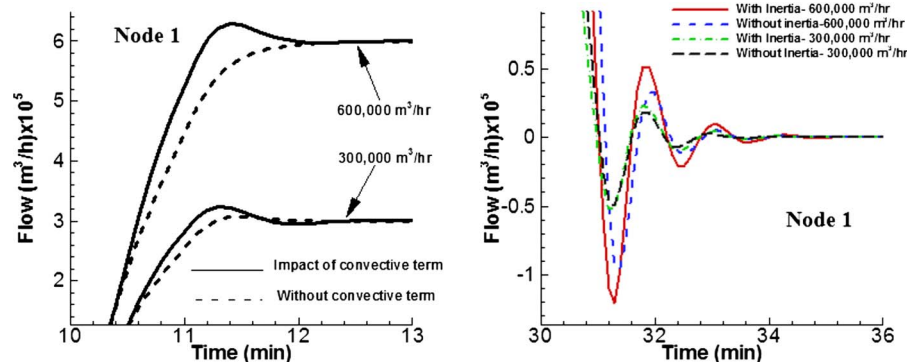


Fig. 10 Comparison of the result for impact of convective inertia term and without convective inertia for different flow rates and $\Delta t=0.1$ min

Impact of Convective Inertia Term

The Kiuchi study neglected the convective inertia term of Eq. (21) and assumed a uniform and constant friction factor. The present study investigates the effect of the convective inertia term as well as friction term variation. Figure 8 shows the effect of the time step on the computed flow rate at Node 1 when the convective inertia term is included. As depicted in Fig. 8, the computed flow behaves differently at different time steps until the flow reaches its steady-state conditions. For a large time step, some of the governing equation terms such as convection inertia term may not be affected by small disturbances during the sudden closing or opening of valves, and the calculation may not capture the fluid flow physics.

In this case, the existence of convection inertia term in momentum equations has no significant effect in the flow, as shown in Fig. 8 ($\Delta t=1.0$ min). On the other hand, for small time step, the effect of this term is significantly important and plays an important role in the fluctuation amplitude and damping prior to reaching the flow to its steady-state condition, as shown in Fig. 8 ($\Delta t=0.1$ min and $\Delta t=0.01$ min).

When the valve at the outlet end of the pipe is suddenly closed, the fluid layer upstream of the valve is brought to rest immediately, which causes the pressure at this point to rise. As the pressure wave gradually travels toward the inlet end, the rate of increase of the pressure at the end point is higher than the locations upstream of it, thus resulting in a reverse flow. The reverse flow in turn causes the pressure to decrease, thereby creating an oscillation in the pressure and flow variation, which depends on the time step used for simulation, as shown in this figure.

Figure 9 clearly shows the impact of the convective inertia term and without convective inertia term. As shown in this figure, the convective inertia term cannot be neglected from the equation during the opening and closing valve (sudden change). During the opening valve event, the gas speed increases suddenly; therefore, the term of convective inertia term in Eq. (21), $\partial/\partial x(m^2 V_w^2/PA^2)$, becomes more significant. But, during closing valve process, the mass flow rate drops down and therefore, the gas velocity suddenly decreases to zero; at this time, the convective inertia term is not more significant than before.

With increasing mass flow rate in the pipeline, the speed of gas increases and therefore, the convective term plays an important role on the result, as shown in Fig. 10. This affects fluctuation amplitude results which for 600,000 m³/h case are more significant than 300,000 m³/h case at closing and opening valve condition.

Impact of Nonisothermal Condition

The energy equation (Eq. (16)) that is coupled to the momentum and continuity equations is used to account for the temperature variation along the pipe. Specifically, the governing equations

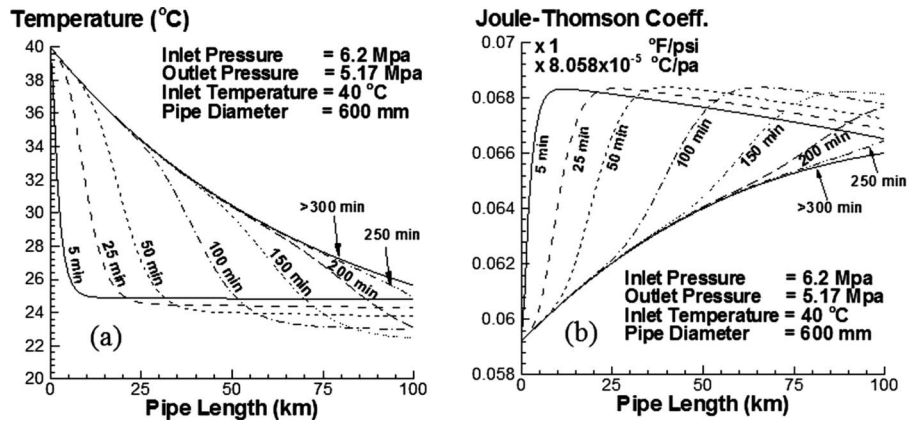


Fig. 11 Variation of temperature (a) and Joule–Thompson coefficient (b) along the pipe with respect to time

are solved using the fully implicit method. To validate the nonisothermal equation, the steady-state result for temperature is compared with the temperature equation developed by Coulter and Bardon [18]. The temperature equation is defined as

$$T_{\text{out}} = (T_{\text{in}} - b)\exp(-aL) + b \quad (22)$$

where

$$a = \frac{\pi DU}{\dot{m} C_p}$$

$$b = T_G + \left(\frac{\eta_J}{a} \right) \frac{P_{\text{out}} - P_{\text{in}}}{L}$$

In this equation, all units are in British unit. A straight pipe with the following specifications is considered for our validation:

- pipe length: 100 km
- pipe diameter: 600 mm
- inlet pressure: 6.3 MPa
- outlet pressure: 5.17 MPa
- inlet temperature: 40 °C
- ground temperature: 25 °C
- overall heat transfer coefficient: 2.84 (W/m² K)

Figure 11(a) shows the transient variation of temperature along the pipe. As shown in this figure, as time increases, the temperature along the pipe becomes more stable where after 300 min reaches the steady-state condition. At time 300 s, the analytical given outlet temperature reached the steady state by numerical calculation (Fig. 12). As illustrated in Fig. 11(a), the outlet tem-

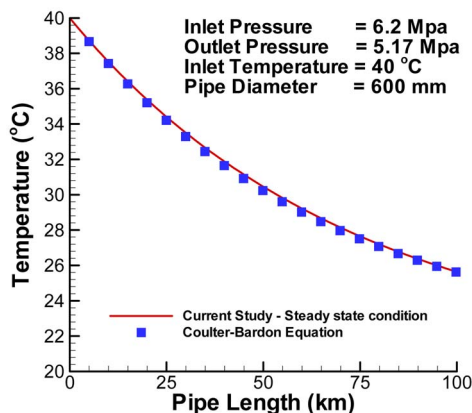


Fig. 12 Temperature comparison for current study and Coulter–Bardon equation

perature is changed with time. This variation of temperature is due to Joule–Thompson effect, which is the temperature change that occurs during isenthalpic expansion or compression of a gas. In a pipeline, it causes an expanding gas to cool and a compressed gas to warm.

The Joule–Thompson coefficient is defined as

$$\eta_J = \frac{R}{C_p} \left(\frac{T^2}{P} \right) \left(\frac{\partial Z}{\partial T} \right)_P \quad (23)$$

Figure 11(b) shows the transient variation of Joule–Thompson coefficient along the pipe. In Eq. (22), an average Joule–Thompson coefficient (0.063 °R/psi–5.08 × 10⁻⁶ °C/Pa) is considered. Based on this assumption, Fig. 12 illustrates the comparison between the temperature result for steady-state condition with Coulter and Bardon equation (Eq. (22)). As depicted in Fig. 12, the result of temperature is quietly matched with the Coulter and Bardon equation.

Now, consider the closure problem discussed earlier to see the impact of nonisothermal condition. The mass flow rate and pressure boundary conditions are identical to the isothermal case and also Fig. 13 shows the temperature boundary condition for the inlet flow as the valve is opened.

The effect of a nonisothermal condition on the flow is illustrated in Fig. 14 for different time steps. This figure shows that the flow rate is significantly affected by the temperature until the flow reaches the steady-state condition. While the mass flow rate of the isothermal condition immediately approaches the steady-state condition, the mass flow rate of the nonisothermal condition gradually increases until it reaches the steady-state condition. This occurs because the density varies with respect to temperature. By opening and closing the valve, the flow, pressure, and temperature change, and this results in a density change. This density change propagates gradually from the left side (upstream) of the pipe to

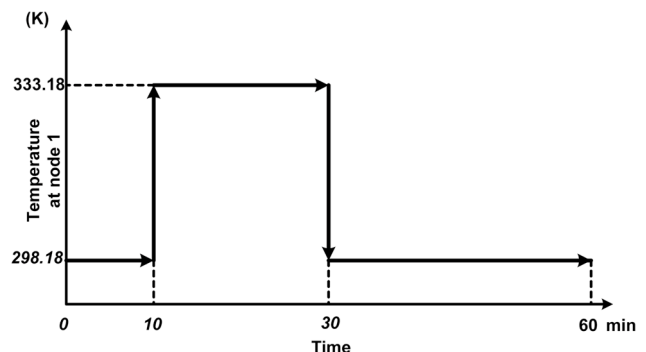


Fig. 13 Temperature boundary condition at Node 1

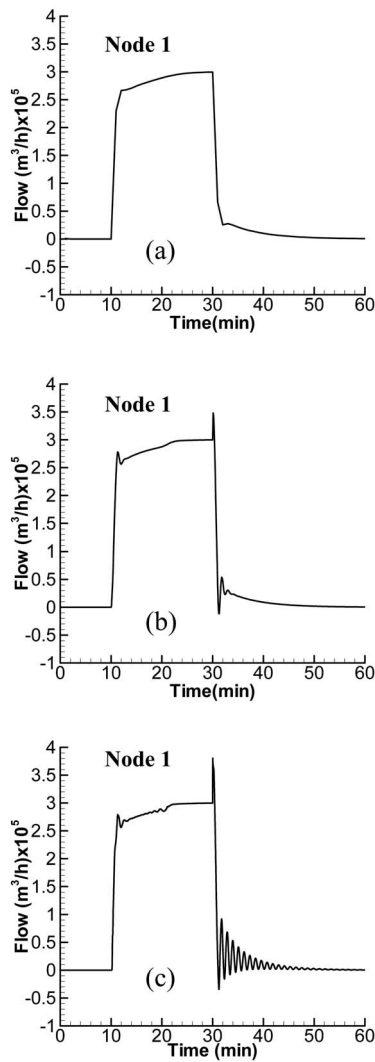


Fig. 14 Solution for nonisothermal condition for (a) $\Delta t = 1.0$ min, (b) $\Delta t = 0.1$ min, and (c) $\Delta t = 0.01$ min

the right side (downstream) and causes the other properties to change. Hence, the temperature effect on the pipeline flow analysis must be taken into account since ambient conditions change with the seasons. Table 2 illustrates the value of Mach and Reynolds numbers for opening and closing valve condition. As shown in this table, the value of Mach and Reynolds numbers increases as the time step decreases. This affects convective inertia term, which causes more fluctuation. Figure 15 shows the effect of using the nonisothermal formulation on mass flow rate. This figure clearly shows that the mass flow rate asymptotically approaches the steady-state value for the nonisothermal condition because of changing density with temperature. As shown in this figure, the

Table 2 Mach and Reynolds numbers for opening and closing valve condition

Time step	Opening valve		Closing valve	
	Max Reynolds	Max Mach	Max Reynolds	Max Mach
1 min	14.24×10^6	0.021	18.61×10^6	0.027
0.1 min	14.89×10^6	0.022	21.63×10^6	0.030
0.01 min	15.78×10^6	0.023	23.66×10^6	0.033

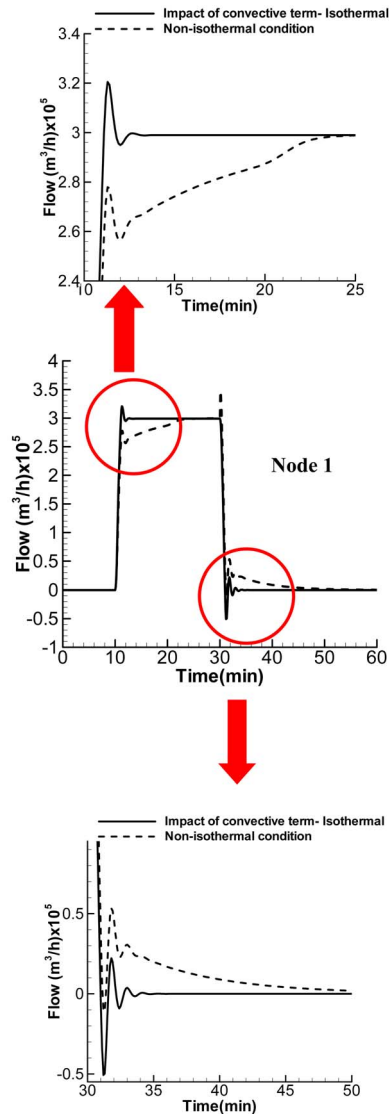


Fig. 15 Comparison of the result for impact of convective inertia term (isothermal) and nonisothermal condition for $\Delta t = 0.1$ min

fluctuation amplitude remained the same as isothermal condition because it is just affected by momentum equation.

Figure 16 shows the same effect on the temperature and compressibility variation for $\Delta t = 0.1$ min. Due to the continuous heat transfer along the pipe, the final temperatures at each node reach a different value but almost constant. The temperature starts to fluctuate at opening valve condition, and as the time increases, the temperature reaches the stable condition. This fluctuation is due to the variation of flow in this period of time. As indicated in Eq. (7), we can expect the same behavior explained for compressibility factor, which is a strong function of temperature and pressure.

Conclusion

This study uses a fully implicit finite-difference method to analyze transient and nonisothermal flow within a gas pipeline. The convective term of the momentum equation is included in the analysis. The numerical results show the following.

- The fully implicit method has advantages, such as the guaranteed stability for large time step, which is very useful for simulating long-term transient in natural gas pipeline.
- As the time step decreases, the ability to capture the physi-

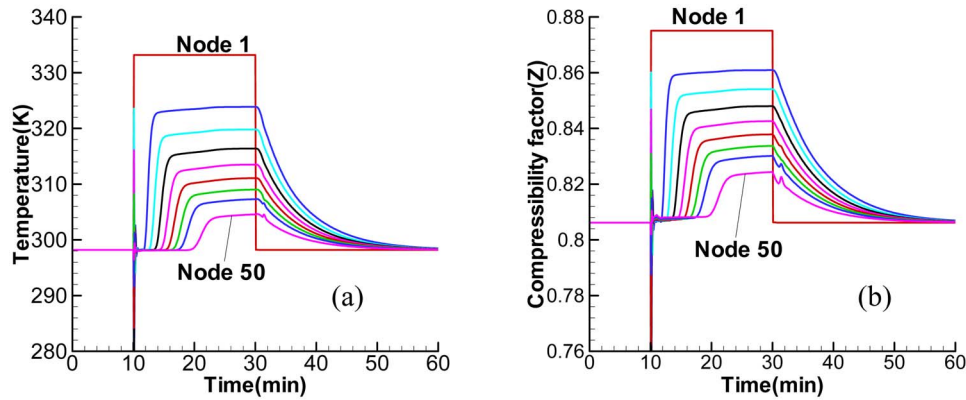


Fig. 16 Temperature (a) and compressibility factor (b) for nonisothermal condition and $\Delta t=0.1$ min

cally existing flow oscillation becomes more apparent during the opening and closing valve condition (sudden change).

- The convective inertia term plays an important role in the gas flow analysis and cannot be neglected from the calculation and the effect of this term is more significant when the mass flow rate increases in the pipe.
- The effect of treating the gas in a nonisothermal manner is very necessary for pipeline flow calculation accuracies and is extremely necessary for rapid transient processes.
- The effect of the Joule–Thompson coefficient is significant on the temperature distribution, and it can be important for a variety of mass flow rates through the pipe.

Acknowledgment

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Nomenclature

- A = cross-sectional area of pipe (m^2, ft^2)
 A_1 – A_6 = constants for compressibility factor (–)
 C_p = specific heat at constant pressure ($\text{J/kg K}, \text{BTU/lbm } ^\circ\text{R}$)
 D = pipe diameter (m, ft)
 g = gravitational acceleration ($\text{m/s}^2, \text{ft/s}^2$)
 f = Darcy friction factor (–)
 h = specific enthalpy ($\text{W/m}^2 \text{K}, \text{BTU/ft}^2 \text{h } ^\circ\text{R}$)
 L = pipe length (m, ft)
 $\dot{m}$ = mass flow rate ($\text{kg/s}, \text{lbm/h}$)
 n = time level (–)
 N = number of node (–)
 P = pressure of the gas (Pa, psi (absolute))
 P_r = reduced pressure (–)
 R = specific gas constant ($\text{J/kg K}, \text{BTU/lbm } ^\circ\text{R}$)
 t = time (s)
 T = temperature (K, $^\circ\text{R}$)
 T_G = ground temperature (K, $^\circ\text{R}$)
 T_r = reduced temperature (–)
 U = overall heat transfer coefficient ($\text{W/m}^2 \text{K}, \text{BTU/ft}^2 \text{h } ^\circ\text{R}$)
 v = velocity of the gas directed along the axis of the pipe ($\text{m/s}, \text{ft/s}$)
 V_w = isentropic wave speed ($\text{m/s}, \text{ft/s}$)
 W = frictional force per unit length of pipe and per unit time ($\text{N/m}, \text{lbf/ft}$)

- x = distance along the pipe (m, ft)
 Z = compressibility factor (–)
 θ = angle of inclination of pipe to the horizontal (radian)
 ρ = density of the gas ($\text{kg/m}^3, \text{lbm/ft}^3$)
 ρ_r = reduced density (–)
 η_J = Joule–Thompson coefficient ($^\circ\text{C/Pa}, ^\circ\text{R/psi}$)
 ε = pipe roughness (mm, ft)
 Δ = difference (–)
 Ω = heat flow ($\text{J/m s}, \text{BTU/ft h}$)

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Spectral and Perturbation Analysis of First-Order Beams With Notch Damage

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The influence of damage on waves propagating in beam structures is investigated through a numerical model formulated by combining spectral finite elements and perturbation techniques. The resulting numerical tool allows for an efficient computation of the wave propagation response and the analysis of the effects of localized damages of various extents and locations. The dynamic behavior of damaged beams is described through a first-order model, which couples bending and axial behavior, thus allowing the prediction of mode conversion phenomena. Damage is modeled as a small, localized reduction of the beam thickness which, allows for an application of perturbation theory. Numerical examples in the time and frequency domains are presented to illustrate the model capabilities. [DOI: 10.1115/1.2839904]

Keywords: damaged beam, notched beam, spectral finite element method, perturbation techniques, first order beam theory

1 Introduction

Damage detection in structures is one of the objectives of health monitoring. Analytical models that incorporate the changes in the structural dynamic characteristic parameters is a very useful tool for damage localization as well as for the determination of the defect magnitude.

Literature offers several models for notched beams. Shen and Pierre [1] proposed a modified Galerkin expansion in order to study the behavior of beams with pairs of symmetric open cracks. They validated their model based on finite element models. Luo and Hanagud [2] and Lestari [3] developed a perturbation technique for damaged beams. They assumed that the behavior of the damaged beam is obtained as a perturbation (over a small parameter) of the behavior of the undamaged beam. Sharma et al. [4] extended the perturbation techniques to plates with localized defects.

The spectral finite element method (SFEM), which is very similar to the finite element method but is formulated in the frequency domain, gives a very accurate solution for the dynamic analysis of structures because it is based on the exact dynamic stiffness matrix by using the exact shape functions (Doyle [5]). Consequently, SFEM implementation does not require any structural discretization to improve the solution accuracy—longer and fewer elements are needed. Lee et al. [6] presented a review of the SFEM in structural dynamics. From the transform function point of view, few types of SFEM were developed: Mahapatra and Gopalakrishnan [7] developed a SFE model based on discrete (fast) Fourier transform (FFT) for laminated composite beams; Mitra and Gopalakrishnan [8] derived a SFE model based on wavelet transform. FFT based SFEM has problems handling finite structures (small dimensions) and is valid only when the initial conditions (displacements and velocities) are zero. Wavelet transform based SFEM overcomes the above two restrictions and do not have signal processing errors due to wraparound or aliasing problems. Kumar et al. [9] developed and implemented a SFE model to describe the behavior of a first-order shear deformation beam with

embedded transverse crack. The cracked region is discretized into few internal elements, and the internal nodes are condensed out based on the continuity of displacements and governing equations. Ostachowicz and Krawczuk (1991) [10] studied the effects of cracks on the natural frequencies of a cantilever beam. They calculated the equivalent stiffness in the crack and the natural vibration frequencies of a beam with two cracks. Krawczuk et al. [11] introduced a finite spectral element of a cracked Timoshenko beam. The crack is substituted by a dimensionless and massless spring, whose bending and shear flexibilities are calculated using Castigliano's theorem and laws of the fracture mechanics.

This paper presents the application of SFEM in conjunction with the perturbation analysis of damaged beams. In contrast to the approach of Krawczuk et al. [11], damage is modeled as a thickness reduction of small extent, which allows the introduction of a perturbation parameter ϵ . The application of perturbation techniques yields a set of differential equations, corresponding to increasing orders of ϵ , which are solved through the application of SFEM. The model considers a first-order deformation theory, with strain varying linearly across the thickness. This leads to a set of two equations governing bending and axial motions of the beam. In the absence of damage, the two equations are completely uncoupled as predicted by the elementary beam theory, whereas damage causes coupling and mode conversion phenomena to occur. This is a noteworthy aspect of the study, which shows how a simple beam formulation can be employed to predict and analyze mode conversions caused by damage. The solution technique is validated by comparing its predictions with those of a model developed in the commercial code ABAQUS and with solutions from the modal superposition (SM) approach. The SFEM is introduced as a general framework that combines the advantages of conventional finite elements with the computational efficiency of analytical techniques. The modal superposition technique can easily handle only simple geometries with reasonable computational costs, which limits its applicability. For this reason, it is here only used for validation purposes on simple geometries and cannot be considered as a general tool for the simulation of wave propagation in damaged structures. In contrast, the combination of SFEM and perturbation analysis lends itself to the analysis of complex waveguides affected by small defects.

The paper is organized as follows. The brief introduction presented in this section is followed by the derivation of the governing equations and corresponding boundary conditions for the

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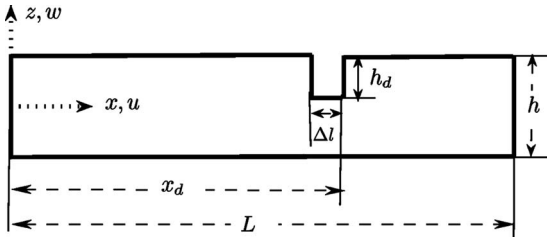


Fig. 1 Beam geometry

notched beam based on the Hamilton principle and perturbation techniques (Sec. 2). Section 3 presents the methodology followed in solving the perturbation equations through SFEM, while Sec. 4 presents numerical results obtained from the analysis. Validation results, as well as time domain and frequency domain simulations, illustrate the effect of the considered type of damage on the dynamic behavior of the beam. Finally, Sec. 5 summarizes the main results of the work and outlines current and future research directions.

2 Perturbation Equations for a Beam With Notch Damage

The dynamic behavior of the notched beam shown in Fig. 1 is described by a set of governing equations derived through the Hamilton principle. The defect is modeled as a reduction in thickness of depth h_d , extending over a length Δl , placed at the distance x_d . According to Fig. 1, $x \in [0, L]$ denotes the horizontal coordinate, whereas the vertical coordinate z varies in the following interval:

$$z \in \left[-\frac{h}{2}, \frac{h}{2}(1 - 2\varepsilon\gamma_d(x)) \right] \quad (1)$$

where $\varepsilon = h_d/h$ and $\gamma_d(x)$ is a damage function defined as

$$\gamma_d(x) = H(x - (x_d - \Delta l)) - H(x - x_d) \quad (2)$$

with H denoting the Heaviside function.

The governing equations for the notched beam and the appropriate set of boundary conditions are derived using the Hamilton principle. The required kinetic and strain energies and the work of external forces are formulated using the following kinematic assumptions:

$$\begin{aligned} u(x, z, t) &= u(x, t) - z \frac{\partial w}{\partial x} \\ w(x, z, t) &= w(x, t) \end{aligned} \quad (3)$$

where $u(x, t)$ and $w(x, t)$ are the axial and transverse displacements in the reference plane $z=0$, respectively. The linear strain-displacement relations are

$$\begin{aligned} \varepsilon_{xx}(x, z, t) &= u_{,x}(x, t) - z w_{,xx}(x, t) \\ \varepsilon_{zz}(x, z, t) &= 0, \quad \gamma_{xz}(x, z, t) = 0 \end{aligned} \quad (4)$$

where the subscript $,x$ denotes a partial derivative with respect to x . The constitutive relation is assumed to be of the well known form

$$\sigma_{xx}(x, z, t) = E \varepsilon_{xx}(x, z, t) \quad (5)$$

where σ_{xx} is the normal stress in the x direction and E is the Young's modulus. Accordingly, the axial force resultant and bending moment resultant are expressed as

$$\begin{aligned} N_{xx}(x, t) &= b \int_{-h/2}^{h/2(1-2\varepsilon\gamma_d(x))} \sigma_{xx}(x, z, t) dz = Ehb u_{,x} + \left[-u_{,x} \right. \\ &\quad \left. + w_{,xx} \frac{h}{2} \right] Ebh \gamma_d(x) \varepsilon \end{aligned} \quad (6)$$

$$\begin{aligned} M_{xx}(x, t) &= b \int_{-h/2}^{h/2(1-2\varepsilon\gamma_d(x))} z \sigma_{xx}(x, z, t) dz = -E \frac{bh^3}{12} w_{,xx} + \left[-u_{,x} \right. \\ &\quad \left. + w_{,xx} \frac{h}{2} \right] E \frac{bh^2}{2} \gamma_d(x) \varepsilon \end{aligned} \quad (7)$$

Hamilton's principle

$$\int_{t_1}^{t_2} \delta(U - T + V) dt = 0 \quad (8)$$

requires the derivation of the first variation of the beam's strain and kinetic energies and of the work of the external forces, which, in this case, are given by

$$\begin{aligned} \delta T &= -b \int_0^L \int_{-h/2}^{h/2(1-2\varepsilon\gamma_d(x))} \rho(\ddot{u} \delta u + \ddot{w} \delta w) dz dx \\ &= -b\rho \int_0^L \left\{ \ddot{u} h [1 - \varepsilon\gamma_d(x)] + \ddot{w}_x \frac{h^2}{2} \varepsilon \gamma_d(x) \right\} \delta u dx \\ &\quad - b\rho \int_0^L \left\{ - \left[\ddot{u} \frac{h^2}{2} \varepsilon \gamma_d(x) + \ddot{w}_x \frac{h^3}{12} [1 - \varepsilon\gamma_d(x)] \right] \right. \\ &\quad \left. + \ddot{w} h [1 - \varepsilon\gamma_d(x)] \right\} \delta w dx \\ &\quad - b\rho \left\{ \ddot{u} \frac{h^2}{2} \varepsilon \gamma_d(x) + \ddot{w}_x \frac{h^3}{12} [1 - \varepsilon\gamma_d(x)] \right\} \delta w \Big|_{x=0}^{x=L} \end{aligned} \quad (9)$$

$$\begin{aligned} \delta U &= b \int_0^L \int_{-h/2}^{h/2(1-2\varepsilon\gamma_d(x))} \sigma_{xx}(x, z, t) \delta \varepsilon_{xx}(x, z, t) dz dx \\ &= b \int_0^L \int_{-h/2}^{h/2(1-2\varepsilon\gamma_d(x))} \sigma_{xx} [(\delta u)_{,x} - z(\delta w)_{,xx}] dz dx \\ &= b \int_0^L [N_{xx}(x, t)(\delta u)_{,x} - M_{xx}(x, t)(\delta w)_{,xx}] dx = b N_{xx}(x, t) \delta u \Big|_{x=0}^{x=L} \\ &\quad - b \int_0^L N_{xx,x}(x, t) \delta u dx - b [M_{xx}(x, t)(\delta w)_{,x} - M_{xx,x}(x, t) \delta w] \Big|_{x=0}^{x=L} \\ &\quad - b \int_0^L M_{xx,xx}(x, t) \delta w dx \end{aligned} \quad (10)$$

and

$$\begin{aligned} \delta V &= - \int_0^L [n(x, t) \delta u + q(x, t) \delta w + m(x, t) (\delta w)_{,x}] dx \\ &\quad - \sum_{j=1}^N \int_0^L [N_j(t) \delta u + Q_j(t) \delta w + M_j(t) (\delta w)_{,x}] \delta(x - x_j) dx \\ &= - \int_0^L \left[n(x, t) + \sum_{j=1}^N N_j(t) \delta(x - x_j) \right] \delta u dx \\ &\quad - \int_0^L \left[q(x, t) - m_{,x}(x, t) + \sum_{j=1}^N Q_j(t) \delta(x - x_j) \right] \delta w dx \end{aligned}$$

$$- \sum_{j=1}^N M_j(t) (\delta(x-x_j))_{,x} \Big] \delta w dx - \left[m(x,t) + \sum_{j=1}^N M_j(t) \delta(x-x_j) \right] \delta w \Big|_{x=0}^{x=L} \quad (11)$$

where, for simplicity, we assume that the loads are applied along the reference plane $z=0$. Also, in Eq. (11), $n(x,t)$ and $q(x,t)$ respectively denote axial and transverse distributed external loads, $m(x,t)$ denotes a distributed bending moment distribution, while $N_j(t)$, $Q_j(t)$, and $M_j(t)$ are external concentrated longitudinal and vertical loads, and bending moment applied at N locations $x=x_j$. Finally, δ is the Dirac delta function and $\ddot{}$ denotes the second order partial derivative with respect to t .

The application of Hamilton's principle yields the following set of differential equations:

$$N_{xx,x}(x,t) - \rho h [1 - \varepsilon \gamma_d(x)] \ddot{u} - \rho \frac{h^2}{2} \varepsilon \gamma_d(x) \ddot{w}_{,x} = f_1(x,t) \\ M_{xx,xx}(x,t) + \rho \left\{ \ddot{u} \frac{h^2}{2} \varepsilon \gamma_d(x) + \ddot{w}_{,x} \frac{h^3}{12} [1 - \varepsilon \gamma_d(x)] \right\} \\ - \rho h [1 - \varepsilon \gamma_d(x)] \ddot{w} = f_2(x,t) \quad (12)$$

where $f_1(x,t)$ and $f_2(x,t)$ are respectively defined as

$$f_1(x,t) = -n(x,t) - \sum_{j=1}^N N_j(t) \delta(x-x_j) \\ f_2(x,t) = -q(x,t) + m_{,x}(x,t) - \sum_{j=1}^N Q_j(t) \delta(x-x_j) \\ + \sum_{j=1}^N M_j(t) \{ \delta(x-x_j) \}_{,x} \quad (13)$$

The associated boundary conditions, at $x=0$ and $x=L$, are

$$N_{xx}(x,t) = 0 \quad \text{or } u(x,t) \text{ given} \\ bM_{xx,x}(x,t) + \rho I_0 \ddot{w}_{,x}(x,t) - m(x,t) \\ - \sum_{j=1}^N M_j(t) \delta(x-x_j) = 0 \quad \text{or } w(x,t) \text{ given} \quad (14)$$

$$M_{xx}(x,t) = 0 \quad \text{or } w_{,x}(x,t) \text{ given}$$

Equations (12) can be conveniently expressed in the frequency domain through the Fourier transform (FT) of the applied generalized loads $f_j(x,t)$ (with $j=1,2$), which can be expressed as

$$f_j(x,t) = \sum_k \hat{f}_{jk}(x, \omega_k) e^{i\omega_k t} \quad (15)$$

where $i=\sqrt{-1}$ and $\hat{f}_{jk}(x, \omega_k)$ denotes the harmonic component of the generalized load at frequency ω_k [5]. Accordingly, the beam's displacements can be written as

$$u(x,t) = \sum_k \hat{u}_k(x, \omega_k) e^{i\omega_k t} \quad (16)$$

$$w(x,t) = \sum_k \hat{w}_k(x, \omega_k) e^{i\omega_k t} \quad (17)$$

where $\hat{u}_k(x, \omega_k)$, and $\hat{w}_k(x, \omega_k)$ are the displacements corresponding to the k th harmonic component of the load. For simplicity, in the remainder of the paper, the subscript k is dropped so that $\omega_k = \omega$, $\hat{u}_k(x, \omega_k) = \hat{u}(x, \omega)$, and $\hat{w}_k(x, \omega_k) = \hat{w}(x, \omega)$ are adopted.

Next, the axial and vertical displacements of the beam in the reference plane are considered as perturbations (over the small parameter ε) of the axial and vertical displacements of the undamaged beam,

$$\begin{Bmatrix} \hat{u}(x, \omega) \\ \hat{w}(x, \omega) \end{Bmatrix} = \begin{Bmatrix} \hat{u}^{(0)}(x, \omega) \\ \hat{w}^{(0)}(x, \omega) \end{Bmatrix} - \varepsilon \begin{Bmatrix} \hat{u}^{(1)}(x, \omega) \\ \hat{w}^{(1)}(x, \omega) \end{Bmatrix} - O(\varepsilon^2) \quad (18)$$

Replacing Eqs. (6), (7), and (18) into the differential system Eq. (12) and collecting the coefficients of ε^0 and ε^1 yields the following set of differential equations:

$$\varepsilon^0: \begin{bmatrix} m\omega^2 & 0 \\ 0 & m\omega^2 \end{bmatrix} \begin{Bmatrix} \hat{u}^{(0)}(x, \omega) \\ \hat{w}^{(0)}(x, \omega) \end{Bmatrix} + \begin{bmatrix} EA & 0 \\ 0 & -\rho I_0 \omega^2 \end{bmatrix} \begin{Bmatrix} \hat{u}_{,xx}^{(0)}(x, \omega) \\ \hat{w}_{,xx}^{(0)}(x, \omega) \end{Bmatrix} \\ + \begin{bmatrix} 0 & 0 \\ 0 & -EI_0 \end{bmatrix} \begin{Bmatrix} \hat{u}_{,4x}^{(0)}(x, \omega) \\ \hat{w}_{,4x}^{(0)}(x, \omega) \end{Bmatrix} = \begin{Bmatrix} \hat{f}_1(x, \omega) \\ \hat{f}_2(x, \omega) \end{Bmatrix} \quad (19)$$

$$\varepsilon^1: \begin{bmatrix} m\omega^2 & 0 \\ 0 & m\omega^2 \end{bmatrix} \begin{Bmatrix} \hat{u}^{(1)}(x, \omega) \\ \hat{w}^{(1)}(x, \omega) \end{Bmatrix} + \begin{bmatrix} EA & 0 \\ 0 & -\rho I_0 \omega^2 \end{bmatrix} \begin{Bmatrix} \hat{u}_{,xx}^{(1)}(x, \omega) \\ \hat{w}_{,xx}^{(1)}(x, \omega) \end{Bmatrix} \\ + \begin{bmatrix} 0 & 0 \\ 0 & -EI_0 \end{bmatrix} \begin{Bmatrix} \hat{u}_{,4x}^{(1)}(x, \omega) \\ \hat{w}_{,4x}^{(1)}(x, \omega) \end{Bmatrix} = \begin{Bmatrix} \hat{g}_1(x, \omega) \\ \hat{g}_2(x, \omega) \end{Bmatrix} \quad (20)$$

where $I=bh^3/12$, $A=bh$, ρ is the density per unit area, m is the beam mass, and

$$\begin{Bmatrix} \hat{g}_1(x, \omega) \\ \hat{g}_2(x, \omega) \end{Bmatrix} = \begin{bmatrix} -m\omega^2 \gamma_d(x, \omega) & 0 \\ -\frac{mh}{2} \omega^2 \gamma_{d,x}(x, \omega) & -m\omega^2 \gamma_d(x, \omega) \end{bmatrix} \begin{Bmatrix} \hat{u}^{(0)}(x, \omega) \\ \hat{w}^{(0)}(x, \omega) \end{Bmatrix} \\ + \begin{bmatrix} -EA \gamma_{d,x}(x, \omega) & \frac{mh}{2} \omega^2 \gamma_d(x, \omega) \\ -EA \frac{h}{2} \gamma_{d,xx}(x, \omega) - \frac{mh}{2} \omega^2 \gamma_d(x, \omega) & 3\rho I_0 \omega^2 \gamma_{d,x}(x, \omega) \end{bmatrix} \\ \times \begin{Bmatrix} \hat{u}_{,x}^{(0)}(x, \omega) \\ \hat{w}_{,x}^{(0)}(x, \omega) \end{Bmatrix} \\ + \begin{bmatrix} -EA \gamma_d(x, \omega) & EA \frac{h}{2} \gamma_{d,x}(x, \omega) \\ -EA h \gamma_{d,x}(x, \omega) & 3EI_0 \gamma_{d,xx}(x, \omega) + \rho I_0 \omega^2 \gamma_d(x, \omega) \end{bmatrix} \\ \times \begin{Bmatrix} \hat{u}_{,xx}^{(0)}(x, \omega) \\ \hat{w}_{,xx}^{(0)}(x, \omega) \end{Bmatrix} + \begin{bmatrix} 0 & EA \frac{h}{2} \gamma_d(x, \omega) \\ -EA \frac{h}{2} \gamma_d(x, \omega) & 6EI_0 \gamma_{d,x}(x, \omega) \end{bmatrix} \\ \times \begin{Bmatrix} \hat{u}_{,3x}^{(0)}(x, \omega) \\ \hat{w}_{,3x}^{(0)}(x, \omega) \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 3EI_0 \gamma_{d,x}(x, \omega) \end{bmatrix} \begin{Bmatrix} \hat{u}_{,4x}^{(0)}(x, \omega) \\ \hat{w}_{,4x}^{(0)}(x, \omega) \end{Bmatrix} \quad (21)$$

Equations (19) and (20) can be solved for an assigned set of loads in terms of the unknown displacements $\hat{\mathbf{u}}^{(0)}(x, \omega) = [\hat{u}^{(0)}(x, \omega) \ \hat{w}^{(0)}(x, \omega)]^T$ and their first-order perturbation $\hat{\mathbf{u}}^{(1)}(x, \omega) = [\hat{u}^{(1)}(x, \omega) \ \hat{w}^{(1)}(x, \omega)]^T$.

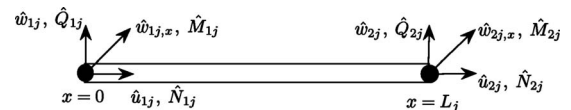


Fig. 2 Spectral finite element with nodal displacements and loads

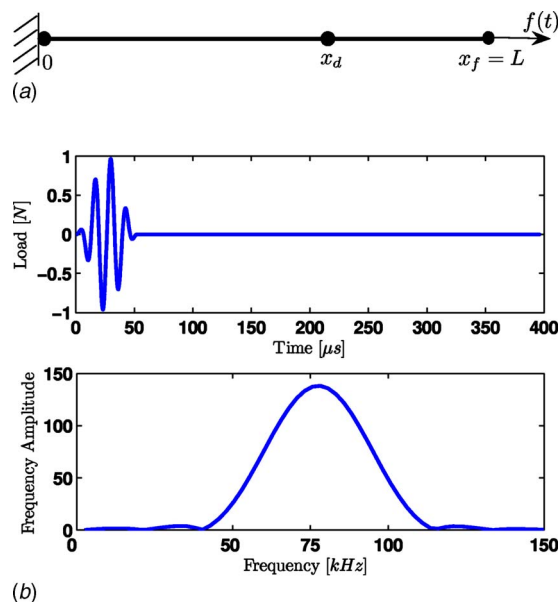


Fig. 3 (a) Schematic of the clamped-free beam with a longitudinal tip load, modeled using two spectral elements. (b) Modulated sinusoidal pulse load in time and frequency domains

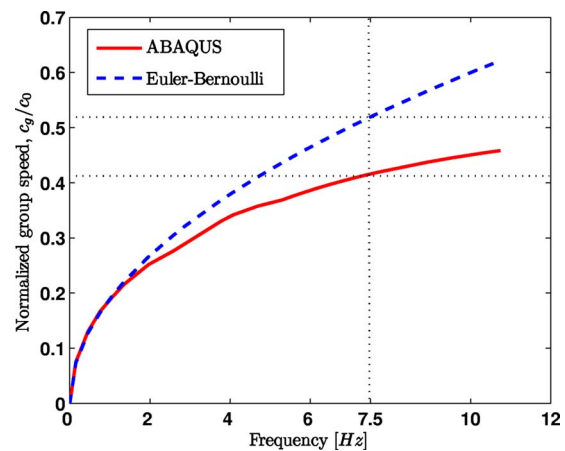


Fig. 5 Comparison of ABAQUS and SFEM group velocities

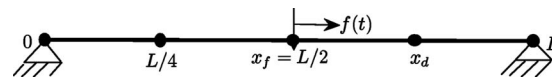


Fig. 6 Schematic of the simply supported beam with a longitudinal load at the middle, used to compare the superposition of modes and SFEM results

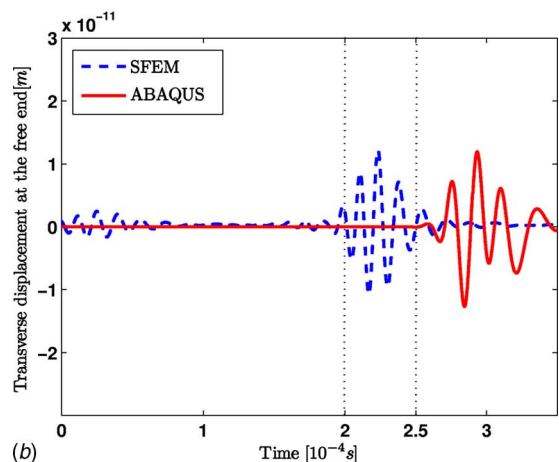
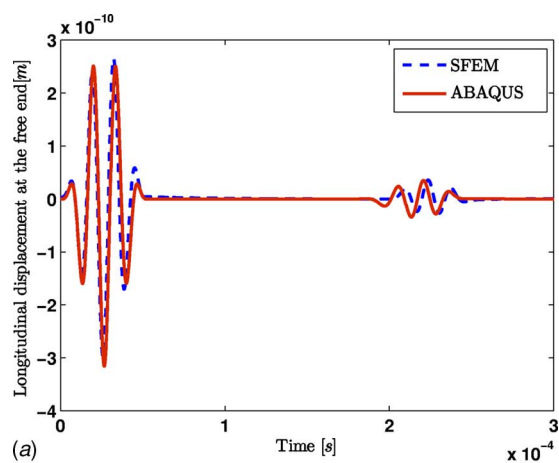


Fig. 4 Comparison between FEM and SFEM results: (a) longitudinal and (b) transverse displacements at the free end of the notched beam with a defect at $x_d = L/2$

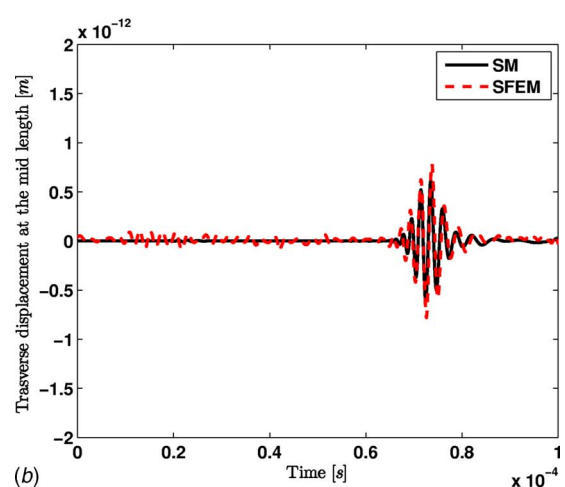
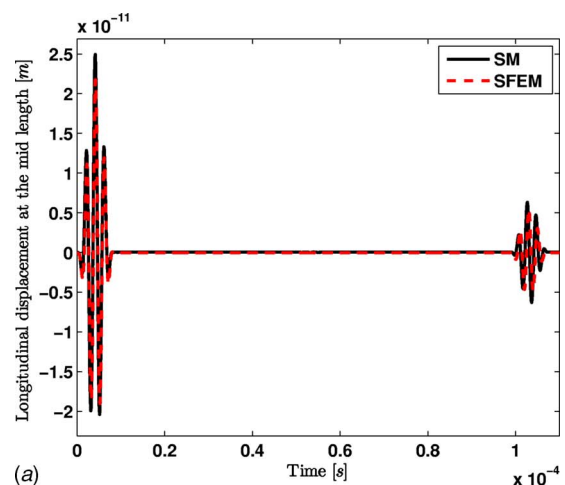


Fig. 7 (a) Longitudinal and (b) transverse displacements at the midlength of notched beams with a defect at $x_d = 3L/4$

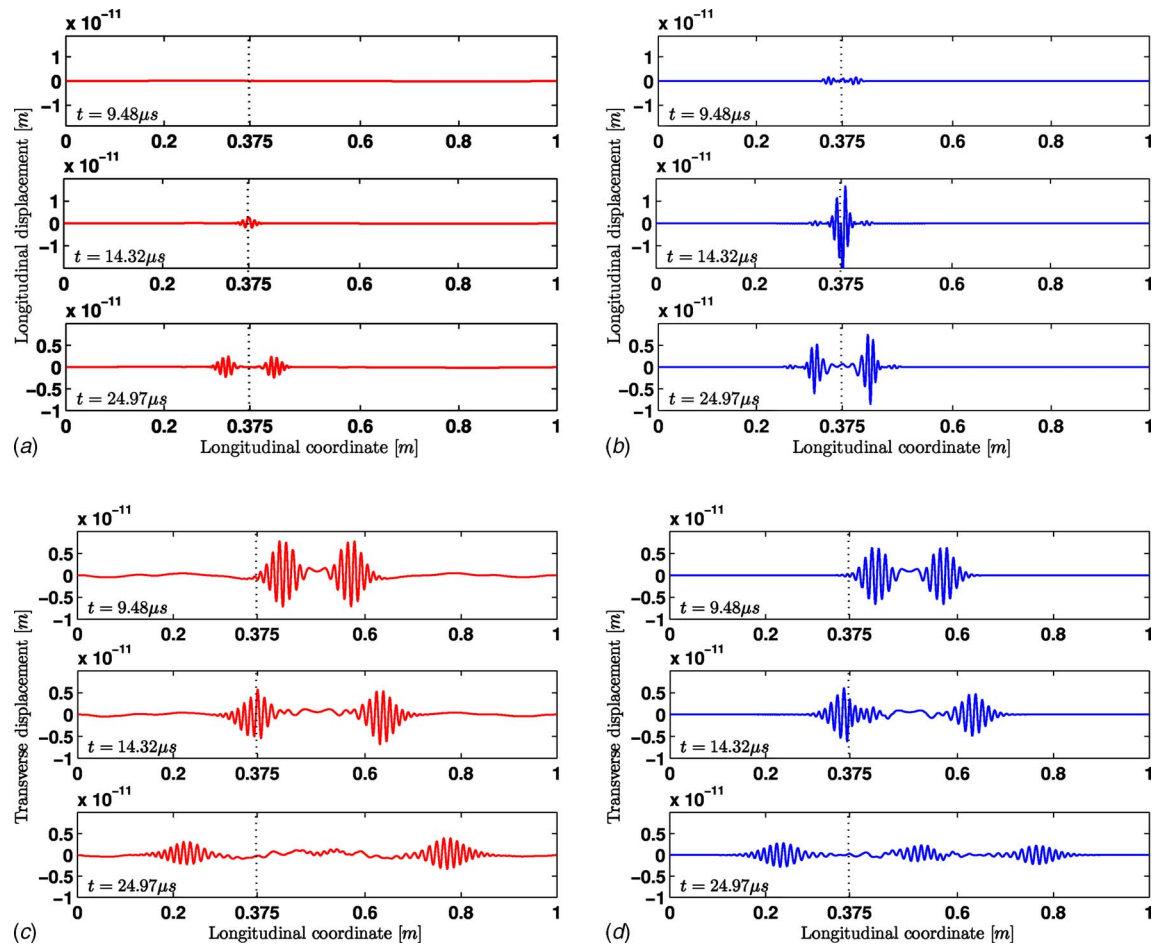


Fig. 8 Displacements as a function of longitudinal coordinate at the same moments: (a) SFEM longitudinal displacement, (b) SM longitudinal displacement, (c) SFEM transverse displacement, and (d) SM transverse displacement

3 Spectral Finite Element Discretization

The equation for the ε^0 term corresponds to the governing equation for the undamaged beam; the first-order perturbation equation has the same form and features an applied generalized load that is a function of the solution of the ε^0 equation. A common strategy for the solution of the two equations (Eqs. (19) and (20)) deriving from the expansion of the beam's displacements in terms of the perturbation parameter can be adopted based on their formally identical form. Each of the equations can, in fact, be written in the following matrix form:

$$\begin{bmatrix} m\omega^2 & 0 \\ 0 & m\omega^2 \end{bmatrix} \begin{Bmatrix} \hat{u}(x, \omega) \\ \hat{w}(x, \omega) \end{Bmatrix} + \begin{bmatrix} EA & 0 \\ 0 & -\rho I \omega^2 \end{bmatrix} \begin{Bmatrix} \hat{u}_{,xx}(x, \omega) \\ \hat{w}_{,xx}(x, \omega) \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -EI \end{bmatrix} \begin{Bmatrix} \hat{u}_{,4x}(x, \omega) \\ \hat{w}_{,4x}(x, \omega) \end{Bmatrix} = \begin{Bmatrix} q_1(x, \omega) \\ q_2(x, \omega) \end{Bmatrix} \quad (22)$$

or

$$\mathbf{M}\hat{\mathbf{u}}(x, \omega) + \mathbf{E}_1\hat{\mathbf{u}}_{,xx}(x, \omega) + \mathbf{E}_2\hat{\mathbf{u}}_{,4x}(x, \omega) = \mathbf{q}(x, \omega) \quad (23)$$

The weak form solution of Eqn. (23) can be sought through multiplication by a suitable test function $\mathbf{v}(x, \omega)^T$,

$$\begin{aligned} \int_0^{L_j} \mathbf{v}^T(x, \omega) \mathbf{M}\hat{\mathbf{u}}(x, \omega) dx - \int_0^{L_j} \mathbf{v}_{,x}^T(x, \omega) \mathbf{E}_1 \hat{\mathbf{u}}_{,x}(x, \omega) dx \\ + \int_0^{L_j} \mathbf{v}_{,xx}^T(x, \omega) \mathbf{E}_2 \hat{\mathbf{u}}_{,xx}(x, \omega) dx = \int_0^{L_j} \mathbf{v}^T(x, \omega) \mathbf{q}(x, \omega) dx \end{aligned} \quad (24)$$

where L_j is the length of an element j that connects two nodes (Fig. 2). The behavior of each node is described by three degrees of freedom (DOFs), so that the element's vector of DOFs is defined as $\mathbf{d}_j = \{\hat{u}_{1j}, \hat{w}_{1j}, \hat{w}_{1j,x}, \hat{u}_{2j}, \hat{w}_{2j}, \hat{w}_{2j,x}\}^T$. The displacement $\hat{\mathbf{u}}(x, \omega)$ within element j is obtained as an interpolation of the nodal DOFs $\mathbf{d}_j$,

$$\hat{\mathbf{u}}(x, \omega) = \mathbf{N}_j(x, \omega) \mathbf{d}_j(\omega) \quad (25)$$

where $\mathbf{N}_j(x, \omega)$ is the matrix of the dynamic shape functions, which is obtained from the solution of the homogeneous governing equation

$$\mathbf{N}_j(x, \omega) = \mathbf{\Theta}(\omega) \mathbf{G}_j(x, \omega) \mathbf{T}_j^{-1}(\omega) \quad (26)$$

where $\mathbf{\Theta}(\omega)$ is an amplitude ratio matrix,

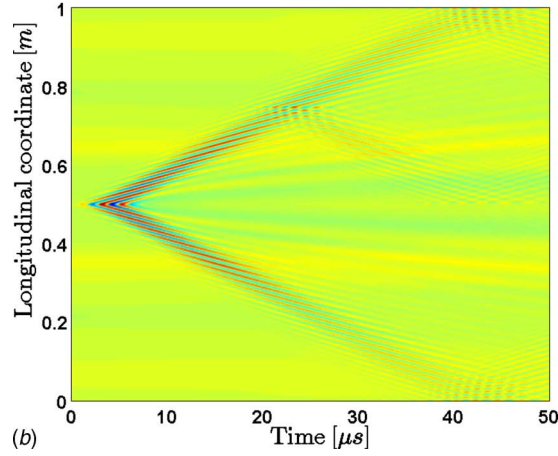
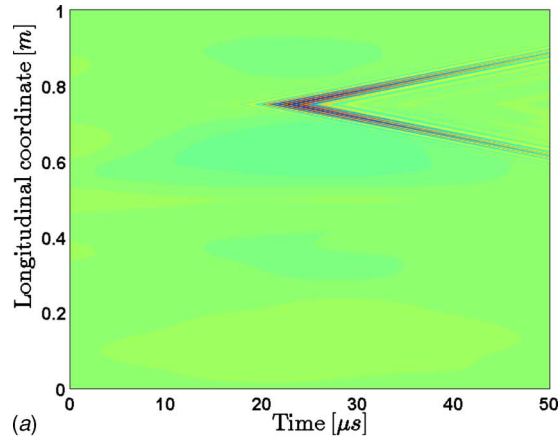


Fig. 9 Displacements as a function of time and horizontal coordinates: (a) longitudinal displacement and (b) transverse displacement. The length of the notch is $\Delta l = 0.01$ m.

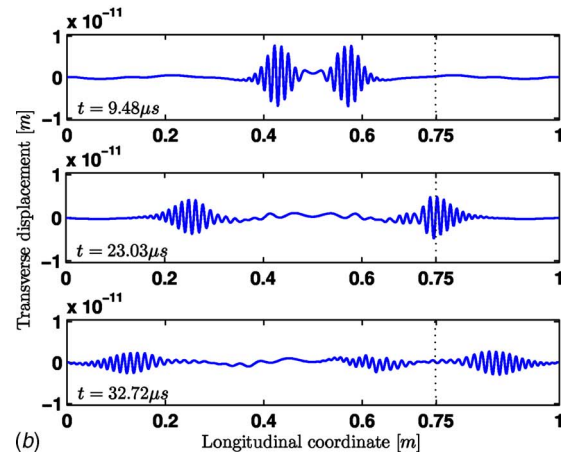
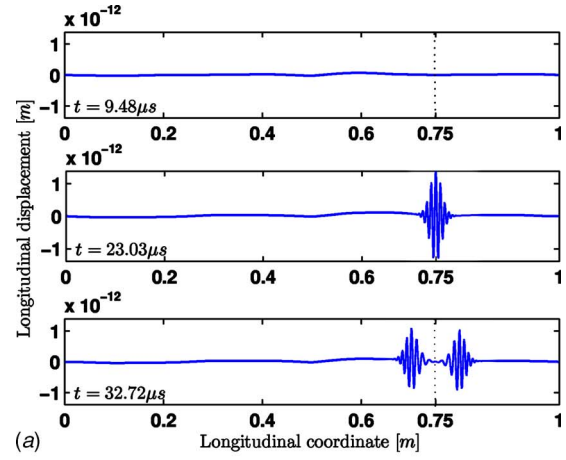


Fig. 10 Displacements as a function of horizontal coordinate at the same moments: (a) longitudinal displacement and (b) transverse displacement

$$\Theta(\omega) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \quad (27)$$

$G_j(x, \omega)$ is defined as

$$G_j(x, \omega) = \begin{bmatrix} e^{-ikx} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\beta x} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-\beta(L_j-x)} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-ik(L_j-x)} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-i\beta x} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-i\beta(L_j-x)} \end{bmatrix} \quad (28)$$

whereas $T_j(\omega)$ is obtained by imposing the displacements at the nodes,

$$T_j(\omega) = \begin{bmatrix} 1 & 0 & 0 & e^{-ikL_j} & 0 & 0 \\ 0 & 1 & e^{-\beta L_j} & 0 & 1 & e^{-i\beta L_j} \\ 0 & -\beta & -\beta e^{-\beta L_j} & 0 & -i\beta & i\beta e^{-i\beta L_j} \\ e^{-ikL_j} & 0 & 0 & 1 & 0 & 0 \\ 0 & e^{-\beta L_j} & 1 & 0 & e^{-i\beta L_j} & 1 \\ 0 & -\beta e^{-\beta L_j} & \beta & 0 & -i\beta e^{-i\beta L_j} & i\beta \end{bmatrix} \quad (29)$$

with $k^2 = \omega^2 / c^2 = \omega^2 m / EA$ and $\beta^4 = \omega^2 m / EI_0$.

The dynamic shape functions provide the exact displacement variation along the beam if the external loads are concentrated at the nodal locations [5]. In the case considered here, it can be shown that the generalized load in the first-order perturbation equations reduces to a concentrated nodal load if a node is placed

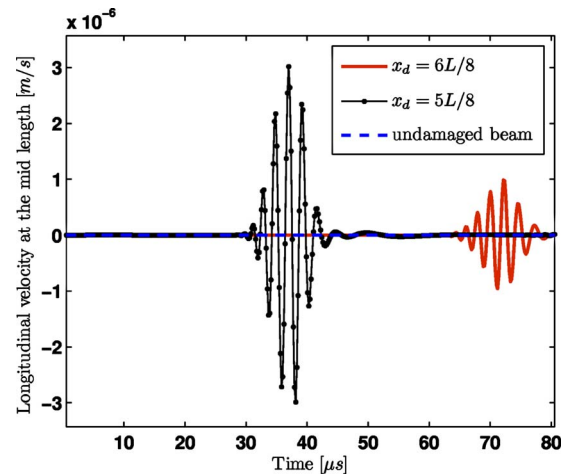


Fig. 11 Longitudinal velocity at the midlength for undamaged beam and notched beams with a defect at $x_d = 5L/8$ and $x_d = 6L/8$

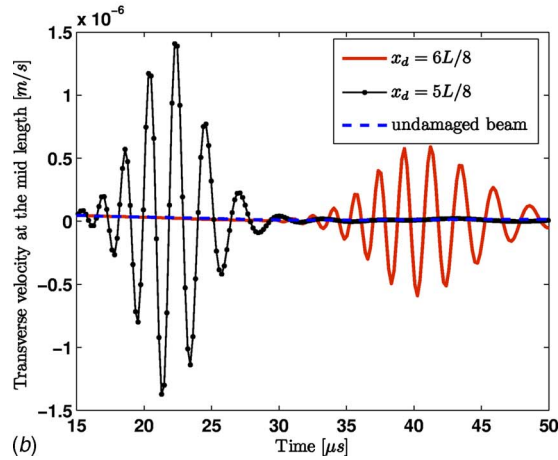
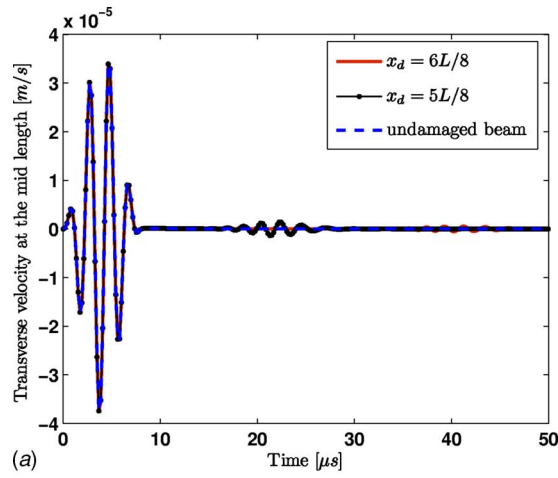


Fig. 12 (a) Transverse velocity at the midlength of an undamaged beam and notched beams with a defect at $x_d=5L/8$ and $x_d=3L/4$. (b) Details of reflections caused by damage.

at the damage location. Accordingly, the solution of homogeneous beam equations and proper description of nodal loads corresponding to the presence of damage based on the formulation presented above can be used to obtain exact dynamic shape functions and accurate representations of the beam's displacements in the frequency range corresponding to the applied load. This approach can also be applied when loads are generally distributed along the element length. In this case, the dynamic shape functions do not reproduce exactly the displacement field within the element, and some approximation is introduced. The application of nodes at damage and load locations does not cause a dramatic increase in the computational time, and the presented modeling approach still represents an efficient tool for the analysis of wave propagation in the considered class of damaged structures. Refinements of the formulation, allowing the accurate representation of general load distributions and of damage locations within the element, are under development and will be discussed in future papers.

The dynamic interpolation functions can also be used for the test function $\mathbf{v}$. Substitution in the weak form of the equation yields the following algebraic equation:

$$\mathbf{K}_j(\omega) \mathbf{d}_j(\omega) = \mathbf{f}_j(\omega) \quad (30)$$

where $\mathbf{K}_j(\omega)$ is the element stiffness matrix at frequency ω , defined as,

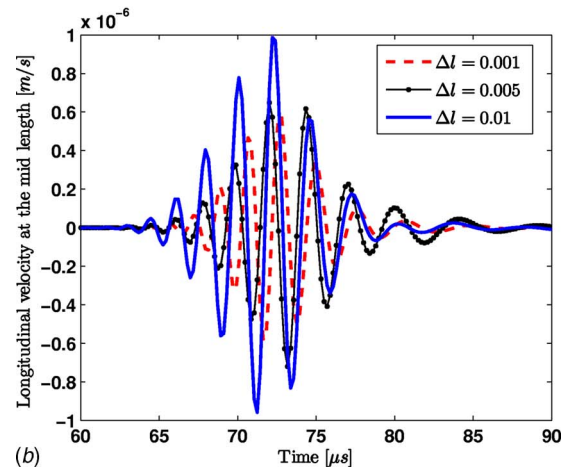
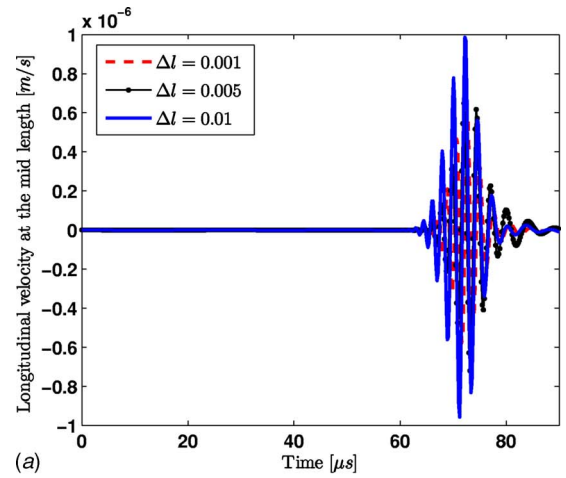


Fig. 13 (a) Longitudinal velocity at the midlength of notched beams with a defect length, $\Delta l=0.001$ m, 0.005 m, and 0.01 m. (b) Details of reflections caused by damage.

$$\begin{aligned} \mathbf{K}_j(\omega) = & \int_0^{L_j} \{ \mathbf{N}_j^T(x, \omega) \mathbf{M} \mathbf{N}_j(x, \omega) - \mathbf{N}_{j,x}^T(x, \omega) \mathbf{E}_1 \mathbf{N}_{j,x}(x, \omega) \\ & + \mathbf{N}_{j,xx}^T(x, \omega) \mathbf{E}_2 \mathbf{N}_{j,xx}(x, \omega) \} dx \end{aligned} \quad (31)$$

and where $\mathbf{f}$ is the vector of applied nodal loads

$$\mathbf{f}_j(\omega) = \int_0^{L_j} \mathbf{N}_j^T(x, \omega) \mathbf{q}(x, \omega) dx \quad (32)$$

4 Numerical Examples

In this section, the developed technique is applied to evaluate longitudinal and transverse wave propagations in damaged beams. The solution based on SFEM is first validated through comparisons with the predictions of a model developed in ABAQUS. The case of a simply supported beam solved through the modal superposition approach is then used as a base line for comparison. Upon assessment of the accuracy of the procedure, simulations in the time and frequency domains are performed for various sets of boundary conditions, excitation configurations, and damage extent and location to show the potential of the technique as a general simulation tool and to highlight interesting phenomena related to the interaction of propagating waves with damage.

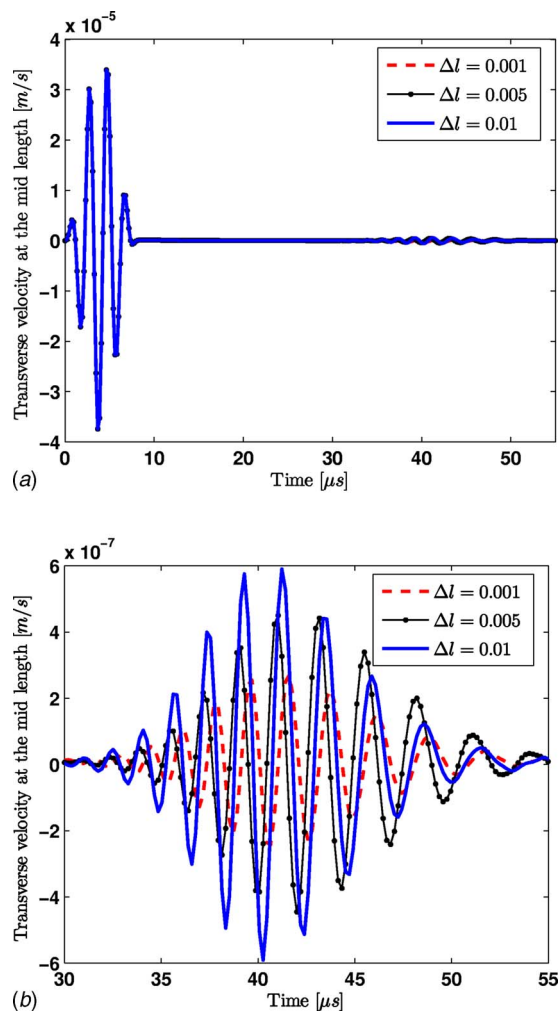


Fig. 14 (a) Transverse velocity at the middle point of the beam for three values of damage length, $\Delta l = 0.001$ m, 0.005 m, and 0.01 m. (b) Details of reflections caused by damage.

Technique Validation

SFEM Versus ABAQUS Predictions. A detailed finite element model of the damaged beam is developed using the commercial software ABAQUS. The considered beam is assumed in a clamped-free configuration, and has length $L = 1$ m, thickness $h = 10 \times 10^{-3}$ m, and width $b = 50 \times 10^{-3}$ m. The beam is made of aluminum (Young's modulus $E = 70$ GPa and density $\rho = 2750$ kg/m³) and features a notch of length $\Delta l = 1 \times 10^{-3}$ m and depth $h_d = h/2$ at $x_d = L/2$. The beam is modeled using 10,000 four-node bilinear plane stress quadrilateral elements, and its response is computed through an explicit dynamic analysis. The same beam is modeled using two spectral finite elements, as shown in Fig. 3(a). The considered excitation is a four-cycle sinusoidal burst at 75 kHz, modulated by a Hanning window (Fig. 3(b)), applied at the free end on the beam in the longitudinal direction according to the configuration shown in Fig. 3(a).

Figures 4(a) and 4(b) compare longitudinal and transverse displacements at the free end of the beam as obtained using SFEM and ABAQUS. The longitudinal responses in Fig. 4(a) show an excellent agreement, both in terms of amplitude of the incident wave and of the reflected wave produced by the damage and in terms of time of arrival of the reflected waves. Figure 4(b) presents the comparison between corresponding transverse displacements. Both SFEM and ABAQUS models predict the generation of a transverse displacement component upon interaction of the lon-

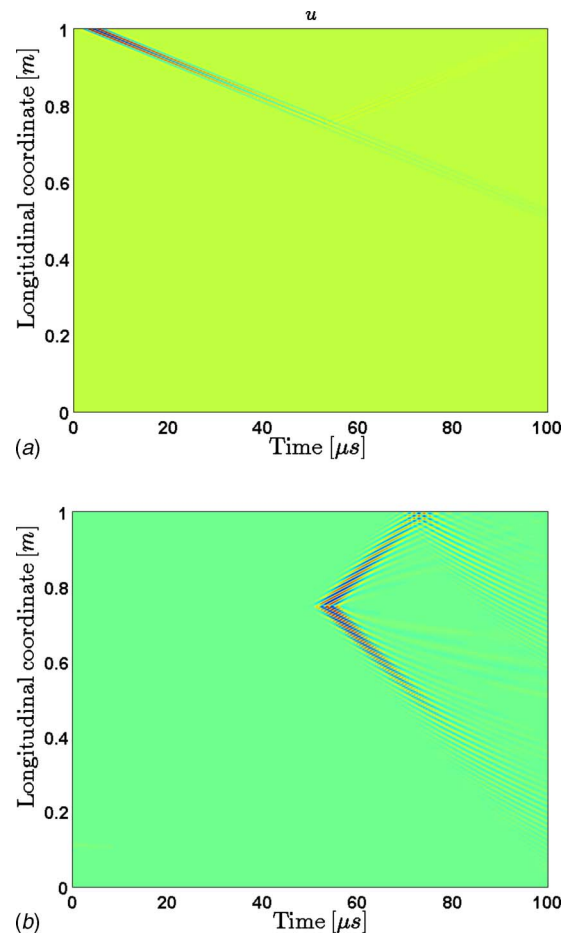


Fig. 15 Displacements as a function of time and longitudinal coordinates: (a) longitudinal displacement and (b) transverse displacement. The length of the notch is $\Delta l = 0.001$ m.

gitudinal wave with the defect, which indicates that mode conversion has taken place. The time lag between the models can be explained by the fact that a simple Euler–Bernoulli formulation has been employed for SFEM. It is well known that the Euler–Bernoulli theory overestimates the wave speeds in comparison with more refined beam theories, such as the Timoshenko formulation [5]. The ABAQUS model does not rely on beam theory and can be considered as a more accurate description of the dynamic behavior due to the highly refined mesh employed for the analysis. The discrepancies in terms of wave velocities is estimated from the dispersion relations predicted by the two models. The dispersion relation in SFEM follows the Euler–Bernoulli relation $k = (\omega^2 EI / \rho A)^{1/4}$, while that of the ABAQUS model needs to be evaluated by means of a numerical experiment. Specifically, the beam is excited by a broadband pulse, and its response in the time domain is recorded at all the nodal points available along the beam span. This allows the computation of two-dimensional FT in space and time, which provides wave number and frequency information over the considered frequency range. These data are then used to estimate the group velocity variation in terms of frequency and to compare it with the SFEM one. The result of this analysis is presented in Fig. 5: The mismatch in group velocities observed at the excitation frequency of 75 kHz corresponds to the time delay observed in the time plots of Fig. 4(b).

SFEM and Modal Superposition Results. A second validation is carried out through comparisons with the modal superposition solution of the perturbation equations (Eqs. (20) and (21)). The modes of the considered structure are used to decouple the equa-

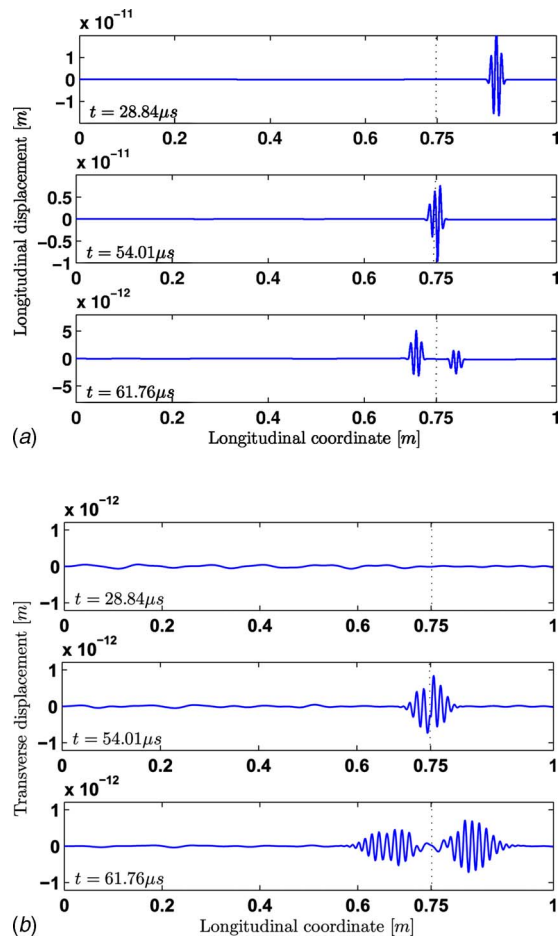


Fig. 16 Displacements as a function of longitudinal coordinate at the same moments: (a) longitudinal displacement and (b) transverse displacement

tions of motion and to obtain time-domain ordinary differential equations (ODEs) in terms of the modal coordinates. The convolution integral is then employed for the time-domain solution, given the assigned excitation time history and the corresponding modal loads. This approach, very well known and established, does not provide the generality of the SFEM and can only be conveniently applied in the case of simple geometries. The considered configuration is that of the beam in Fig. 6, which is excited longitudinally at midspan by a four-cycle modulated sinusoidal burst at 500 kHz. The corresponding longitudinal and transverse midspan displacements in Fig. 7 show the agreement between the solutions and confirm that the interaction with the damage partially converts the longitudinal wave into a transverse one.

The second validation example considers a notch placed at $x_d = 3L/8$ and a transverse load at midspan. Figure 8 compares snapshots of the beam deflected configuration (longitudinal and bending components) at various instants of time. The plots show how longitudinal displacements are produced by the interaction of the bending wave with the defect and confirm the good agreement between SFEM and mode superposition solutions.

Time-Domain Results. The first example in the time domain considers a simply supported beam with a notch at $x_d = 3L/4$ and a vertical load at $x_f = L/2$. The beam has the same geometry and material properties as described in the previous section. The beam is modeled using four spectral elements, as shown in Fig. 6. The beam response computed through the SFEM model is presented in Fig. 9 both in time and space as a color map plot, while Fig. 10

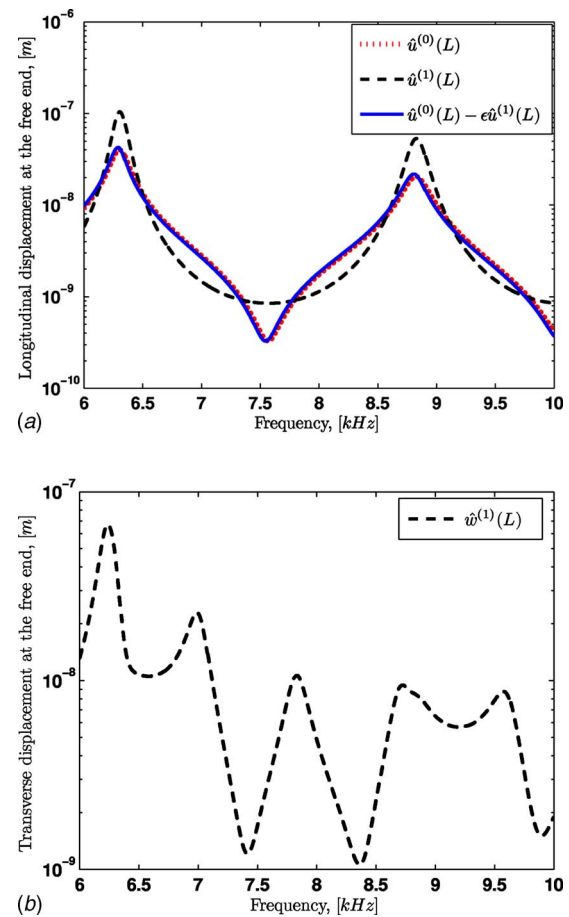


Fig. 17 Displacements at the tip of a cantilever beam in the frequency domain. Case I: horizontal load. (a) Longitudinal displacement and (b) transverse displacement.

shows snapshots of the displacements' variation along the beam at three instants of time. In both figures, the axial displacement is plotted in the subplot (a), and the transverse displacement is displayed in subplot (b). The applied transverse load generates a transverse wave, which propagates from the middle of the beam in both directions. When the wave reaches the notch, it is partially reflected and partially gets converted into a longitudinal wave originating at the notch location. Figures 11 and 12 show the time variation of longitudinal and transverse velocities at the middle of the beam for two different defect positions ($x_d = 5L/8$ and $x_d = 3L/4$), and compare them directly with the velocity of the undamaged beam. As expected, the arrival time of the wave that is reflected from a defect closer to the applied load is smaller, and the amplitude of the wave is higher due to the dissipation added to the model. Details of the reflected transverse waves for different damage locations are shown in Fig. 12(b). The influence of the notch length on the axial and transverse velocities is shown in Figs. 13 and 14, which illustrate how the arrival time of both reflected waves does not change with the damage axial length and how the amplitude of the waves instead increases proportionally with the notch length.

A second problem considers a clamped-free beam with a notch at $x_d = 3L/4$ and a horizontal load at $x_f = L$. The beam is modeled using two spectral elements with a total of nine DOFs. (Fig. 3(a)). The considered applied load is again a modulated sine burst at 500 kHz. Figure 15 presents 3D surfaces of the displacements (as functions of time and longitudinal coordinate), whereas Fig. 16 presents snapshots of displacement variations along the beam length at three instants of time. In both cases, the axial displace-

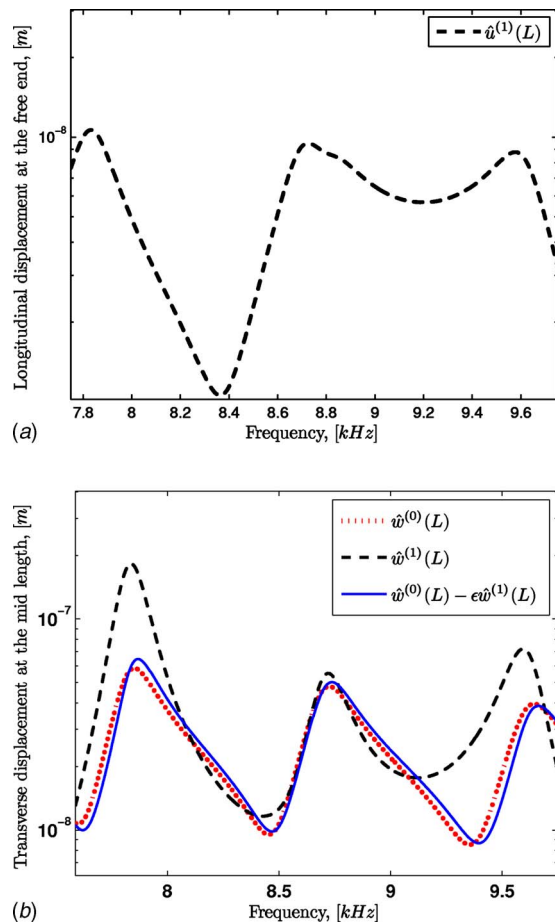


Fig. 18 Displacements at the tip of a cantilever beam in the frequency domain. Case II: vertical load. (a) Longitudinal displacement and (b) transverse displacement.

ment is plotted in the subplot (a) and the transversal displacement is plotted in subplot (b). A longitudinal load causes a longitudinal wave to propagate from the tip of the beam. When the wave reaches the notch (in this case at $x=3L/4$), it is partially reflected and partially converted into a transversal wave originating at the notch location.

Frequency Domain Results. The SFEM can be conveniently used to obtain results in the frequency domain, upon transformation of the applied load and direct solution for the nodal amplitudes at each frequency. Frequency sweeps of unit amplitude loads are considered to obtain frequency response function (FRF) predictions. Examples of this kind of analyses are presented in this section.

The frequency response of a clamped-free beam with a unit harmonic tip load is evaluated in the presence of a notch at $x_d=3L/4$, of length $\Delta l=1 \times 10^{-2}$ m, and depth $h_d=h/10$. Figure 17 shows FRFs corresponding to a longitudinal load of frequency varying in the 5–10 kHz range. Both longitudinal and transverse response components resulting from a longitudinal load are presented. Specifically, Fig. 17(a) compares the responses for damaged and undamaged beams and shows how the small localized notch produces small changes in the frequency domain and, par-

ticularly, how the location of the resonant peaks is shifted by a negligible amount. This confirms the notion that the considered type of damage does not significantly modify the natural frequencies of the structure, even in a high frequency range as considered here. The changes in natural frequencies due to notch damage in beams and plates are quantified respectively in Lestari [3] and Sharma et al. [4], where it is essentially shown how perturbations $\mathcal{O}(\epsilon)$ in the mode shapes correspond to the $\mathcal{O}(\epsilon^2)$ change in the natural frequencies. The transverse response in Fig. 17(b) again demonstrates the intermodal coupling between longitudinal and transverse motions. The peaks in the plot correspond both to the bending frequencies of the beam and to the longitudinal ones, the latter being excited by the considered axial excitation. The results for a transverse tip load shown in Fig. 18 lead to similar conclusions and confirm the observations made in commenting on the previous figure.

5 Conclusions

The dynamic behavior of the damaged beams is investigated through the perturbation techniques. The governing equations of a first-order coupled beam are derived using the Hamilton principle. A spectral element formulation based on the exact dynamic stiffness matrix is used to determine the solution. The SFEM solution is validated by a comparison with a superposition of mode model. The influence of geometry parameters on the results are investigated. The results show reflections of the waves and coupling between the longitudinal and transversal displacements and velocities due to the notch presence.

Acknowledgment

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Optimum Location of an Internal Hinge of a Uniform Column on an Elastic Foundation

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The buckling of a column affected by both an internal hinge and an elastic foundation is studied analytically. It is found that the elastic foundation raises the buckling force, but nonlinearly. The optimum locations for the hinge are determined.
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Introduction

Buckling of columns is basic in elastic stability [1]. However, in some cases the column may have an added interior joint or hinge due to the ease of construction/transportation or an equivalent hinge due to a through crack. The buckling of uniform columns with an internal hinge was first considered by Wang [2] and recently by Wang [3]. The present paper studies the buckling of a column with an interior hinge and in addition supported by a continuous elastic foundation. The results are important for the design of embedded pilings, drill strings, and structures on foundations, such as railroad tracks.

Although the formulation and the method of solution are simple, the results are complex. This paper contributes to the determination of optimum hinge locations for various foundation stiffnesses and will be useful in the design of load bearing columns.

Formulation

The equation governing small deflections of a column under axial load and supported by a linear elastic (Winkler) foundation is [4,5]

$$EI \frac{d^4 y'}{dx'^4} + N' \frac{d^2 y'}{dx'^2} + k' y' = 0 \quad (1)$$

Here, EI is the flexural rigidity of the beam, N' is the axial compressive force, k' is the elastic constant of the foundation, and (x', y') are axial and lateral coordinates. Normalize all lengths by the length of the column L and drop primes. Equation (1) becomes

$$\frac{d^4 y}{dx^4} + N \frac{d^2 y}{dx^2} + ky = 0 \quad (2)$$

where $N = N'L^2/EI$ is the normalized force and $k = k'L^4/EI$ is the normalized foundation elastic constant. The general solution to Eq. (2) can be classified into three types.

1. If $N^2 > 4k$, then the solution is a linear combination of the functions

$$y = \{\sin(\alpha x), \cos(\alpha x), \sin(\beta x), \cos(\beta x)\} \quad (3)$$

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where

$$\alpha = \sqrt{\frac{N - \sqrt{N^2 - 4k}}{2}} \quad \beta = \sqrt{\frac{N + \sqrt{N^2 - 4k}}{2}} \quad (4)$$

2. If $N^2 = 4k$, then the solution is composed of

$$y = \{\sin(\alpha x), \cos(\alpha x), x \sin(\alpha x), x \cos(\alpha x)\} \quad (5)$$

where $\alpha = \sqrt{N/2}$.

3. If $N^2 < 4k$, the solution is

$$y = \{e^{-ax} \cos(bx), e^{-ax} \sin(bx), e^{ax} \cos(bx), e^{ax} \sin(bx)\} \quad (6)$$

where

$$a = k^{1/4} \cos(\theta/2), \quad b = k^{1/4} \sin(\theta/2) \quad (7)$$

and

$$\theta = \pi - \tan^{-1} \left(\frac{\sqrt{4k - N^2}}{N} \right) \quad (8)$$

The ends of the column may be clamped, pinned, or free. Let the internal hinge be at $x=c$ and the subscripts 1, 2 denote the segments $0 \leq x \leq c$ and $c \leq x \leq 1$, respectively. At the hinge, the deflections are continuous, the moments are zero, and the shears are equal, i.e.,

$$y_1 = y_2 \quad (9)$$

$$\frac{d^2 y_1}{dx^2} = 0 \quad (10)$$

$$\frac{d^2 y_2}{dx^2} = 0 \quad (11)$$

$$\frac{d^3 y_1}{dx^3} + N \frac{dy_1}{dx} = \frac{d^3 y_2}{dx^3} + N \frac{dy_2}{dx} \quad (12)$$

Solution

There are six combinations of end conditions. These are clamped-clamped, clamped-pinned, clamped-free, pinned-pinned, pinned-free, and free-free.

If the end at $x=0$ is clamped, the displacement and slope are zero there. The solution is different for $N^2 > 4k$, $N^2 = 4k$, $N^2 < 4k$, respectively,

$$y_1 = \begin{cases} C_1 [\beta \sin(\alpha x) - \alpha \sin(\beta x)] + C_2 [\cos(\alpha x) - \cos(\beta x)] \\ C_1 x \sin(\alpha x) + C_2 [\sin(\alpha x) - \alpha x \cos(\alpha x)] \\ C_1 \sinh(\alpha x) \sin(bx) \\ + C_2 [b \sinh(\alpha x) \cos(bx) - a \cosh(\alpha x) \sin(bx)] \end{cases} \quad (13)$$

If the end is pinned, the displacement and the moment are zero. The solution is

$$y_1 = \begin{cases} C_1 \sin(\alpha x) + C_2 \sin(\beta x) \\ C_1 \sin(\alpha x) + C_2 x \cos(\alpha x) \\ C_1 \sinh(\alpha x) \cos(bx) + C_2 \cosh(\alpha x) \sin(bx) \end{cases} \quad (14)$$

If the end is free, the moment and shear are zero. After some work, we find

$$y_1 = \begin{cases} C_1[\beta(N - \beta^2)\sin(\alpha x) - \alpha(N - \alpha^2)\sin(\beta x)] + C_2[\beta^2 \cos(\alpha x) - \alpha^2 \cos(\beta x)] \\ C_1[(N - 3\alpha^2)\sin(\alpha x) - \alpha(N - \alpha^2)x \cos(\alpha x)] + C_2[2 \cos(\alpha x) + \alpha x \sin(\alpha x)] \\ C_1[b(N - b^2 + 3a^2)\sinh(ax)\cos(bx) - a(N + a^2 - 3b^2)\cosh(ax)\sin(bx)] + C_2[2ab \cosh(ax)\cos(bx) - (a^2 - b^2)\sinh(ax)\sin(bx)] \end{cases} \quad (15)$$

For the end at $x=1$, one can replace x by $(x-1)$ in the above forms. For example, if the $x=1$ end is clamped, form Eq. (13),

$$y_2 = \begin{cases} C_3\{\beta \sin[\alpha(x-1)] - \alpha \sin[\beta(x-1)]\} + C_4\{\cos[\alpha(x-1)] - \cos[\beta(x-1)]\} \\ C_3(x-1)\sin[\alpha(x-1)] + C_4\{\sin[\alpha(x-1)] - \alpha(x-1)\cos[\alpha(x-1)]\} \\ C_3 \sinh[a(x-1)]\sin[b(x-1)] + C_4\{b \sinh[a(x-1)]\cos[b(x-1)] - a \cosh[a(x-1)]\sin[b(x-1)]\} \end{cases} \quad (16)$$

The other forms are similarly constructed.

Given the location of the hinge c , the conditions Eqs. (9)–(12) are applied. For nontrivial solutions, a 4×4 characteristic determinant is obtained. The lowest eigenvalue N is the buckling load.

Results

Figure 1 shows the buckling force as a function of hinge location for the clamped-clamped case. When the foundation is absent, the buckling condition is governed by the equation [2]

$$\tan(c\sqrt{N}) + \tan[(1-c)\sqrt{N}] - \sqrt{N} = 0 \quad (17)$$

The maximum buckling force is 31.324 at $c=0.2193$, and due to the symmetry, also at $c=0.7807$. These are optimum hinge locations where the stability of the column is maximized. The minimum is 9.870 when the hinge is at the midpoint. When a foundation exists, the buckling force is increased, but unevenly. The optimum location shifts to the side, and the midpoint is no longer the weakest location. Table 1 shows the optimum location (one side only) for various foundation stiffnesses k .

Also shown in Fig. 1 are the mode shapes. Due to the hinge, the modes are more difficult to define, and change subtly as k is increased. In comparison, for a column without a hinge, there are distinct mode changes in integer wave numbers.

The clamped-pinned case does not have midpoint symmetry. If there is no foundation, the buckling force is from the smaller root

of the following equations:

$$\begin{cases} \tan(c\sqrt{N}) - \tan\sqrt{N} = 0 \\ \tan(c\sqrt{N}) - \sqrt{N} = 0 \end{cases} \quad (18)$$

The maximum buckling force is 20.1907 at $c=0.30084$, where the two forms of Eq. (18) intersect. The minimum is zero at $c=1$ where an infinitesimal swivel is created by the hinge and the pinned end. This phenomenon, the swivel of two closely spaced hinges, was also noted by Sawyer [6], who studied the stability of a column caused by a short hinged link. Figure 2 shows the increase of buckling force due to the foundation. Note that for $k < 100$, the optimum location stays at $c=0.301$ with no change in maximum buckling force. For $k > 100$, the optimum location is closer to the clamped end.

Figure 3 shows the results for the clamped-free case. Without a foundation, the buckling force is zero since a hinge and the free end would rotate freely. With a foundation, the buckling force is nonzero except when $c=1$. The optimum location is either at the clamped end or at some interior location, which moves toward the free end as foundation stiffness increases.

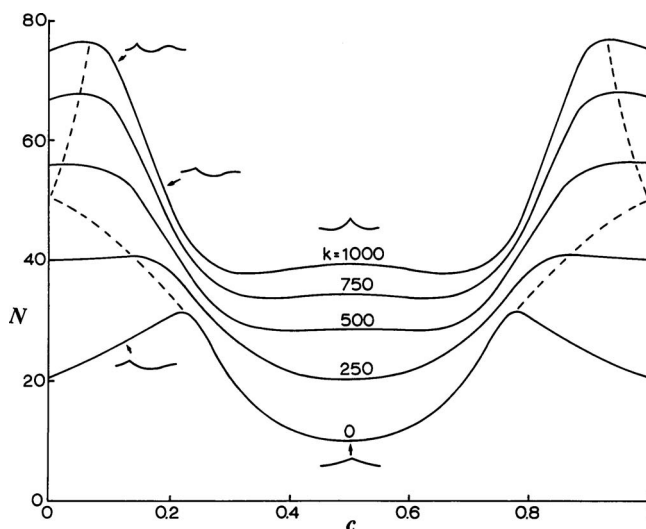


Fig. 1 Buckling force as a function of hinge location for the clamped-clamped case. The dashes lines show locus of maximum buckling forces and optimum locations. Typical buckling shapes are illustrated.

Table 1 Optimum locations for the clamped-clamped case

k	0	250	404	500	750	1000
c_{opt}	0.219	0.123	0	0.022	0.053	0.066
N_{max}	31.32	40.77	50.25	55.86	67.65	76.35

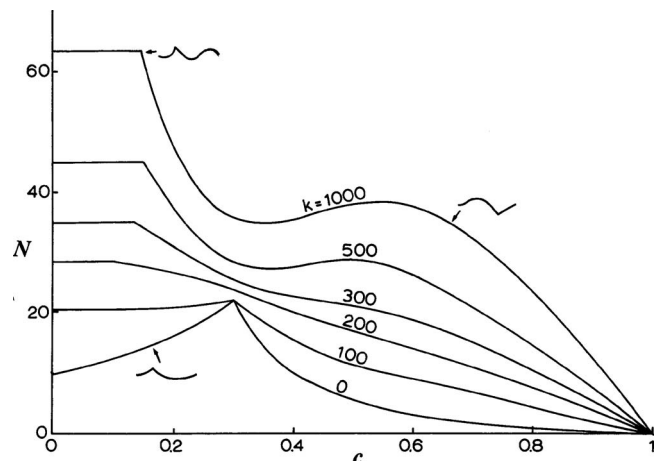


Fig. 2 The clamped-pinned case. Legend same as Fig. 1.

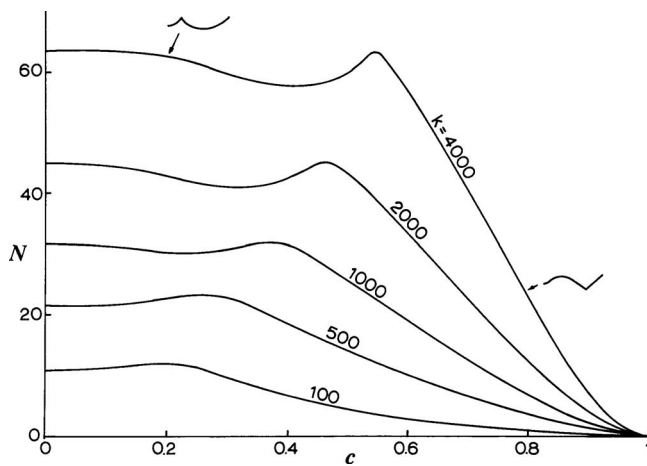


Fig. 3 The clamped-free case. Legend same as Fig. 1.

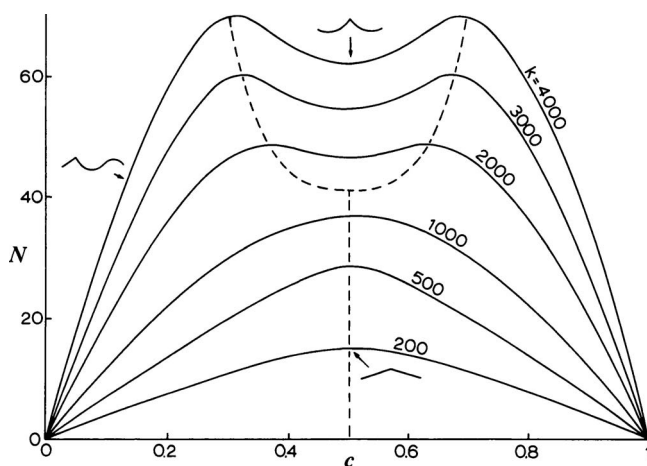


Fig. 4 The pinned-pinned case. Legend same as Fig. 1.

Table 2 Optimum locations for the pinned-pinned case

k	200	500	1000	1399	2000	3000	5000
c_{opt}	0.5	0.5	0.5	0.5	0.365	0.324	0.305
N_{max}	14.91	28.14	36.81	41.05	48.12	59.78	69.62

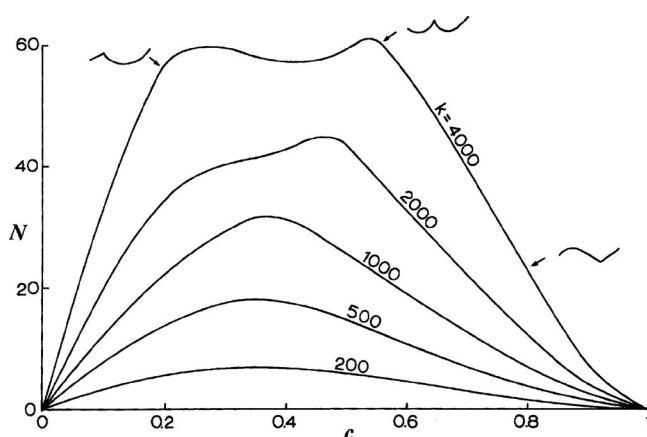


Fig. 5 The pinned-free case. Legend same as Fig. 1.

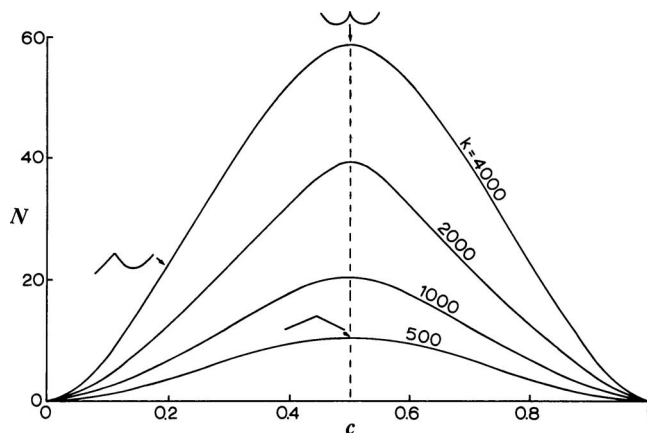


Fig. 6 The free-free case. Legend same as Fig. 1.

The pinned-pinned case is again symmetric with respect to the midpoint. Figure 4 shows that the optimum location for maximum strength is at the midpoint for $k < 1399$, then bifurcates to two branches. Since the pinned-pinned case is fundamental, the numerical values (for one side) are given in Table 2.

Figure 5 shows the pinned-free case, which is not symmetric. For low k , there is only one maximum, while for large k , two maxima appeared.

Figure 6 shows the free-free case. The optimum location for maximum strength is always at the midpoint. The corresponding buckling forces are 10.28, 20.27, 29.92, 39.19, 48.04, and 58.39 for $k = 500, 1000, 1500, 2000, 2500$, and 4000 , respectively.

The free-free case is particularly important since the finite beam can be reflected about the ends to model an infinite beam with an infinite series of hinges. If the hinges are evenly spaced with a distance of $L/2$, the buckling force is given as above.

Conclusions

We mention the related problem of shape optimization, which sometimes results in equivalent interior hinges as the area at certain locations is found to be zero [7–9]. However, these nonuniform columns differ from the uniform columns studied in this paper. For example, for the clamped-hinged case without foundation, we found that the optimum location of an interior hinge on a uniform column is at $c = 0.3008$, while for the shape optimized column, the hinge is at $c = 0.2261$ [7].

Our analytic solutions of the governing equations lead to exact characteristic determinants. These exact solutions are useful for checking numerical results for more complicated models of either the beam or the foundation [5]. All three forms of the analytic solutions (Eqs. (3), (5), and (6)) have been utilized in the determination of the buckling force.

The buckling of a column with an internal hinge is greatly influenced by an elastic foundation. The results are complex due to the interaction of the hinge and the foundation. We see large changes in the buckling characteristics, which include the uneven (nonlinear) increase of the buckling force. The optimum locations of the hinge for maximal buckling force are determined for the first time. Our graphs and tables will be helpful in the design of such load bearing columns. It is hoped that some future experiments would confirm our interesting results.

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Vibration Tailoring of a Polar Orthotropic Circular Plate With a Translational Spring

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In this study, the vibration tailoring problem is analytically solved for the polar orthotropic circular plate with translational spring along its circumference. By using the semi-inverse method and postulating the mode shape as a polynomial, we derive a closed-form solution. [DOI: 10.1115/1.2839896]

Keywords: polar orthotropic circular plates, semi-inverse method, closed-form solutions

1 Introduction

This paper deals with inverse dynamic problems in a continuous system, namely, a polar orthotropic circular plate with a spring. Whereas there are several papers on analytical and numerical evaluation of the natural frequencies of polar orthotropic circular plates, there appears to be no work devoted to vibration tailoring, i.e., the inverse problem. The latter consists in designing a plate that possesses the desired vibration characteristics. Such characteristics may include having a prespecified fundamental natural frequency, or possessing some prespecified natural frequencies, say, the first several natural frequencies, or having a response to a specified excitation such that it does not reach a specified frequency level. This paper addresses the design of a plate with specified fundamental vibration mode, as well as a preselected natural frequency value. Research in plate vibrations, in direct eigenvalue problems (when the plate's characteristics are given and determination of the spectrum is required), is given in the monograph by Leissa [1]. Since then, several papers have been written on vibration characteristics of polar orthotropic circular plates. For example, Gupta et al. [2] studied plates with linearly varying thickness. Chen [3] investigated the case of plates with arbitrarily varying thickness. Axisymmetric vibrations of a plate with a parabolically varying profile with restrained elastic edge were studied by Gupta and Ansari [4] and Luisoni and Laura [5]. The former studied the polar orthotropic plate, whereas the latter investigated the rectangular orthotropy. Other pertinent papers include those by Narita and Leissa [6], where plates elastically constrained along parts of the edge were studied, and Avalos and Laura [7], who investigated elastic rotational restraint. Closely related papers include those of Elishakoff [8], Gunaratnam and Bhattacharya [9], Elishakoff and Meyer [10], and Pardoen [11]. The superiority of the formulation in this paper lies in the fact that we obtain a *closed-form* solution to the vibration tailoring problem.

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2 Basic Equations

The governing differential equation of the polar orthotropic circular plate of varying flexural rigidity that undergoes axisymmetric vibrations reads (Leissa [1]) in nondimensional form

$$zD_r \frac{d^4 V}{dz^4} + \left(2D_r + 2z \frac{dD_r}{dz} + \nu_\theta D_r - \nu_r D_\theta \right) \frac{d^3 V}{dz^3} - \left[\frac{d}{dz} \left(-D_r - z \frac{dD_r}{dz} - \nu_\theta D_r + \nu_r D_\theta \right) - \nu_\theta \frac{dD_r}{dz} + \frac{1}{z} D_\theta \right] \frac{d^2 V}{dz^2} - \left[\frac{d}{dz} \left(-\nu_\theta \frac{dD_r}{dz} + \frac{1}{z} D_\theta \right) \right] \frac{dV}{dz} - z m \omega^2 V(z) R^4 = 0 \quad (1)$$

where z is the nondimensional radial coordinate equals r/R , $V(z) = W(r)/R$ is the nondimensional displacement, $W(r)$ is the mode shape, $\tilde{D}_r(r)$ and $\tilde{D}_\theta(r)$ are, respectively, radial and circumferential flexural rigidities, ν_r, ν_θ are Poisson's ratios of the orthotropic plate, assumed to be constants, r is the radial coordinate, ρ is the material density, h is the thickness, m is the ρh mass per unit area, and ω is the natural frequency. Note that in new circumstances, the flexural rigidities are functions of z , and $D_r(z) = \tilde{D}_r(Rz)$, $D_\theta(z) = \tilde{D}_\theta(Rz)$. Since Eq. (1) is a fourth-order differential equation with respect to z , it appears appropriate to look for the mode shape as a fourth-order polynomial:

$$V(z) = \alpha_0 + \alpha_2 z^2 + z^4 \quad (2)$$

where α_0 and α_2 are unknown coefficients. The coefficients α_0 and α_2 will be determined at a later stage.

The flexural rigidity D_r is sought as a polynomial of the fourth order since all terms in Eq. (1) are then represented as polynomials of the same order:

$$D_r(z) = b_0 + b_1(z-1) + b_2(z-1)^2 + b_3(z-1)^3 + b_4(z-1)^4 \quad (3)$$

which implies that coefficients b_j have the same dimension. We consider the case in which $D_\theta(z)$ is proportional to $D_r(z)$, i.e., $D_\theta(z) = k^2 D_r(z)$, where k is taken as a constant.

3 Boundary Conditions

We study a circular plate with a translational spring since practical situations can better be modeled by the presence of the spring at the boundary. The boundary conditions are (Leissa [1])

$$M_r(1) = 0, \quad Q_r(1) + PV(1) = 0 \quad (4)$$

where $P = pR^3$, and

$$M_r(z) = -\frac{D_r}{R} \left(V'' + \frac{\nu_\theta}{z} V' \right), \quad Q_r = -\frac{1}{R} \left(\frac{M_r}{z} + \frac{dM_r}{dz} - \frac{M_\theta}{z} \right), \quad (5)$$

$$M_\theta(z) = -\frac{D_\theta}{R} \left(\nu_r V'' + \frac{1}{z} W' \right)$$

M_r and M_θ denote the radial bending moment and the circumferential bending moment, respectively, Q_r symbolize the shear force, p is the stiffness per unit of length of the translational spring, and prime denotes differentiation with respect to the z coordinate.

4 Method of Solution

First of all, we need to calculate the coefficients α_0 and α_2 of the mode shape given in Eq. (3). These coefficients can be found by applying the boundary conditions given in Eq. (4), which become

$$2\alpha_2 + 12 + k^2 \nu_r (2\alpha_2 + 4) = 0 \quad (6)$$

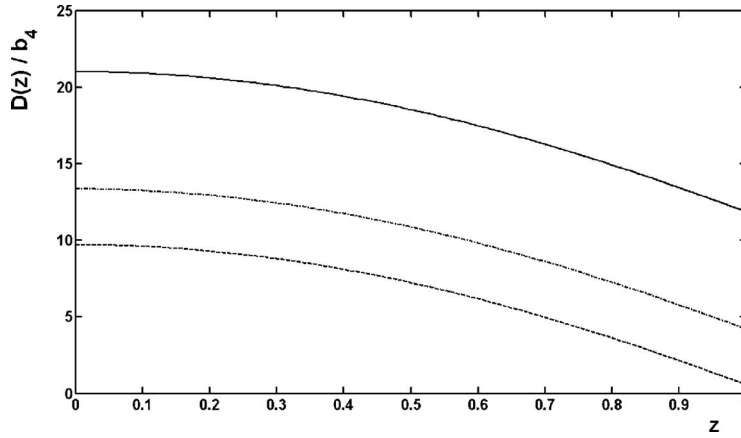


Fig. 1 Variation of $D(z)$ versus nondimensional radial coordinate z for $k=1$ and Poisson's ratio $\nu_r=0.3$ (--- $\beta=10$, -.- $\beta=100$, — $\beta=1000$)

$$24b_0 + 32b_0k^2\nu_r + 8b_0k^2 + P\alpha_0 + P\alpha_0k^2\nu_r - 5P + Pk^2\nu_r = 0 \quad (7)$$

Solution of Eqs. (6) and (7) reads

$$\alpha_0 = (-8b_0k^2 - 24b_0 - 32b_0k^2\nu_r + 5P + Pk^2\nu_r)/(1 + k^2\nu_r)P, \quad (8)$$

$$\alpha_2 = -2(3 + k^2\nu_r)/(1 + k^2\nu_r)$$

Substituting Eq. (8) into Eq. (2), we get the equation of mode shape as depending on the coefficient b_0 and the translational spring constant P :

$$V(z)(1 + k^2\nu_r)P = -8b_0k^2 - 24b_0 - 32b_0k^2\nu_r + 5P + Pk^2\nu_r - 2(3 + k^2\nu_r)Pz^2 + P(1 + k^2\nu_r)z^4 \quad (9)$$

It is natural to expect the dependence on the mode shape of the spring constant P as well as on the flexural rigidity, which is represented here by coefficient b_0 . We are solving a semi-inverse problem in which the flexural rigidities play the role of the output that should match the input represented by the mode shape.

It appears instructive to investigate limiting cases, namely, when P approaches zero or infinity. For the case $P=0$, Eq. (7) reduces to

$$24b_0 + 32b_0k^2\nu_r + 8b_0k^2 = 0 \quad (10)$$

Solution of Eq. (10) yields $b_0=0$. The equation for a_2 remains the same as in Eq. (8). Thus, the mode shape reduces to

$$V(z) = \alpha_0 - 2(3 + k^2\nu_r)z^2/(1 + k^2\nu_r) + z^4 \quad (11)$$

For P becoming unbounded, Eq. (8) becomes $\alpha_0 + \alpha_0k^2\nu_r - 5 + k^2\nu_r = 0$ from which we obtain α_0 :

$$\alpha_0 = (5 + k^2\nu_r)/(1 + k^2\nu_r) \quad (12)$$

For the coefficient α_2 , we get the same expression as given in Eq. (8). Hence, the mode shape remains as in case $P=0$ in Eq. (11). It must be noted that Eq. (11) corresponding to unbounded P can be directly obtained from Eq. (9) with P set to approach infinity. The first three terms in parentheses in Eq. (9) are canceled out, whereas the remaining three terms in parentheses coincide with Eq. (11). We first consider the semi-inverse method of solution associated with $k=1$. This case is associated with the isotropic plate $D_\theta(z)=D_r(z)=D(z)$, hence $\nu_\theta=\nu_r$. The substitution of Eq. (3), $D_\theta(z)=k^2D_r(z)$, and (9) into the governing differential equation, where $k=1$, leads to the following polynomial equation: $A_0 + A_1z + A_2z^2 + A_3z^3 + A_4z^4 + A_5z^5 = 0$, where A_j depends on b_j , ν_r , m , P , and R , as well as the sought frequency ω . To save space, the coefficients are not reproduced here except the coefficient A_5

$= 480Pb_4 + 96Pb_4\nu_r^2 + 576Pb_4\nu_r - m\omega^2R^4 - m\omega^2R^4P\nu_r = 0$. From it, we get the relationship between the natural frequency squared ω^2 and the coefficient b_4 :

$$\omega^2 = 96(5 + \nu_r)b_4/mR^4 \quad (13)$$

We obtain the final expression of the radial flexural rigidity as follows:

$$\frac{D(z)}{b_4} = \frac{1}{2} \frac{(35 + 12\nu_r + \nu_r^2)\beta}{(1 + \nu_r)(\beta + 240b_4 + 48b_4\nu_r)} + \frac{11 + \nu_r}{1 + \nu_r} - \frac{(12 + 4\nu_r)}{1 + \nu_r} z^2 + z^4 \quad (14)$$

The mode shape in Eq. (13) becomes

$$V(z) = -16 \frac{(35 + 12\nu_r + \nu_r^2)\beta}{(1 + \nu_r)(240 + \beta + 48\nu_r)} + \frac{5 + \nu_r}{1 + \nu_r} - 2 \frac{(3 + \nu_r)}{1 + \nu_r} z^2 + z^4 \quad (15)$$

Note that Eq. (15) coincides with Eq. (30) in the study by Elishakoff and Meyer [10]. Figure 1 depicts the variation of flexural rigidity $D(z)/b_4$ versus the radial coordinate z for different values of β and Poisson's ratio $\nu_r=0.3$. For the free plate ($\beta=0$), the flexural rigidity vanishes at $z=1$. For the plate with nonzero translational spring coefficient β , the flexural rigidity $D(z)$ does not vanish in the region $0 \leq z \leq 1$.

5 Semi-Inverse Method of Solution Associated With $k=2$ or $k=3$

In the particular case of $k=2$, the substitution of Eqs. (3), $D_\theta(z)=k^2D_r(z)$, and (9) into the governing differential equation leads to the following fifth-order polynomial equation:

$$B_0 + B_1z + B_2z^2 + B_3z^3 + B_4z^4 + B_5z^5 = 0 \quad (16)$$

where

$$B_0 = -16b_1\nu_r - 96b_4 - 48b_2 - 48b_3\nu_r - 192b_3\nu_r^2 + 32b_2\nu_r + 24b_1 + 128b_2\nu_r^2 + 256b_4\nu_r^2 + 72b_3 - 64b_1\nu_r^2 + 64b_4\nu_r \quad (17)$$

$$B_1 = 40Pb_0 - 112Pb_3 + 160Pb_0\nu_r + 64Pb_2 - 800b_4\nu_r - 256Pb_2\nu_r^2 - 160Pb_1\nu_r + 184Pb_4 - 40Pb_1 + 320Pb_3\nu_r + 128m\omega^2b_0R^4\nu_r + 768Pb_3\nu_r^2 - 4m\omega^2PR^4\nu_r - 5m\omega^2R^4P + 56m\omega^2R^4b_0 - 1536Pb_4\nu_r^2 \quad (18)$$

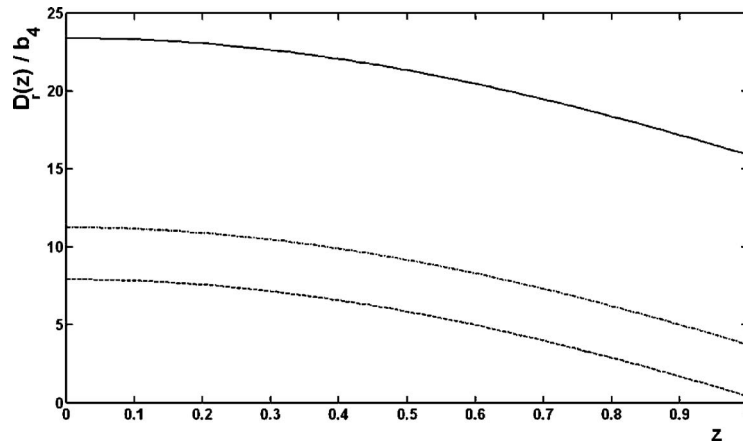


Fig. 2 Variation of $D_r(z)$ versus nondimensional radial coordinate z for $k=2$ and Poisson's ratio $\nu_r=0.3$ (--- $\beta=10$, -.- $\beta=100$, — $\beta=1000$)

$$B_2 = 288b_3 - 384b_4 - 864b_2\nu_r + 432b_1\nu_r + 192b_1\nu_r^2 + 1536b_4\nu_r^2 + 864b_3\nu_r + 96b_1 - 384b_2\nu_r^2 - 192b_2 \quad (19) \quad \omega^2 = 24(17 + 16\nu_r)b_4/mR^4 \quad (23)$$

$$B_3 = 1008b_4 + 8m\omega^2R^4\nu_r + 512b_2\nu_r^2 + 2048b_4\nu_r^2 + 6m\omega^2R^5 + 832b_2\nu_r + 4160b_4\nu_r - 1536b_3\nu_r^2 - 2496b_3\nu_r + 176b_2 - 528b_3 \quad (20) \quad b_3 = 4b_4 \quad (24)$$

$$B_4 = -5440b_4\nu_r - 1120b_4 + 960b_3\nu_r^2 + 1360b_3\nu_r - 3840b_4\nu_r^2 + 280b_3 \quad (21) \quad b_2 = -4b_4(21 - 4\nu_r - 16\nu_r^2)/(1 + 4\nu_r)(11 + 8\nu_r) \quad (25)$$

$$B_5 = 2016b_4\nu_r + 1536b_4\nu_r^2 + 408b_4 - m\omega^2R^4 - 4m\omega^2\nu_rR^4 \quad (22) \quad b_1 = -128b_4(2 + 3\nu_r + \nu_r^2)/(1 + 4\nu_r)(11 + 8\nu_r) \quad (26)$$

Solution of Eq. (22) for the natural frequency squared ω^2 results in

The equation resulting from substitution of Eqs. (24) and (25) into Eq. (18) leads to the formula for b_0 :

$$b_0 = \frac{256(10 + 47\nu_r + 69\nu_r^2 + 40\nu_r^3 + 8\nu_r^4)P}{(1 + 4\nu_r)(11 + 8\nu_r)(2856b_4 + 5P + 20P\nu_r + 9216b_4\nu_r + 6144b_4\nu_r^2)}b_4 \quad (27)$$

The expression for the flexural rigidity becomes

$$\frac{D_r(z)}{b_4} = \frac{1}{(1 + 4\nu_r)(8\nu_r + 11)} \left[\frac{256(10 + 47\nu_r + 69\nu_r^2 + 40\nu_r^3 + 8\nu_r^4)P}{(2856b_4 + 5P + 20P\nu_r + 9216b_4\nu_r + 6144b_4\nu_r^2)} + 139 + 244\nu_r + 96\nu_r^2 + (84 - 16\nu_r - 64\nu_r^2)z - (150 + 296\nu_r + 68\nu_r^2)z^2 + z^4 \right] \quad (28)$$

The mode shape is expressed as

$$V(z) = \frac{1}{1 + 4\nu_r} \left[-\frac{(10 + 47\nu_r + 69\nu_r^2 + 40\nu_r^3 + 8\nu_r^4)(14,336 + 32,768\nu_r)}{(1 + 4\nu_r)(11 + 8\nu_r)(2856 + 5\beta + 9216\nu_r + 20\beta\nu_r + 6144\nu_r^2)} + 5 + 4\nu_r - 2(3 + 4\nu_r)z^2 \right] + z^4 \quad (29)$$

Figure 2 represents the graph of $D_r(z)/b_4$ for different values of β and Poisson's ratio $\nu_r=0.3$.

For another particular case, namely, $k=3$, the analogous procedure yields natural frequency squared as follows:

$$\omega^2 = 288(1 + 3\nu_r)b_4/mR^4 \quad (30)$$

The expression for the flexural rigidity reads

$$\frac{D_r(z)}{b_4} = \frac{1}{1 + 9\nu_r} \left[\frac{1}{32} \frac{(5 + 14\nu_r + 9\nu_r^2)\beta}{(1 + 3\nu_r)} + 19 + 27\nu_r - (20 - 36\nu_r)z^2 \right] + z^4 \quad (31)$$

whereas the mode shape is

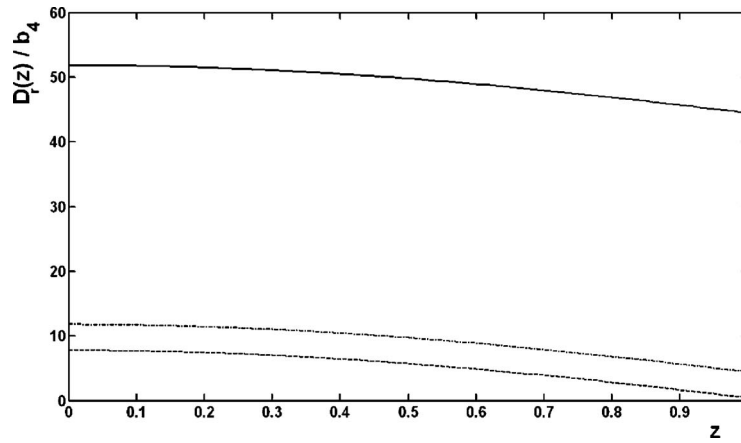


Fig. 3 Variation of $D_r(z)$ versus nondimensional radial coordinate z for $k=3$ and Poisson's ratio $\nu_r=0.3$ (--- $\beta=10$, -.- $\beta=100$, — $\beta=1000$)

$$V(z) = \frac{1}{1+9\nu_r} \left[-3 \frac{(5+14\nu_r+9\nu_r^2)}{(1+3\nu_r)} + 5+9\nu_r - 2(3+9\nu_r)R^2z^2 \right] + z^4 \quad (32)$$

The graph of $D_r(z)/b_4$ is shown in Fig. 3 for different values of β , while Poisson's ratio is fixed at $\nu_r=0.3$.

6 Conclusion

In this paper, we deal with vibration tailoring with the specified mode shape of the polar orthotropic plate. Like in the classical works of Newton's (as quoted in Ref. [12]) and Bertrand's [13] problems, the *motion* of the system is postulated and the *cause* is looked after. In the case of Ref. [13], the cause is the force, whereas in our study, the cause is represented by the flexural rigidity.

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Flow Through a Lens-Shaped Duct

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The flow through a symmetric, lens-shaped duct is solved by accurate Ritz and perturbation methods. The flow rate (resistance) is found for various thickness ratios. The flow rate is much better than friction factor–Reynolds number product as an index for duct flows, especially for the lens duct studied in this paper. The results are also important for the torsion of lens-shaped bars.
[DOI: 10.1115/1.2840045]

Introduction

Laminar, fully developed flow in ducts has a wide range of applications in thermofluid transport processes. Solutions have been found for a variety of cross sectional shapes [1–3]. The governing equation is the Poisson equation, which also describes the torsion of elastic rods and the deformation of uniformly loaded membranes. See Ref. [4] and the elasticity literature listed in Ref. [5]. The present paper considers the flow through a duct with a cross section bounded by two symmetric circular arcs. There are two reasons for this study. First, the flow properties of this simple fundamental shape have not been reported before, although some resistance values were obtained in Ref. [3] by a low accuracy graphical method. Second, there are some numerical difficulties for this shape (for example, direct finite difference or finite element integration) especially when the thickness of the lens duct shrinks to zero. To circumvent the deficiencies of direct numerical integration, we shall use an efficient Ritz method and analytic perturbation methods.

Ritz Method

Consider a lens-shaped duct of width $2L$ and thickness ratio $b < 1$. Normalize all lengths by L and the velocity by (pressure gradient G) L^2 (viscosity μ). The inset of Fig. 2 shows the normalized cross section and the Cartesian axes. The Navier–Stokes reduce to the Poisson equation,

$$\nabla^2 w = -1 \quad (1)$$

where w is the longitudinal velocity. Let

$$c = \frac{1-b^2}{2b} \quad (2)$$

The boundary conditions are that w is zero on the arcs

$$x^2 + (y \pm c)^2 = c^2 + 1, \quad -1 \leq x \leq 1 \quad (3)$$

We shall use the Ritz method, which is most suited to this problem. The method and its convergence have been known in elasticity (e.g., Ref. [6]), but its applications to fluid mechanics are less frequent. In essence, the solution to Eq. (1) is equivalent to minimizing

$$I = \iint (w_x^2 + w_y^2 - 2w) dx dy \quad (4)$$

where the double integration is over the whole cross sectional area. Approximate w by the expansion

$$\begin{aligned} w(x, y) = & [x^2 + (y - c)^2 - c^2 - 1][x^2 + (y + c)^2 - c^2 - 1] \\ & \times (a_1 + a_2 x^2 + a_3 y^2 + a_4 x^4 + a_5 x^2 y^2 + a_6 y^4 + a_7 x^6 \\ & + a_8 x^4 y^2 + a_9 x^2 y^4 + a_{10} y^6 + a_{11} x^8 + a_{12} x^6 y^2 + a_{13} x^4 y^4 \\ & + a_{14} x^2 y^6 + a_{15} y^8 + \dots) \\ = & \sum_{i=1}^N a_i \phi_i(x, y) \end{aligned} \quad (5)$$

Here, w satisfies the boundary conditions exactly and the series is even in x and y , complete and convergent in the circle of radius 1. The number of terms N can be taken as 1, 3, 6, 10, 15, 21, 28, etc., containing the highest homogenous powers. Setting the derivatives of Eq. (4) to zero yields the linear algebraic equation

$$\sum_{j=1}^N a_j \Gamma_{ij} = \Lambda_i, \quad i = 1 \text{ to } N \quad (6)$$

where

$$\Gamma_{ij} = \iint \left(\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right) dx dy \quad (7)$$

$$\Lambda_i = \iint \phi_i dx dy \quad (8)$$

We solve for the coefficients a_i from Eq. (6) and the velocity from Eq. (5). The flow rate, normalized by $L^4 G / \mu$, is then simply

$$Q = \iint w dx dy = \sum_{i=1}^N a_i \Lambda_i \quad (9)$$

Let us compare the results of the Ritz method with an exact solution. Sokolnikoff and Sokolnikoff [7] used conformal mapping to solve the torsion of a bar whose cross section is bounded by two symmetric circular arcs intersecting at a right angle (the thickness ratio is specific, $b = \sqrt{2} - 1 = 0.41421$). Their solution in terms of fluid flow is

$$\begin{aligned} w = & \frac{1}{4}(1 - x^2 - y^2) + \frac{1}{2\pi(x^2 + y^2)[(1 + x^2 + y^2)^2 - 4y^2]} \\ & \times \{ \pi(x^2 + y^2)[(x^2 + y^2)^2 - 1] + x(1 + x^2 + y^2) \\ & \times [4y^2 - (1 - x^2 - y^2)^2] \ln R \\ & + y(1 - x^2 - y^2)[(1 + x^2 + y^2)^2 + 4x^2] S \} \end{aligned} \quad (10)$$

where

$$R = \frac{[(1 - x^2 - y^2)^2 + 4y^2]^{1/2}}{(1 + x)^2 + y^2}, \quad S = \tan^{-1} \left(\frac{2y}{1 - x^2 - y^2} \right) \quad (11)$$

The maximum velocity is $w(0, 0) = 0.06831$. Table 1 shows the convergence of the maximum velocity to the exact value using the Ritz method. The error alternates in sign as in a truncated Fourier series. We see that in taking ten terms, the error is less than 0.1%. A comparison of velocities elsewhere in the region shows similar error magnitudes.

However, the Ritz method is poor when the cross section is close to a circle ($b \approx 1$). This is because the exact solution for the circle is quadratic and is ill represented by the approximation of Eq. (5). On the other hand, there are also convergence problems for very thin cross-sections ($b \approx 0$). For these extremes, we shall use perturbation methods.

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Table 1 Convergence of the Ritz method to the exact solution

N	3	6	10	15	Exact
$w(0,0)$	0.06756	0.06851	0.06824	0.06834	0.06831

Almost Circular Duct

Since $b \approx 1$ and $c \approx 0$, we perturb about $c=0$, which is the circle. For $y \geq 0$, Eq. (3) gives

$$y^2 + 2cy + x^2 = 1 \quad (12)$$

or, in cylindrical coordinates (r, θ) ,

$$r = -c \sin \theta + \sqrt{1 + c^2 \sin^2 \theta}, \quad 0 \leq \theta \leq \pi \quad (13)$$

The entire boundary is thus described by

$$\begin{aligned} r &= -c |\sin \theta| + \sqrt{1 + c^2 \sin^2 \theta} \approx 1 - c |\sin \theta| + c^2 \sin^2 \theta / 2 + O(c^3) \\ &\equiv 1 + \delta \end{aligned} \quad (14)$$

The velocity is also expanded

$$w = w_0 + cw_1 + c^2w_2 + O(c^3) \quad (15)$$

Equation (1) gives

$$\nabla^2 w_0 = -1, \quad \nabla^2 w_1 = 0, \quad \nabla^2 w_2 = 0 \quad (16)$$

The boundary condition are that w is bounded at the origin and $w=0$ on $r=1+\delta$ or

$$\begin{aligned} w|_{1+\delta} &= w|_1 + \delta w_r|_1 + \delta^2 w_{rr}|_1 / 2 + \dots \\ &= w_0|_1 + c(w_1|_1 - |\sin \theta| w_{0r}|_1) \\ &\quad + c^2[w_2|_1 - |\sin \theta| w_{1r}|_1 + \sin^2 \theta (w_{0r} + w_{0rr})|_1 / 2] \\ &\quad + \dots = 0 \end{aligned} \quad (17)$$

The zeroth order is the Poiseuille flow in a circular tube,

$$w_0 = (1 - r^2)/4 \quad (18)$$

Form Eq. (17), the first order boundary condition is

$$w_1|_1 = -\frac{1}{2} |\sin \theta| = -\frac{1}{2} \sum_{n=0}^{\infty} A_n \cos(2n\theta) \quad (19)$$

Here, A_n are the Fourier coefficients

$$A_0 = \frac{2}{\pi}, \quad A_n = \frac{-4}{(4n^2 - 1)\pi}, \quad n = 1, 2, \dots \quad (20)$$

The bounded harmonic solution for w_1 is

$$w_1 = -\frac{1}{2} \sum_{n=0}^{\infty} A_n r^{2n} \cos(2n\theta) \quad (21)$$

The second order is more involved. From Eq. (17),

$$\begin{aligned} w_2|_1 &= \frac{1}{2} \sin^2 \theta - \sum_0^{\infty} A_n \cos(2n\theta) \sum_1^{\infty} A_n n \cos(2n\theta) \\ &= \sum_0^{\infty} B_n \cos(2n\theta) \end{aligned} \quad (22)$$

Here, B_n are the coefficients of each individual harmonic. The solution is

$$w_2 = \sum_0^{\infty} B_n r^{2n} \cos(2n\theta) \quad (23)$$

The maximum velocity is at the origin

$$w|_0 = \frac{1}{4} - c \frac{A_0}{2} + c^2 B_0 + O(c^3) \quad (24)$$

Let

$$K(s) = \int_0^s w r dr \quad (25)$$

Then, the flow rate is

$$\begin{aligned} Q &= 2 \int_0^{\pi} \int_0^{1+\delta} w r dr d\theta \\ &= 2 \int_0^{\pi} K(1+\delta) d\theta \\ &= 2 \int_0^{\pi} [K(1) + \delta K'(1) + \delta^2 K''(1)/2 + \dots] d\theta \\ &= 2 \int_0^{\pi} \left(\int_0^1 w_0 r dr + c \int_0^1 w_1 r dr \right. \\ &\quad \left. + c^2 \int_0^1 w_2 r dr + c^2 \sin^2 \theta / 4 + \dots \right) d\theta \end{aligned} \quad (26)$$

Due to the integration in θ , only the constant terms of the Fourier expansions are left. From Eq. (22), we find

$$B_0 = \frac{1}{4} - \frac{1}{2} \sum_1^{\infty} n A_n^2 = \frac{1}{4} - \frac{8}{\pi^2} \sum_1^{\infty} \frac{n}{(4n^2 - 1)^2} = \frac{1}{4} - \frac{1}{\pi^2} \quad (27)$$

From Eq. (26), the flow rate is

$$\begin{aligned} Q &= 2\pi \left(\frac{1}{16} - c \frac{A_0}{4} + c^2 \frac{B_0}{2} + \frac{c^2}{8} + \dots \right) \\ &= \frac{\pi}{8} \left[1 - \frac{8}{\pi} c + \left(4 - \frac{8}{\pi^2} \right) c^2 + O(c^3) \right] \end{aligned} \quad (28)$$

For the circle ($b=1$ or $c=0$), Eq. (28) gives the correct value of $\pi/8=0.3927$ while the Ritz method does not converge. In fact, the Ritz method is inadequate for $b>0.9$.

Thin Duct

When the thickness of the duct is small, one can approximate the flow locally by a parallel plate solution. This has been suggested in Ref. [1] for small aspect ratio ducts. We present here a systematic perturbation solution such that higher order corrections can be obtained. The natural expansion parameter is b ; thus, let

$$y = b\eta \quad (29)$$

where $b \ll 1$ and $\eta = O(1)$. Now, the velocity is no longer $O(1)$. A balance of leading terms in Eq. (1) shows that the proper expansion is

$$w = b^2 [w_0(x, \eta) + b^2 w_2(x, \eta) + \dots] \quad (30)$$

Then, Eq. (1) gives

$$w_{0\eta\eta} = -1 \quad (31)$$

$$w_{2\eta\eta} + w_{0xx} = 0 \quad (32)$$

Form Eqs. (2) and (3), the top boundary is given by

$$y = \sqrt{\left(\frac{1+b^2}{2b} \right)^2 - x^2} - \frac{1-b^2}{2b} \sim (1-x^2)b + x^2(1-x^2)b^3 + O(b^5) \quad (33)$$

or the velocity is zero at

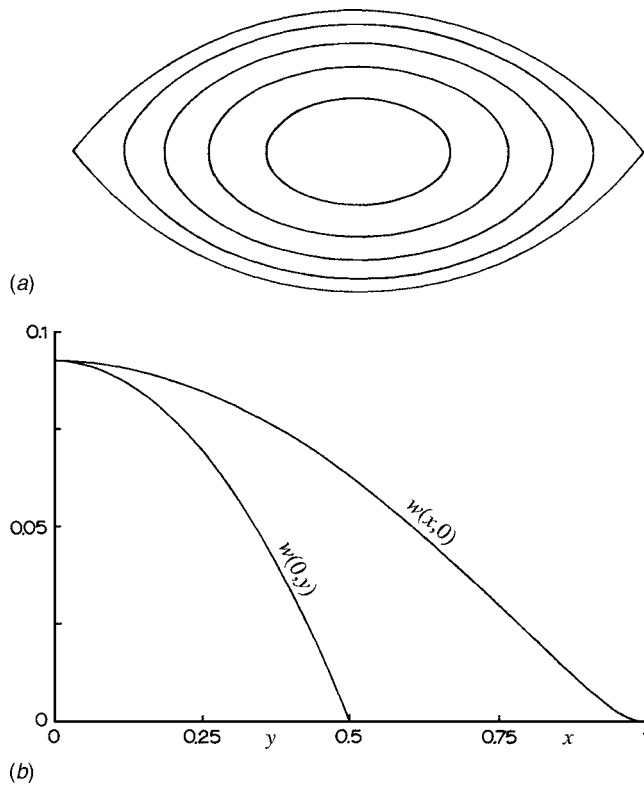


Fig. 1 (a) Constant velocity lines for a lens duct with a thickness ratio of $b=0.5$. See the inset of Fig. 2 for dimensions. Values of constant velocity from the boundary: $w=0, 0.02, 0.04, 0.06$ and 0.08 . (b) Velocity profiles $w(x,0)$ along the x axis and $w(0,y)$ along the y axis.

$$\eta = \eta_1 = (1-x^2) + x^2(1-x^2)b^2 + \dots \quad (34)$$

An expansion similar to Eq. (17) gives

$$w_0(x, 1-x^2) = 0 \quad (35)$$

$$w_2(x, 1-x^2) = -x^2(1-x^2) \frac{\partial}{\partial \eta} w_0(x, 1-x^2) \quad (36)$$

The symmetric solution to Eqs. (31) and (35) is

$$w_0 = [(1-x^2)^2 - \eta^2]/2 \quad (37)$$

After some work, the solution to Eqs. (32) and (36) is

$$w_2 = -(1-3x^2)[(1-x^2)^2 - \eta^2] + x^2(1-x^2)^2 \quad (38)$$

Similar to Eq. (26), the flow rate is

$$\begin{aligned} Q &= 4b^3 \int_0^1 \int_0^{\eta_1} (w_0 + b^2 w_2 + \dots) d\eta dx \\ &= 4b^3 \int_0^1 \left[\int_0^{1-x^2} w_0 d\eta + b^2 x^2 (1-x^2) w_0(x, 1-x^2) \right. \\ &\quad \left. + b^2 \int_0^{1-x^2} w_2 d\eta + \dots \right] dx \\ &= \frac{64}{105} b^3 [1 - b^2 + O(b^4)] \end{aligned} \quad (39)$$

Results and Discussions

Figure 1(a) shows typical constant velocity lines for a lens duct with a thickness ratio of 0.5. The flow rate is 0.06360, somewhat

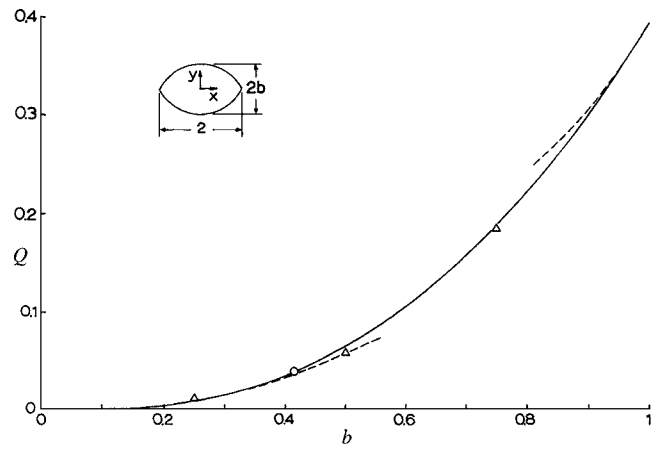


Fig. 2 The normalized flow rate Q as a function of thickness ratio b . Dashed lines are approximations of Eq. (28) or Eq. (39). The small circle is the exact solution (Sokolnikoff and Sokolnikoff [7]). Small triangles are from Shah and Bhatti [3].

less than the value of 0.07854 for an elliptic duct with the same thickness ratio. Figure 1(b) shows the velocity profiles along the x and y axes. Note that at the corner ($x=1$), the slope of the velocity (and shear stress) is zero. This can be explained as follows. Let ρ, φ be cylindrical coordinates placed at a corner. Then, for small ρ , Eq. (1) reduces to the Laplace equation. If the opening angle is 2β , the solution is dominated by $\rho^\lambda \cos(\lambda\varphi)$, where $\lambda = \pi/2\beta$. The shear stress at the corner is then zero if $\lambda > 1$ or $\beta < \pi/2$ (interior corner). This behavior at the corner is absent for an elliptic duct. We note that the lens duct, with two circular arcs, is much easier to fabricate than the elliptic duct or the rectangular duct.

Figure 2 shows that the flow rate Q increases with the thickness (ratio) b . The small circle is the exact solution of Ref. [7], which confirmed the results of our Ritz method. The small triangles are the results of Ref. [3] by a less accurate graphical method. Also shown are the analytical perturbation results, which are more accurate than the Ritz method for extreme values $b \approx 0$ or $b \approx 1$. Combining both our methods, the results are shown in Table 2.

On the other hand, both our methods are superior to direct finite elements or finite differences. This is because for the latter methods, both the curved boundary and the sharp corners need to be compromised. Furthermore, the double numerical integration for the flow rate introduces additional errors. In contrast, the flow can be exactly integrated from the perturbation solutions. Since ϕ_i are polynomials, Eqs. (7) and (8) of the Ritz method can also be exactly integrated. The flow rate (Eq. (9)) is then a simple sum.

The friction factor-Reynolds number product, much used in engineering, is related to Q by

$$f \text{Re} = \frac{8A^3}{P^2 L^4 Q} \quad (40)$$

where A is the cross-sectional area and P is the perimeter length,

Table 2 The flow rate Q as a function of thickness b

b	0	0.1	0.2	0.3	0.4	0.5
Q	0	0.0006034	0.004698	0.01523	0.03437	0.06364
b	0.6	0.7	0.8	0.9	1	
Q	0.1041	0.1564	0.2214	0.3000	0.3927	

Table 3 Friction factor—Reynolds number products for the lens duct

b	0	0.1	0.2	0.3	0.4	0.5
$f Re$	15.56	15.60	15.69	15.80	15.90	15.98
b	0.6	0.7	0.8	0.9	1	
$f Re$	16.00	16.03	16.03	16.01	16.00	

$$A = 2L^2[(c + b)^2 \tan^{-1}(1/c) - c], \quad P = 4L(c + b)\tan^{-1}(1/c) \quad (41)$$

Table 3 shows the friction factor—Reynolds number product fRe of the lens duct. The value of 140/9 for zero thickness ratio is obtained from asymptotic expansions of Eqs. (39)–(41).

The flow rate Q seems to be much better than fRe as an index for duct flows, especially for the lens duct studied in this paper. The reasons are as follows. First, cross-sectional area and perimeter (Eq. (41)) are tedious to calculate. Second, the Reynolds

number is immaterial for laminar duct flows. Third, the value of fRe (Table 3) is insensitive to thickness ratio, shows a maximum, and has little physical meaning.

In conclusion, we have successfully determined the flow properties through a simple lens duct, which may be applied especially to microfluidics. The results are also useful for the torsion of lens-shaped rods. Our Ritz method and our systematic modified perturbation methods are more efficient than direct numerical integration for this geometry.

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A Comparison of Coulomb Friction and Friction Stress Models Based on Multidimensional Nanocontact Experiments

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The accuracy of the Oliver–Pharr approach for nanoindentation experiments critically depends on the interfacial friction condition. Although Coulomb friction is often used in finite element simulations for the correction, the friction stress model may give a more appropriate physical scenario. The measurement of the tangential contact stiffness by a recently developed multidimensional nanocontact system provides a direct verification of these two friction models. Both friction models will predict the tangential stiffness reduction as the consequence of interface microslip, but quantitative comparison to the experiments supports the friction stress model. [DOI: 10.1115/1.2871022]

1 Introduction

The Oliver–Pharr approach is frequently used for the measurement of elastic modulus and material hardness using load and displacement sensing nanoindentation techniques [1,2]. The central idea is based on Sneddon's solution, i.e., the unloading contact stiffness S_z is given by $S_z = dP/d\delta_z = \beta(2/\sqrt{\pi})E^*\sqrt{A}$, where P is the normal indentation force, δ_z is the normal penetration depth, A is the contact area, $E^* = [(1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2]^{-1}$ is the effective modulus, E_1 and E_2 are Young's moduli, and ν_1 and ν_2 are Poisson's ratios of the indenter and the substrate, respectively. The numerical prefactor β is equal to 1 for frictionless contact. In practice, β can deviate from unity due to the interfacial friction; for Berkovich indenter, the difference is about 5%, but for sharper indenters, the difference can be as large as about 20% [2,3]. In finite element simulations for the evaluation of this correction factor β , Coulomb friction is often adopted. However, Coulomb friction is usually interpreted as a statistical average of multiple asperities in sliding contact [4,5], so that it does not formally apply to the single-asperity nanocontact experiment. On the other

hand, the friction stress model may give a more appropriate physical picture, in which the friction force is given by $\tau_0 A$, where τ_0 is called the friction stress or interfacial shear strength [4,6]. This model is essentially similar to the cohesive interface model used in fracture mechanics [7,8]. In this paper, the two friction models are compared to tangential contact stiffness measurements by a multidimensional nanocontact system [9], and it is found that the friction stress model gives better agreement, suggesting the use of this model for the evaluation of correction factor β .

2 Experimental Observation and Theory

Combining three nanoindentation actuators in orthogonal directions, a multidimensional nanocontact system has been developed to quantitatively examine tangential mechanical properties at the nano- and mesoscopic length scales [9]. Since the objective of this work is to use the stiffness measurements to examine the friction model, the readers are referred to Ref. [9] for experimental details. The continuous stiffness measurement technique is employed [1,2], and the amplitude of the displacement oscillation is subnanometer, much smaller than the typical indentation depth. Therefore, the normal contact stiffness should be a constant value given by elastic Sneddon's solution. Similarly, the elastic tangential contact stiffness is given by $S_x^{\text{elastic}} = dT_x/d\delta_x = 8a\mu^*$, where T_x and δ_x are the tangential force and displacement (applied at distant reference points), respectively, a is the contact radius for an arbitrary axisymmetric indenter (Fig. 1(a)), and the effective shear modulus is $\mu^* = [(2-\nu_1)/\mu_1 + (2-\nu_2)/\mu_2]^{-1}$ with shear moduli μ_1 and μ_2 for the two contacting solids [9–12]. Therefore, the stiffness ratio $S_x/S_z = 4\mu^*/E^*$ can be used to evaluate Poisson's ratio. Using a diamond Berkovich indenter (i.e., a three-sided pyramid, which is often approximated by a cone with half apex angle $\alpha = 70.3^\circ$ deg in numerical simulations) in this multidimensional nanocontact system, we have measured the normal and tangential contact stiffnesses for fused silica and several other materials. The normal contact stiffness agrees with the elastic prediction over a wide regime of indentation depths, from a few to hundreds of nanometers. As shown in Fig. 1(b), the plateau value of the stiffness ratio agrees with the elastic prediction $4\mu^*/E^*$, while the transition at small indentation depths is due to the tangential stiffness reduction.

The significant reduction of the tangential contact stiffness relative to the elastic prediction in Fig. 1(b) occurs when the contact radius is below about 50–200 nm for the aluminum single crystal, fused silica, and several other materials. It is rather counterintuitive that a small oscillation in the tangential direction (i.e., 0.7 nm) causes such a significant deviation from the elastic solution for a contact radius of hundreds of nanometers. This paradox can be understood by interface microslip phenomenon [10,12]. For elastic tangential contact, the interface shear stress field, $\sigma_{xz}^{\text{elastic}} = T_x/2\pi a\sqrt{a^2-r^2}$, i.e., Mindlin solution, shows an inverse-square-root singularity at the contact edge, where $r = \sqrt{x^2+y^2}$ is the radial coordinate. Consequently, the two contacting surfaces may start to slip at the contact perimeter, and the annular slip zone propagates toward the center with an increase of the applied tangential force. The tangential shear stress at the contact edge may be limited by the Coulomb friction (i.e., $|\sigma_{xz}| \leq q|\sigma_{zz}|$ with the friction coefficient q) or limited by the shear strength τ_0 (i.e., the friction stress). An important question then arises as to whether we can validate the friction model from measurement of tangential contact stiffness, despite that the two models have different physical origins. In other words, will the Coulomb friction model and the friction stress model give a noticeable difference in terms of the tangential contact stiffness?

As shown in Fig. 1(a), the boundary value problem to be solved is an axisymmetric contact of two solids subjected to a remote tangential force, where the interface shear stress in the annular slip zone $c \leq r \leq a$ is either $\sigma_{xz}^{\text{Coulomb}}(r) = q|\sigma_{zz}|\text{sgn}(T_x)$ for the Coulomb friction model or $\sigma_{xz}^{\text{cohesive}}(r) = \tau_0$ for the friction stress

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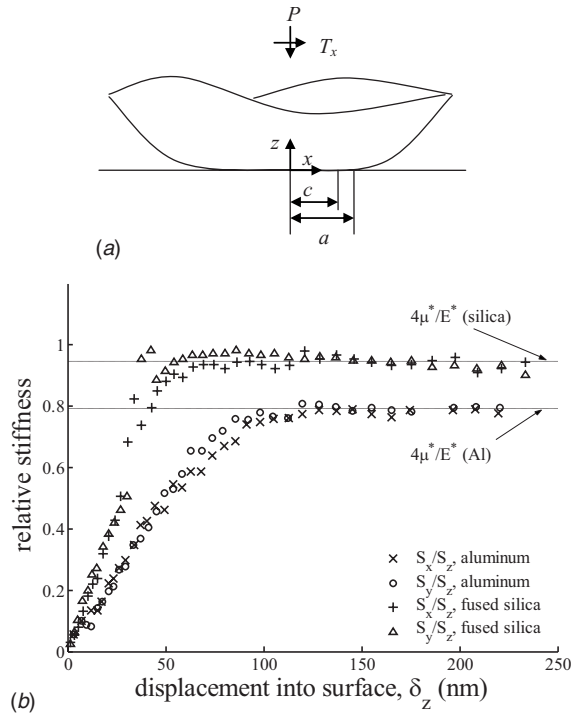


Fig. 1 (a) Geometric conventions used in the axisymmetric contact problem. The contact radius is a and the stick zone radius is c . The slip zone is assumed to be annular. (b) The ratio of measured tangential to normal contact stiffness is plotted against the penetration depth of a Berkovich diamond indenter into the surface of bulk aluminum single crystal and fused silica (original data from Dr. B. N. Lucas). Data at large indentation depths agree with the elastic contact prediction, and the transient region at small indentation depths will be used to evaluate the interfacial friction condition.

model. In the latter model, the interface shear stress distribution is independent of the normal stress distribution, while in the former model we need to know the relationship between P and a . The normalized mean pressure p_m/σ_Y for a Berkovich contact is a function of the effective strain $(E^*/\sigma_Y)\cot\alpha$ with the equivalent half apex angle α (page 176 in Ref. [12]). For fully developed plastic contact, $p_m = H\sigma_Y$ where the dimensionless prefactor H depends weakly on the apex angle and material sink-in or pileup. Finite element simulations show that the contact pressure distribution is qualitatively similar to the elliptical distribution when pileup occurs for soft materials [2]. Therefore, we can approximate this problem by the Cattaneo–Mindlin solution [12]. The interface shear stress distribution and tangential displacement at the reference point are therefore given by

$$\sigma_{xz}^{\text{Coulomb}}(r) = \begin{cases} \frac{3qP}{2\pi a^2} \left\{ \sqrt{1 - \left(\frac{r}{a}\right)^2} - \frac{c}{a} \sqrt{1 - \left(\frac{r}{c}\right)^2} \right\}, & r \leq c \\ \frac{3qP}{2\pi a^2} \sqrt{1 - \left(\frac{r}{a}\right)^2}, & c \leq r \leq a \end{cases} \quad (1a)$$

$$\delta_x^{\text{Coulomb}} = \frac{3qP}{16\mu^*a} \left[1 - \left(\frac{c}{a}\right)^2 \right] \quad (1b)$$

$$T_x^{\text{Coulomb}} = qP \left[1 - \left(\frac{c}{a}\right)^3 \right] \quad (1c)$$

for the Coulomb friction model, and

$$\sigma_{xz}^{\text{cohesive}}(r) = \frac{\tau_0}{\pi} \cos^{-1} \left\{ \frac{2c^2 - a^2 - r^2}{a^2 - r^2} \right\}, \quad r \leq c \quad (2a)$$

$$\delta_x^{\text{cohesive}} = \frac{\tau_0 a}{2\mu^*} \sqrt{1 - \left(\frac{c}{a}\right)^2} \quad (2b)$$

$$T_x^{\text{cohesive}} = 2\tau_0 a^2 \left[\cos^{-1} \left(\frac{c}{a}\right) + \frac{c}{a} \sqrt{1 - \left(\frac{c}{a}\right)^2} \right] \quad (2c)$$

for the friction stress model.

For both friction models, the tangential contact stiffness is found to be $S_x = dT_x/d\delta_x = 8c\mu^*$. The slip zone size c can be determined by the applied tangential load from Eq. (1c) or Eq. (2c), depending on which friction model is used, or by the applied tangential displacement from Eq. (1b) or Eq. (2b). In order to compare these results to the experiments, the tangential contact stiffness is given in normalized form:

$$\frac{S_x}{S_{x,\text{crt}}} = \begin{cases} \frac{a}{a_{\text{crt}}} \sqrt{1 - \left(\frac{a_{\text{crt}}\delta_x}{a\delta_{\text{max}}}\right)^\gamma}, & \frac{a\delta_{\text{max}}}{a_{\text{crt}}\delta_x} \geq 1 \\ 0, & \frac{a\delta_{\text{max}}}{a_{\text{crt}}\delta_x} < 1 \end{cases} \quad (3)$$

where a_{crt} is a length parameter to normalize the contact radius a , and $S_{x,\text{crt}} = 8a_{\text{crt}}\mu^*$ is used to normalize the tangential contact stiffness. From Eqs. (1b) and (2b), a_{crt} and γ can be derived, being $a_{\text{crt}}^{\text{Coulomb}} = 16\mu^*\delta_{\text{max}}/3\pi qp_m$ and $\gamma^{\text{Coulomb}} = 1$ for the Coulomb friction model, and $a_{\text{crt}}^{\text{cohesive}} = 2\mu^*\delta_{\text{max}}/\tau_0$ and $\gamma^{\text{cohesive}} = 2$ for the friction stress model. From Eq. (3), when $a\delta_{\text{max}}/a_{\text{crt}}\delta_x \gg 1$, the tangential contact stiffness is indistinguishable from the elastic contact solution and the size of the stick zone approaches the contact radius. As $a\delta_{\text{max}}/a_{\text{crt}}\delta_x$ decreases, the slip zone size increases, and the tangential contact stiffness can be significantly lower than the elastic solution. The tangential contact stiffness becomes zero when $a\delta_{\text{max}}/a_{\text{crt}}\delta_x < 1$. The dependence of the normalized tangential contact stiffness on this dimensionless parameter $a\delta_{\text{max}}/a_{\text{crt}}\delta_x$ has different power exponents γ in the two friction models. Consequently, the difference can be evaluated by fitting the theoretical prediction to the experimental measurements in Fig. 1.

In order to describe the stiffness measured by the continuous stiffness measurement method (in which a small harmonic force is applied to the indenter and the harmonic response of the displacement and phase angle shift are measured), we approximate it by a weighted average over $\delta_x \in [0, \delta_{\text{max}}]$, namely, $\bar{S}_x = (1/\delta_{\text{max}}) \int_0^{\delta_{\text{max}}} w S_x d\delta_x$, where the weight function can be chosen as $w=1$ (uniform weighted average) or $w=(\pi/2)\sin(\pi\delta_x/2\delta_{\text{max}})$ (biased weighted average). The weighted average is used as an approximation in light of the vibration analysis.

3 Discussion

The composite shear modulus μ^* and Poisson's ratio ν can be obtained from the data in the large indentation-depth range in Fig. 1(b). The amplitude of the displacement oscillation in continuous stiffness measurements is $\delta_{\text{max}} = 0.7$ nm. The relationship between the contact radius and the indentation depth can be calibrated using the Oliver–Pharr approach and the materials with known elastic constants [1,2,9]. Consequently, the only adjustable parameter in Eq. (3) is a_{crt} , which is fitted so as to minimize the mean square value of the difference between measurements in Fig. 1(b) and the two weighted averages for Eq. (3). The results for the two friction models are presented in Fig. 2. It is seen that Fig. 2(a) gives much better agreement than Fig. 2(b), suggesting the valid-

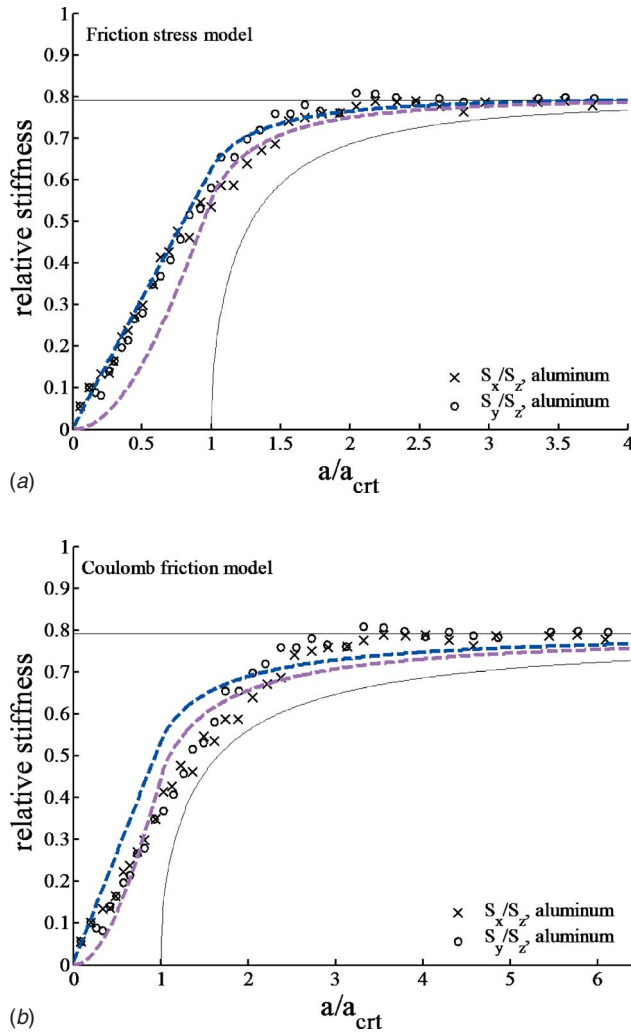


Fig. 2 Stiffness measurements of aluminum single crystal are compared to the predictions by the two friction models. Least squares fittings give rise to $a_{crt}^{cohesive}=192$ nm for the friction stress model in (a), and $a_{crt}^{Coulomb}=118$ nm for the Coulomb friction model in (b). The straight and curved solid lines correspond to stiffness ratio at $\delta_x=0$ and $\delta_x=\delta_{max}$, respectively. The two dashed lines are computed from the uniform weighted average (top curve, blue color online) and biased weighted average (bottom curve, magenta color online). The friction stress model leads to better agreement with the experiments.

ity of the friction stress model. This can be understood from the different exponents γ associated with $a_{crt}\delta_x/a\delta_{max}$ in Eq. (3). For Coulomb friction, the tangential stiffness varies slowly with respect to the contact size in Eq. (3), but the experimental data agree better with a rapidly varying relationship. The best fit in Fig. 2(a) gives an error of about 1%, and that in Fig. 2(b) about 8%. The best fit in Fig. 2(b) for the Coulomb friction model gives $a_{crt}^{Coulomb}=118$ nm. Using $\mu^*=15$ GPa, $\delta_{max}=0.7$ nm, and $p_m=105$ MPa (assuming $H=3$ and $\sigma_y=35$ MPa), we get a friction coefficient $q=0.7$, which is much higher than the typical value

(0.1–0.15 for the friction between diamond and most metals). In contrast, Fig. 2(a) gives $a_{crt}=192$ nm and $\tau_0=116$ MPa, which is a reasonable value for the interfacial shear strength in most metals. If we assume that the contact pressure is uniform and equal to a fraction of the hardness, the stress field solutions of the Coulomb friction model will be the same as that of the friction stress model, with the substitution of τ_0 by $qH\sigma_y$. Least squares fitting will give rise to a friction coefficient $q\approx 1.0$. Even if we consider strain gradient effects so that σ_y can be several times larger than the bulk value, the fitted friction coefficient is still unreasonably high. Consequently, we conclude that the friction stress model gives a better prediction than the Coulomb friction model. Moreover, the interfacial shear strength between diamond indenter and fused silica is about $\tau_0=295$ MPa, so that both materials in Fig. 1 give rise to a ratio μ^*/τ_0 of about 100.

In summary, based on the tangential and normal contact stiffness measurements from a multidimensional nanocontact system, this paper compares two friction models. Interface microslip is responsible for the reduction of tangential contact stiffness, while quantitative comparison favors the friction stress model than the Coulomb friction model. This suggests that the friction stress model be used to evaluate the correction factor in the contact stiffness equation.

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